DECOMPOSITION OF UNCERTAINTY IN RELATIONAL SYSTEMS

MARCIN PĘSKI

Abstract. A joint distribution of an infinite collection of random variables $\theta(x), x \in X$ is *exchangeable*, if the joint distributions of any two finite tuples of variables of the same length are equal. A famous result by de Finetti shows that each random variable $\theta(x)$ can be decomposed as an outcome of two kinds of independent shocks: an aggregate shock that affects all variables in the same way and a collection of i.i.d. idiosyncratic shocks that affect each variable separately.

In this paper, we present a generalization of the de Finetti's Theorem. We assume that all tuples of variables of a given length are divided into finitely many *classes of analogy*. A joint distribution of all random variables is *invariant* if the distributions of analogous tuples of variables are equal. Under the finite dimensionality assumption on the system of analogies, we show that each random variable $\theta(x)$ can be decomposed into finitely many independent shocks. These may include the aggregate shock that affects all variables, idiosyncratic shocks that affect each variable separately, and shocks that affect the non-trivial subset of variables.

1. Introduction

An infinite collection of random variables $\theta(x), x \in X$ is *exchangeable,* if the marginal distribution over any finite tuple of variables is equal to the marginal distribution over any other tuple of the same length. For instance, let η and η_x for each x be i.i.d. random variables uniformly distributed on the interval $[0,1)$ and take any measurable function $f:[0,1)^2 \rightarrow Y$, where Y is a Borel space that contains values of variables $\theta(x)$. Then, the joint distribution of variables

$$\theta(x) := f(\eta, \eta_x) \text{ for each } x \in X. \tag{1.1}$$

is exchangeable. In fact, a famous result by de Finetti shows that each exchangeable collection has representation (1.1).[1] The interpretation is that uncertainty about the value of $\theta(x)$ can be decomposed into two different sources: an *aggregate shock* η that affects all variables

Department of Economics, University of Texas at Austin, 1 University Station #C3100, Austin, Texas 78712. . *Email*: mpeski@gmail.com. I am grateful for the hospitality of the Department of Economics at Princeton University where parts of this research were carried out.

[1]The original (and arguably, better known) version of de Finetti's Theorem says that any exchangeable sequence can be represented as a two-stage lottery: first, a distribution over space ΔY is drawn, and then each variable $\theta(x)$ is drawn independently from the distribution obtained in the first step. That version and representation (1.1) are equivalent.

$\theta(x')$ for all x', including $x' = x$, and an *idiosyncratic shock* η_x that uniquely affects variable $\theta(x)$.

The de Finetti's result is one of the most important ideas of the statistical decision theory. Exchangeability captures a simple assumption about the environment: the names of the variables or the order in which they are observed does not affect their distribution. Because of that, it has a wide range of applications (testing for product quality, marketting research, etc ...) The de Finetti's Theorem provides an easy-to-interpret representation of exchangeable distributions. Additionally, decomposition (1.1) has implications for learning, Bayesian decision theory (see, for example, Kreps (1988)) as well as a wide range of more immediate applications. Unfortunately, the decomposition is limited to the situations where exchangeability applies.

The literature (including de Finetti himself[2]) noticed that exchangeability has natural extensions. As an example, consider two infinite sequences of tosses with two different coins. It is reasonable to assume that any two tosses from the first coin have the same distribution as any other two tosses from the same coin. Because it is also reasonable to suspect that such two tosses might have a different distribution than a toss with one coin and a toss with another, the situation cannot be accurately described by exchangeability. In order to deal with such situations, de Finetti suggested a weaker notion of *partial exchangeability* and provided an appropriate representation result (de Finetti (1980)).

In this paper, we show that de Finetti's type of decomposition holds under a broad class of assumptions that are weaker than exchangeability. The primitive of the model is the binary relation of *analogy* between pairs of equal length tuples of elements of X. We treat two tuples as analogous if they are conceptually indistinguishable, i.e., there is no reason to think that the joint distribution of variables over the first tuple is different from the joint distribution over the variables over the second tuple. In the two-coin example, any two tosses of the same coin are analogous, but they are not analogous to the two tosses from two different coins. We say that the distribution ω of variables $\theta(x)$ is *invariant* (with respect to the analogy relation), if the marginal distributions over two tuples of random variables indexed with analogous tuples are equal. Consider the following examples:

[2]"But the case of exchangeability can only be considered as a limiting case: the case in which this 'analogy' is, in a certain sense, absolute for all the events under consideration .... To get from the case of exchangeability to other cases which are more general but still tractable, we must take up the case where we still encounter 'analogies' among the events under consideration, but without attaining the limiting case of exchangeability." (de Finetti (1980))

- *Exchangeability*: Let $\theta(n)$ be an outcome of the nth coin toss, and the order of tosses does not affect the distribution. Here, any two equal-length tuples of distinct elements are analogous.
- *de Finetti's partial exchangeability*: Let $\theta(i, n)$ be an outcome of nth toss with coin $i = 1, 2$ and the order of tosses of any coin does not affect the joint distribution. Two tuples are analogous if they have the same length and each toss of coin i in the first tuple corresponds to a toss of the same coin in the second tuple.
- *Row-column exchangeability* (Aldous (1981)): Suppose that X is an infinite matrix of customer-good pairs such that any two customers or any two goods are exchangeable. For each customer c and good p, let $\theta(c, p)$ be the (random) utility of customer c from good p. Two tuples are analogous if and only if one can be obtained from the other by exchanging the names of customers and/or goods. See Figure 1. Two tuples (x, z) and (w, u) can be obtained from each other by exchanging the names of goods p and p'; hence they are analogous. On the other hand, tuples (x, z), and (x, w) are not analogous.

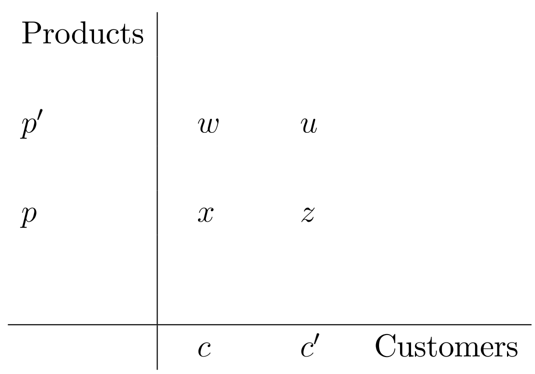


Figure 1

- *Time invariance*: Suppose that X is the set of integers interpreted as different periods. Two tuples of elements of X are analogous if and only if one can be obtained from the other by adding an integer.

We assume that the system of analogies satisfies natural consistency requirements. Additionally, we assume that the complexity of the system is bounded by a certain compactness assumption. The assumption says that the infinite relational system can be approximated by finite systems that grow at a sufficiently slow rate. The assumption is satisfied by the first three examples above, but not by the last example of time invariance. Given the assumptions, Theorem 1 shows that each invariant distribution is equal to the joint distribution of random variables defined as

$$\theta(x) := f_{k(x)}(\eta_{x,1}, ..., \eta_{x,n_0}), \tag{1.2}$$

where

- $\mathcal{U}$ is a collection of independent and uniformly distributed random shocks drawn the interval $[0,1)$,
- $\eta_{x,m} \in \mathcal{U}$ for each $x \in X$, $m \leq m_0 < \infty$,
- $k(x) \leq k_0 < \infty$ for each x, and
- $f_1, ..., f_{k_0} : [0,1]^{m_9} \to Y$ are finitely many measurable functions that may depend on the distribution ω.

In particular, each invariant distribution admits a de Finetti type of decomposition. Additionally, Theorem 3 shows that if functions f^{ω} are to satisfy some additional symmetry restrictions, then the existence of representation (1.2) is necessary and sufficient for invariance.

We discuss some implications of the main result. First, decomposition (1.2) reduces potentially complicated uncertainty about variables $\theta(x)$ to much simpler uncertainty about independent shocks η. It provides information about the correlations between individual variables $\theta(x)$ and $\theta(x')$. In particular, the two random variables $\theta(x)$ and $\theta(x')$ are correlated only insofar they are affected by the same shocks.

Second, we say that the set of elements affected by the same shock is a *domain* of the shock. It turns out that not all subsets of X can be domains; a domain must satisfy a certain stability property that can be stated purely in terms of the analogy relation. Because in typical applications that property is satisfied by sets that share common (example-specific) features, we refer to such sets as *concepts*. The number and the type of concepts depend on a particular example. In the exchangeability case, there are two types of concepts: the entire set X that forms a domain of the aggregate shock (i.e., the shock that determines the idiosyncratic distributions) and single-element concepts $\{x\}$ that form domains of the idiosyncratic shocks. In the case of row-column exchangeability, there are two additional types of concepts: the set of all observations associated with the same customer and the set of observations associated with the same good.

There is a natural interpretation of shock η as a variable that aggregates the (subjective or objective) properties of the domain. Then, representation (1.2) decomposes the uncertainty over $\theta(x)$ into the (independent) uncertainty about the properties of concepts that contain x.

Finally, decomposition (1.2) has implications for learning theory. Recall first that de Finetti's theorem is widely interpreted as the simplest model of induction. Suppose that a statistician uses past data $\theta(x'), x' \neq x$ to predict the value of yet unobserved variable $\theta(x)$. Because of representation (1.1), one can divide the prediction into two stages:

- *induction*, in which the past data are used to infer the value of the aggregate shock. Its value can be inferred from the past observations because it has the same impact on all past observations,
- *deduction*, in which the value (or the distribution) of variable $\theta(x)$ is predicted as a function of the aggregate shock.

In the general case of representation (1.2), the induction stage may involve inference of additional shocks that have smaller domains than the aggregate one. Moreover, prior to the induction stage, one can distinguish

- *conceptualization*, in which element $\theta(x)$ is identified as a member of larger sets of variables.

There is a substantial literature on various extensions of exchangeability (for overviews, see Diaconis (1988), Kallenberg (2005)). Row-column exchangeability and related cases are discussed in Aldous (1981). Hoover (1982) and Kallenberg (2005) contain further extensions. Other notions of exchangeability that are not covered by the present model include the extension to Markov chains presented in Diaconis and Freedman (1980).

Al-Najjar (1995) studies general (not necessarily invariant) distributions ω of random variables $\theta(x)$, $x \in X$. Space X is assumed to be a measurable continuum space with non-atomic measure $\mu \in \Delta X$. He shows that, up to zero μ-probability events, distribution ω decomposes into aggregate and idiosyncratic shocks. In this paper, X is discrete, there is no measure μ, and we find many different types of shocks. Jackson, Kalai, and Smorodinsky (1999) assume that $X = \mathbf{N}$, and that ω satisfies reverse mixing condition. They show that any such distribution can be decomposed into long-run (learnable) and short-run (unpredictable in the long-run) effects.

Section 2 defines the relation of analogy and states main assumptions. Section 3 shows that any invariant distribution in relational systems that satisfy appropriate compactness assumption has decomposition (1.2). Section 4 defines concepts and shows that decomposition (1.2) can be chosen so that all domains of shocks are concepts. Section 5 discusses some examples. The necessary and sufficient conditions for invariance are presented in section 6. Section 7 discusses the main ideas behind the proofs. Section 8 uses an example to show that without the compactness assumption the results of this paper may fail. The proofs can be found in the appendix.

2. Relational systems

Let X be a countably infinite with typical elements $x, x' \in X$. A typical k-tuple of the elements of X is denoted by $\bar{x} = (x_1, ..., x_k) \in X^k$. Let $\bar{x}\hat{\ }\bar{x}'$ denote a concatenation of tuples $\bar{x}$ and $\bar{x}'$. For any set $S \subseteq X$, any tuple $\bar{x} = (x_1, ..., x_k)$, we write $\bar{x} \subseteq S$ if $\{x_1, ..., x_k\} \subseteq S$.

An *enumeration* of set $S \subseteq X$ is a (possibly infinite) tuple $\bar{s} = (s_1, s_2, ...)$ that contains exactly each element of set S once. The enumeration is infinite, if set S is infinite. Whenever we want to fix the enumeration of set S, but the choice of enumeration is not important, we write $\bar{S}$.

Let $\sim$ be an equivalence relation on $\bigcup_k X^k$ such that, for any two tuples $\bar{x}, \bar{x}' \in \bigcup_k X^k$, if $\bar{x} \sim \bar{x}'$, then $\bar{x}, \bar{x}' \in X^k$ for some k. Relation $\sim$ is called an *analogy relation*, if it is reflexive, i.e., $\bar{x} \sim \bar{x}$ for each tuple $\bar{x}$, and satisfies the following axioms: for any k, any two tuples $(x_1, ..., x_k) \sim (x'_1, ..., x'_k)$,

- *invariance to permutations*: $\left(x_{\pi(1)}, ..., x_{\pi(k)}\right) \sim \left(x'_{\pi(1)}, ..., x'_{\pi(k)}\right)$ for any permutation $\pi : \{1, ..., k\} \to \{1, ..., k\}$,
- *internal consistency*: $(x_1, ..., x_{k-1}) \sim \left(x'_1, ..., x'_{k-1}\right)$,
- *external consistency*: for any x, there exists x' such that $(x_1, ..., x_k, x) \sim (x'_1, ..., x'_k, x')$

For any two k-tuples $\bar{x} \sim \bar{x}'$, we say that tuples $\bar{x}$ and $\bar{x}'$ are *analogous.* An equivalence class $[\bar{x}] \subseteq X^k$ of tuple $\bar{x} \in X^k$ is called a *type* of tuple $\bar{x}$. Set X together with analogy relation $\sim$ is called a *relational system* $(X, \sim)$.

The analogy relation encodes prior information about the elements of X. Two analogous tuples are treated as indistinguishable. If two tuples of elements are indistinguishable, then neither the same reordering nor the removal of some elements should make the tuples distinguishable. Similarly, if two tuples are indistinguishable, then one should not be able to tell them apart by looking at their relations to elements outside the tuples.

A relational system has *finitely many types of* 1*-tuples*, if $|\{[x] : x \in X\}| < \infty$. A relational system is *transitive*, if there is only one type of 1-tuples, i.e., $X = [x]$ for each x.

For each $U \subseteq X$, let $\sim_U$ be the restriction of the analogy relation $\sim$ to the tuples of the elements of U. A finite set $U \subseteq X$ is *local* if $\sim_U$ is an analogy relation. In particular, the restriction $\sim_U$ must satisfy external consistency: for all tuples $\bar{x}, \bar{x}' \in U$ if tuples $\bar{x}$ and $\bar{x}'$ are analogous (as elements of X), then for each $x \in U$, there exists $x' \in U$ such that tuples $x\hat{\ }\bar{x}$ and $x'\hat{\ }\bar{x}'$ are analogous.

A relational system $(X, \sim)$ is ψ*-compact*, if there exists finite $U_0 \subseteq X$ such that for each local $U \supseteq U_0$, for each $x \in X$, there exists local set $U' \supseteq U, x$ such that

$$\log |U'| \leq \psi + \log |U| . \tag{2.1}$$

The base of the logarithm is always equal to 2.

We interpret local sets as finite approximations of the infinite relational system. The compactness assumption says that the relational system can be approximated by finite relational systems and the cardinality of approximations does not grow too quickly. The condition puts a bound on the complexity of relations between the elements of X : the more complex are relations between x and elements of a local set U, the less likely that there is a small set $U' \supseteq U, x$ that satisfies external consistency. All but one example of relational systems (including the one discussed immediately below) are ψ-compact for each $\psi \leq \frac{1}{20}$.[3] Section 8 presents an example of a 1-compact relational system.

2.1. **Example: Multiple customers and goods.** We use the row-column exchangeability example mentioned in the introduction to illustrate the definitions and the results of the paper. Consider a statistician who studies purchases in a population of customers. Let $X = C \times P$, where C is an infinite set of customers, and P is an infinite set of goods. Let $\theta(c, p)$ be the utility of customer c from purchasing good p. Suppose that the statistician has no prior information that leads to meaningful differences between customers or goods. For instance, the statistician identifies customers by their names, but there are no reasons to believe that the knowledge of names does not help in predicting purchases.

Define two equivalence relations on set X : for each $x = (c, p)$, and $x' = (c', p')$,

$$
\begin{aligned}
xR_C x' &\text{ if and only if } c = c', \\
xR_P x' &\text{ if and only if } p = p'.
\end{aligned}
$$

For each two tuples $\bar{x}$ and $\bar{x}'$ of the same length k, say that the tuples are analogous, $\bar{x} \sim \bar{x}'$ if and only if for each $l, m \leq k$,

$$
\begin{aligned}
x_l R_C x_m &\text{ if and only if } x'_l R_C x'_m, \text{ and} \\
x_l R_P x_m &\text{ if and only if } x'_l R_P x'_m.
\end{aligned}
$$

It is easy to check that so defined relation $\sim$ is reflexive and it satisfies the other properties of the analogy relation.

We show that the relational system $(X, \sim)$ is $\frac{1}{20}$-compact. It is easy to notice that all finite sets $C_0 \subseteq C$, and $P_0 \subseteq P$, set $C_0 \times P_0$ is local. In fact, there is a finite set U_0 such that each local $U \supseteq U_0$ is equal to $C_0 \times P_0$ for some finite C_0 and P_0. For each $x = (c, p)$, and each C_0, P_0, let

$$
C'_0 = C_0 \cup \{c\} \text{ and } P'_0 = P_0 \cup \{c\} .
$$

[3] In fact, it follows from the proof of our main result that if the relational system has finitely many types of 1-tuples and it is $\frac{1}{20}$-compact, then it is ψ-compact for any $\psi > 0$.

Then, for sufficiently large C_0 and P_0,

$$\frac{|C_0' \times P_0'|}{|C_0 \times P_0|} \leq \frac{(|C_0|+1)(|P_0|+1)}{|C_0||P_0|} \leq 2^{\frac{1}{20}}.$$

3. Invariant distributions

Let $\omega \in \Delta(Y)^X$ denote the joint distribution of variables $\{\theta(x)\}_{x\in X}$. Distribution ω is $(X,\sim)$-*invariant* if for any two analogous tuples $\bar{x} \sim \bar{x}' \in X^k$, any Borel sets $U_1, ..., U_k \subseteq Y$,

$$\omega(\theta(x_i) \in U_i, i \leq k) = \omega(\theta(x_i') \in U_i, i \leq k).$$

Invariant distributions can be treated as functions of the tuples of the elements of X into the space of the distributions of the tuples of the elements of Y. Then, loosely speaking, invariant distributions are measurable with respect to partitions of the space of the tuples induced by the classes of analogy.

Let $\mathcal{U} = \{\eta_i\}$ be an infinite collection of i.i.d. random variables, all uniformly distributed on interval $[0,1]$. Take any function $k : X \to \{1, ..., k_0\}$ and $n : X \to \mathcal{U}^{m_0}$ for some finite $k_0, m_0 < \infty$. We refer to k and n as a pair of assignments. Distribution ω *admits* (k,n)-*decomposition,* if there are measurable functions $f_1^\omega, ..., f_{k_0}^\omega : [0,1]^{m_0} \to Y$ such that ω is equal to the joint distributions of variables

$$\theta(x) := f_{k(x)}^\omega(n_1(x), ..., n_{m_0}(x)),\ x \in X. \tag{3.1}$$

In other words, each variable $\theta(x)$ can be decomposed into finitely many independent shocks, and the decomposition uses one of finitely many different functions.

If X is finite, then any (not necessarily invariant) distribution has a finite decomposition. Similarly, for any distribution, there exists a decomposition with one random shock, but infinitely many aggregating functions.[4] The main result of this paper is that under the appropriate compactness assumption, any invariant distribution admits a finite decomposition.

Theorem 1. *Suppose that X is a countably infinite, relational system $(X,\sim)$ finitely many types of 1-tuples and it is $\frac{1}{20}$-compact. Then, there is a pair of assignments k and n, so that each $(X,\sim)$-invariant distribution ω admits (k,n)-decomposition.*

Theorem 1 presents the conditions necessary for invariance on relational systems. Each invariant distribution has a finite decomposition, i.e., (k,n)-decomposition for some assignments k, and n.

[4]Standard results show that for each Borel space $\bar{Y}$ (a category that includes Polish spaces), each distribution $\omega \in \Delta Y$, there exists a measurable function $g^\omega : [0,1] \to Y$ such that ω is equal to the distribution of $g^\omega(\eta)$, where η is uniformly distributed on the interval $[0,1]$. For example, if $\bar{Y} = Y^X$, then distribution ω has a decomposition with infinitely many aggregating functions $f_x^\omega(\eta) := (g(\eta))_x$.

If there are infinitely many types of 1–tuples, then there are distributions without finite decomposition and the thesis of the Theorem does not hold.[5]

Some restrictions on the complexity of the relational system are necessary for finite decompositions. It is not difficult to find examples that are not compact and that do not satisfy the thesis of the Theorem (for instance, the time invariance case from the introduction is one of them). A more narrow question is whether constant $\frac{1}{20}$ in the statement of the Theorem can be increased. Although we suspect that $\frac{1}{20}$ is not the best possible, we show that the constant cannot be chosen too high. Specifically, the Theorem fails, i.e., there are invariant distributions without finite decompositions, in the 1-compact example from Section 8.

The representation 3.1 is not unique. To see why, consider any measure preserving bijection $o : [0,1] \to [0,1]$. Then, the joint distribution of variables (3.1) is equal to the joint distribution of variables

$$\theta'(x) := f^{\omega}_{k(x)}\left(o \circ n_1(x), n_2(x), ..., n_{m_0}(x)\right), \; x \in X.$$

Finally, the existence of finite decomposition is not sufficient for the distribution ω to be invariant. Theorem 3 below presents the necessary and sufficient conditions for invariance. Theorem 1 is a corollary to the more comprehensive Theorem 3 below.

3.1. **Example: Multiple customers and goods.** In our example, distribution ω is invariant if and only if it remains unchanged under (separate) permutations of customers and products. Aldous (1981) calls such distributions row-column exchangeable. He shows that, for any row-column exchangeable distribution ω, there exists a measurable function $f : [0,1)^4 \to \{0,1\}$, such that ω is equal to the joint distribution of variables

$$\theta(c,p) := f\left(\eta_X, \eta_c, \eta_p, \eta_{(c,p)}\right) \text{ for each } (c,p) \in X. \tag{3.2}$$

Each variable $\theta(c,p)$ is a composition of four types of shocks: η_X is the aggregate shock with the domain equal to the entire space X, η_c is the customer-specific shock with the domain equal to set $S_c = \{(c,p') : p' \in P\}$, η_p is the good-specific shock with the domain equal to $S_p = \{(c',p) : c' \in C\}$, and $\eta_{(c,p)}$ is the idiosyncratic shock with the single-element domain (c,p). In particular, decomposition (3.2) is a special case of Theorem 1.

4. Domains of shocks

One of the implications of the decomposition (3.1) is that there are sets of variables that possibly contain more than one element, and that are affected by the same shocks. We call

[5] Suppose that $f_1, f_2 ... : [0,1] \to Y$ is a sequence of pairwise different measurable functions. Let $x_1, x_2, ...$ be a sequence of elements such that no two elements have the same type. For each x that is analogous to x_n, define $\theta(x) := f_n\left(\eta_{[x_n]}\right)$. The joint distribution of $\theta(x)$ is invariant, but the representation requires infinitely many aggregating functions.

such sets the domains of the shock. Formally, say that $x \in X$ *is not affected by shock* $\eta \in \mathcal{U}$, if for almost all realizations of shocks $\eta' \in \mathcal{U} \backslash \{\eta\}$, $f^{\omega}_{k(x)}(n_1(x), ..., n_{m_0}(x))$ is an almost surely constant function of the realization of shock η. In particular, if $\eta \notin \{n_m(x), m \leq m_0\}$, then x is not affected by η. Define the *domain of* η, $D(\eta)$ as the set of elements of X that are affected by η.

Next, we characterize a certain stability of the domain. For any $x \in X$, say that two (possibly infinite) sets S, S' are *analogous relative to* x, if there exist enumerations $s_1, s_2, ...$ and $x_1, x_2, ...$ of sets, respectively, S and $X \backslash S$, and enumerations $s'_1, s'_2, ...$, and $x'_1, x'_2, ...$ of sets, respectively, S' and $X \backslash S'$ such that for any m, tuples $(x, s_1, x_1 ..., s_m, x_m,)$ and $(x, s'_1, x'_1, ..., s'_m, x'_m)$ are analogous. Informally, two sets are analogous relative to x if they have similar positions relative to x.

Say that set $S \subseteq X$ is the *concept* if the number of sets S' that are analogous to S relative to x is finite and uniformly bounded across all $x \in S$,

$$i_S := \sup_{x \in S} |\{S' : S \text{ is analogous relative to } x\}| < \infty.$$

We refer to i_S as the index of concept S.

For example, sets X and $\{x\}$ for each $x \in X$ are concepts with index 1 in any relational system. We refer to such sets as *trivial concepts*. Examples of non-trivial concepts are presented below. In a typical application, concepts consist of elements that "share certain feature." The exact meaning of "sharing a feature" depends on the particular application. In contrast, our definition of a concept applies to all relational systems.

Theorem 2. *Take the same assumptions as in Theorem 1. Then, the pair of assignments k and n can be chosen so that for each shock η, $D(\eta)$ is a concept.*

In the appendix, we show that each element x of a $\frac{1}{20}$-compact relational system with finitely many 1-types belongs to finitely many concepts (see also Section 7.2.5). Thus, there are significantly fewer concepts than all sets. Moreover, the enumeration of all concepts is typically easy. This fact helps the application of Theorem 1. In order to find a decomposition of an invariant distribution, one has to consider all possible domains of the shocks. This task is easier if the domains must belong to a (relatively) small class of concepts.

A simple heuristics explains why the domains should be concepts. Suppose that the domain of shock η is not a concept and there exists $x \in D(\eta)$ such that there are infinitely many sets D that are analogous to $D(\eta)$ relative to x. For each such D, find a shock η'_D such that $D(\eta') = D$. By invariance, $\theta(x)$ must be affected by each shock η_D *in exactly the same way* as by the shock η. If there are infinitely many sets D that are analogous to $D(\eta)$ relative to x, then $\theta(x)$ is affected *in exactly the same way* by infinitely many independent shocks. But this is impossible, unless the effect of η on $\theta(x)$ is equal to 0 and $x \notin D(\eta)$.

4.1. **Example: Multiple customers and goods.** There are only four types of concepts: the trivial concepts X and $\{x\}$ for $x \in X$, the concept of customer S_c for some c, and the concept of good S_p for some p. (Sets S_c and S_p are defined in Section 3.1.) The formal argument is presented in Appendix A. Here, we discuss two examples. See Figure 2. Set S_c consists of all observations associated with customer c. It is easy to check that any set that is analogous to S_c relative to any $x \in S_c$ must be equal to S_c. On the other hand, set $S = S_c \cup S_{c'}$ consists of all observations associated with customers c or $c' \neq c$. Relative to x, S is analogous to any other set $S' = S_c \cup S_{c''}$ that consists of observations associated with customers c or $c'' \neq c$. Because there are infinitely many such sets, S cannot be a concept.

		x	
		S_1	
P/C		c	

,

		x				
		S_2		S_2		
		S_2'				S_2'
P/C		c		c'		c''

Figure 2.

5. Examples

We discuss examples of relational systems. As in the main example, it is often easier to describe the analogies using other, more primitive relations. A k-ary relation $R^{(k)}$ on X is defined as a subset $R^{(k)} \subseteq X^k$. Let $\mathcal{R} = \left\{R_i^{\{k_i\}}\right\}_{i\in I}$ be a collection of relations on X. For any two tuples $\bar{x} = (x_1, ..., x_k), \bar{x}' = (x_1', ..., x_k')$, say that tuples $\bar{x}$ and $\bar{x}'$ are $(\mathcal{R})$-*inner analogous*, if and only if for each $i \in I$, $l_1, ..., l_{k_i} \leq k$,

$$\left(x_{l_1}, ..., x_{l_{k_i}}\right) \in R_i \text{ iff } \left(x_{l_1}', ..., x_{l_{k_i}}'\right) \in R_i.$$

The inner analogy satisfies invariance to permutations and internal consistency. Additionally, if the inner analogy satisfies external consistency, then the inner analogy is an analogy relation; we say that the analogy relation is induced by $\mathcal{R}$ and write $(X, \mathcal{R})$ for the induced relational system.

External consistency is satisfied in all the examples below.

5.1. **Trivial system.** In this example, we formally describe the relational system in which invariance is equivalent to de Finetti's exchangeability. Let $'=' \subseteq R^2$ be the binary relation of equality: for any $x, x' \in X$, $x\hat{\ }x' \in '=' $ iff $x = x'$. In relational system $(X, '=')$, any two tuples of an equal number of distinct elements are analogous. In particular, invariance with respect to analogies induced by $'='$ is equivalent to exchangeability. Additionally, it is easy to check that any finite subset of X is local and the system is $\frac{1}{20}$-compact.

The trivial concepts, the entire space X and one-element sets $\{x\}$ for each $x \in X$, are the only concepts of the system $(X, '=')$. A version of de Finetti's Theorem says that any exchangeable distribution ω is equal to the joint distribution of (1.1) (see Kallenberg (2005)).
.

5.2. **Multiple goods with multiple disconnected customers.** Next, we discuss two versions of the main example from Section 2.1. First, we show that a removal of some relations from the relational system may reduce the collection of concepts. Let $(X, \sim)$ be the relational system from Section 2.1. Let relation R_P be defined as above and define a new analogy relation $\sim^*$ be induced by relation R_P and the equality relation.

The relational system $(X, \sim^*)$ describes analogies in a situation in which the data collected do not ensure that the population of customers of one product is in any way related to the population of the customers of the other product. The customers of two different products may be coded with the same label, but there is no reason to believe that observations (c, p) and (c, p') are associated with the same customer. For example, for any customers c, c' and any products p, p',

$$((c,p),(c,p')) \sim^* ((c,p),(c',p')).$$

To see the difference between analogy relations $\sim^*$ and $\sim$, notice that

$$((c,p),(c,p')) \nsim ((c,p),(c',p')).$$

There is only one type of non-trivial concept in the relational system $(X, \sim^*)$: concept of product $S_p = \{(c,p) : c \in C\}$. (This claim and other similar claims in this Section can be proven with similar methods as those applied in Appendix A. We omit the details.) For each invariant ω, there is a measurable function $f : [0,1]^3 \to R$ such that ω is equal to the joint distribution of

$$\theta(c,p) = f\left(\eta_X, \eta_p, \eta_{(c,p)}\right), \text{ for each } (c,p) \in X, \tag{5.1}$$

where η_X is the aggregate shock, η_p is the shock associated with concept S_p, and $\eta_{(c,p)}$ is the idiosyncratic shock. Notice that the only difference between (5.1) and (3.2) is that the former does not include the customer-specific shock.

5.3. **Customers, goods, incomes, and prices.** Here, we describe a non-transitive version of the example from Section 2.1. Suppose that the statistician studies the distribution of purchases of infinitely many goods together with a distribution of customers' income and the prices of goods. Let $X = C \times P \cup C \cup P$ and let $\theta(c,p) \in \{0,1\}$ be the indicator of a purchase, $\theta(c) \in R$ be the income of customer c, and $\theta(p)$ be the price of good p.

Define relations on X : for each $x, x' \in X$

- unary R^0_C: $R^0_C x$ if $x \in C$,
- unary R^0_P: $R^0_P x$ if $x \in P$,
- binary R_C: $x R_C x'$ if and only if x and x' refer to the same customer (for example, $x = c$ and $x' = (c, p)$ for some $c \in C$ and $p \in P$),
- binary R_P: $x R_P x'$ if and only if x and x' refer to the same good (for example, $x = p$ and $x' = (c, p)$ for some $c \in C$ and $p \in P$).

Let $\mathcal{R}_{CPIP} = \{R^0_C, R_C, R^0_P, R_P\}$. The relational system induced by $\mathcal{R}_{CPIP}$ is not transitive: the unary relations divide space X into three types of 1-tuples: purchase decisions $C \times P$, incomes of customers C, and prices of goods P.

Similarly as in Section 2.1, we show that the relational system $(X, \mathcal{R}_{CPIP})$ is $\frac{1}{20}$-compact. We enumerate all concepts. For each customer c and product p, define

$$S^0_c = \{c\} \times P, S_c = \{c\} \cup S^0_c,$$
$$S^0_p = C \times \{p\}, S_c = \{p\} \cup S^0_p.$$

For example, $S_c = \{c\} \cup S^0_c$ is the set of all observations associated with customer c (including c's income). All concepts are either trivial or belong to one of the above type.

Theorems 1 and 2 show that each invariant distribution has a decomposition into shocks with concept domains. In fact, for each invariant ω, there are measurable functions $f : [0,1)^4 \to \{0,1\}$, and $f_C, f_P : [0,1)^2 \to \mathbf{R}$ such that ω is equal to the joint distribution of variables

$$\theta(c) := f_C\left(\eta_X, \eta_{S(c)}\right) \text{ for each } c \in C,$$
$$\theta(p) := f_P\left(\eta_X, \eta_{S(p)}\right) \text{ for each } p \in P,$$
$$\theta(c, p) := f_{C\times P}\left(\eta_X, , \eta_c, \eta_p, \eta_{(c,p)}\right) \text{ for each } c \times p \in C \times P,$$

where η_X is the aggregate shock, η_c is the customer-specific shock, η_p is the product-specific shock, and $\eta_{(c,p)}$ is the idiosyncratic shock. Note that even if the variables $\theta(c,p)$, $\theta(c)$, and $\theta(p)$ are generated through different aggregating functions $f_{C\times P}$, f_C, and f_P, they can be correlated with each other through common shocks $\eta_X, \eta_{S(c)}$, and $\eta_{S(p)}$.

5.4. **Bundles of goods.** The last two examples introduce two ideas that appear in the necessary and sufficient conditions for invariance: symmetry of the aggregating function $f(.)$ and orientations of shocks. Additionally, the example described in this section has a concept with an index that is higher than 1.

Suppose that a company studies the demand for bundles of goods. Let P be a countable set of goods. Define $X = \{x \subseteq P : |x| = k\}$ as the set of k-element subsets of P. For each

$l \leq k$, define binary relations R_l : for each $x, x' \in X$,

$$xR_l^S x' \text{ iff } |x \cap x'| = l.$$

In other words, two elements of X are in relation R_l to each other, if their intersection has exactly l elements.

For simplicity, we focus on the case of $k = 2$. Consider the relational system generated by relations R_1 and R_2. There is only one type of non-trivial concept: Let $S(p) = \{x \in X : p \in P\}$ be the set of all observations associated with good p. One checks that set $S(p_2)$ is the only set apart from $S(p_1)$ that is analogous to $S(p_1)$ relative to bundle $\{p_1, p_2\}$. In particular, $S(p)$ is a concept with index 2. Any invariant ω is equal to the joint distribution of

$$\theta\{p_1, p_2\} := f\left(\eta_X, \eta_{p_1}, \eta_{p_2}, \eta_{\{p_1,p_2\}}\right), \tag{5.2}$$

where $f : [0,1)^4 \to Y$ is measurable, η_X is the aggregate shock, η_p is the product-specific shock, and $\eta_{\{p_1,p_2\}}$ is the idiosyncratic shock.

In order for the joint distribution of (5.2) to be invariant, it is necessary and sufficient to require that the value of the aggregating function f does not change with a permutation of the second and third coordinates: for all realizations $\eta_X, \eta_{p_1}, \eta_{p_2}, \eta_{(p_1,p_2)} \in [0,1)$,

$$f\left(\eta_X, \eta_{p_1}, \eta_{p_2}, \eta_{(p_1,p_2)}\right) = f\left(\eta_X, \eta_{p_2}, \eta_{p_1}, \eta_{(p_1,p_2)}\right). \tag{5.3}$$

Intuitively, the label of each bundle $\{p_1, p_2\}$ does not depend on the ordering of goods p_1 and p_2, and the realization of variable $\theta\{p_1, p_2\}$ should be the same if p_1 were switched with p_2.

5.5. **Multiple customers and two goods.** Finally, consider yet another version of the example from Section 3.1, but with two goods only, $P = \{p_1, p_2\}$. Suppose that the statistician does not have any prior information that distinguishes between the two goods. The relational system is induced by the same (appropriately restricted) relations R_C and R_P. Then, for each invariant ω, there exists measurable functions $f_1, f_2 : [0,1)^2 \to \{0,1\}$ such that ω is equal to the joint distribution of variables

$$\theta(c, p_i) = f_i(\eta_X, \eta_c), \tag{5.4}$$

where η_c is the customer-specific shock.

Additionally, distribution (5.4) is invariant if that for all η_X and η_c,

$$f_2(\eta_X, \eta_c) = f_1(1 - \eta_X, 1 - \eta_c). \tag{5.5}$$

(The formal derivation can be found in Section 7.1.7.) We say that the shocks come in one of two orientations: η and $1 - \eta$. Each orientation is associated with one of the goods.

Section 6 formally defines the orientations and uses them to state the sufficient and necessary conditions for invariance.

6. System of orientations

In this section, we derive the necessary and sufficient conditions for invariance. We use two ideas that were introduced in the last two examples of Section 5. First, we add a possibility that the shocks come with orientations. Second, we describe a notion of symmetry for aggregating functions.

6.1. **Orientations.** Measurable mapping $q : [0,1) \to [0,1)$ is an *orientation*, if it preserves Lebesgue measure λ, i.e., for each measurable $E \subseteq [0,1)$, $\lambda(q(E)) = q(E)$. A finite set of orientations P is *regular* if $\mathrm{id} \in P$, for each $q, q' \in P$, $q \circ q' \in P$, and there exists an interval $I_0 \subseteq [0,1)$ such that $\{q(I_0), q \in P\}$ is a partition of $[0,1)$ into $|P|$ subintervals.

Example 1. *For each $\eta \in [0,1)$, let $\mathrm{id}(\eta) = \eta$, and $q_0(\eta) = 1 - \eta$. Then $\mathrm{id} = q_0^2$ and q_0 are orientations, and sets $\{\mathrm{id}\}$ and $\{\mathrm{id}, q_0\}$ (but not $\{q_0\}$) are regular sets of orientations with, respectively, intervals $[0,1)$ and $\left[0, \frac{1}{2}\right)$.*

Let $(X, \sim)$ be a relational system. Let $\mathcal{U}$ be a collection of i.i.d. random shocks uniformly distributed on the interval $[0,1)$. A realization of all shocks in $\mathcal{U}$ is denoted as $u \in [0,1)^{\mathcal{U}}$. Let λ^U be the joint distribution of shocks u, i.e., $\lambda^{\mathcal{U}}$ is the product of Lebesgue distributions on the interval $[0,1)$. For each shock $\eta \in \mathcal{U}$, let Q_η be a finite and regular set of orientations. Let $\mathcal{O} = \bigcup_{\eta \in \mathcal{U}} \{\eta\} \times Q_\eta$. Each element $o = (\eta, q) \in \mathcal{O}$ is called an *orientation of shock* η.

A *system with orientations* of $(X, \sim)$ is a relational system $(X \cup \mathcal{O}, \sim)$ such that

- $\sim$ is the extension of the original relation of analogy from set X to $X \cup \mathcal{O}$, and
- for each tuple of orientations $\bar{o} = ((\eta, q_1), ..., (\eta, q_m))$ of shock η, if $\bar{o}$ is analogous to tuple $\bar{o}'$ then there are shocks η' and $q \in Q_{\eta'}$ such that $Q_\eta = Q_{\eta'}$ and $\bar{o}' = ((\eta', q_1 \circ q), ..., (\eta', q_m \circ q))$.

The first part of the definition ensures that the original analogy relation and the analogy relation in the system with orientations agree on X. The second part ensures that (a) tuples of orientations of the same shock η are analogous only to tuples of orientations of the same shock η' (possibly, $\eta' \neq \eta$), and that (b) the regular structure of orientations is preserved by analogies.

6.2. **Symmetric functions.** Orientations of shocks affect their values in the following way. Take any realization of shocks $u \in [0,1)^{\mathcal{U}}$. For each tuple of orientations

$$\bar{o} = ((\eta_1, q_1), ..., (\eta_n, q_n)),$$

define the n-tuple

$$\bar{o}(u) := (q_1(u(\eta_1)), ..., q_n(u(\eta_n))) \in [0,1)^n.$$

In other words, tuple $\bar{o}(u)$ consists of the realization of shocks $\eta_1, ..., \eta_n$ "interpreted" according to orientations $q_1, ..., q_n$.

Fix $x \in X$, $n \in \mathbf{N}$, and a tuple of orientations $\bar{o} \in \mathcal{O}^n$. Measurable function $f : [0,1)^n \to Y$ is $(x, \bar{o})$-*symmetric*, if for each $u \in [0,1)^{\mathcal{U}}$ and for all tuples of orientations $\bar{o}$ such that $x\hat{}\bar{o}$ and $x\hat{}\bar{o}'$ are analogous,

$$f(\bar{o}(u)) = f(\bar{o}'(u)).$$

To see what symmetry means, notice that tuples $\bar{o}$ and $\bar{o}'$ may differ in terms of the order and the orientations of shocks. Then, function $f(.)$ is symmetric if it does not change after certain reorderings of its parameters (as in the example from Section 5.4), and with respect to some changes in the orientations (as in the example from Section 5.5).

6.3. **Decompositions.** Orientations and symmetric functions can be used to construct invariant distributions. For each x' that is analogous to x, find a tuple of orientations $\bar{o}^{x'}$ such that $x'\hat{}\bar{o}^{x'}$ is analogous to $x\hat{}\bar{o}$ (such tuple exists because of the consistency axioms). For each realization of shocks $u \in [0,1)^{\mathcal{U}}$, define

$$\theta^{f,x,\bar{o}}(x; u) := f\left(\bar{o}^{x'}(u)\right). \tag{6.1}$$

It can be shown (Lemma 50 in Appendix G) that symmetry implies that (6.1) does not depend on the choice of orienting tuple $\bar{o}^{x'}$ as long as $x'\hat{}\bar{o}^{x'}$ is analogous to $x\hat{}\bar{o}$. Notice that $\theta^{f,x,\bar{o}}(x; u)$ is a function of the random realization of the shocks, and hence, it is a random variable.

We describe the construction in two cases separately. First, we assume that the relational system $(X, \sim)$ is transitive. Take any $x^* \in X$ and the tuple of orientations $\bar{o}^* \subseteq \mathcal{O}$. Distribution ω *admits* $(x^*, \bar{o}^*)$-*decomposition* if there exists $(x, \bar{o})$-symmetric function $f : [0,1)^n \to Y$ such that ω is equal to the joint distribution of variables

$$\theta^{f,x,\bar{o}}(x; u), x \in X.$$

In the general case, assume that the relational system has finitely many 1-tuples, and let $V = \{[x] : x \in X\}$. Suppose that the tuple $(x^v, \bar{o}^v)$ consists of elements $x^v \in v$ and tuples of orientations $\bar{o}^v$ for each type $v \in V$. Distribution ω *admits* $(x^v, \bar{o}^v)_{v\in V}$-*decomposition*, if there exist $(x^v, \bar{o}^v)$-symmetric functions $f^v : [0,1)^{n^v} \to Y$ for $v \in V$ such that ω is equal to the joint distribution of variables

$$\theta^{f^v,x^v,\bar{o}^v}(x; u), x \in v \in V.$$

6.4. **Sufficient and necessary conditions for invariance.** The next result describes the necessary and sufficient conditions for invariance.

Theorem 3. *Suppose that the relational system $(X, \sim)$ has finitely many types of 1-tuples, $V = \{[x] : x \in X\}$ and $|V| < \infty$. For each system with orientations $(X \cup \mathcal{O}, \sim)$, elements x^v, and tuples of orientations $\bar{o}^v$, $v \in V$, if distribution ω admits $(x^v, \bar{o}^v)$-decomposition, then it is $(X, \sim)$-invariant.*
Additionally, if $(X, \sim)$ is $\frac{1}{20}$-compact, then there exists a relational system with orientations $(X \cup \mathcal{O}, \sim)$, elements x^v, and tuples of orientations $\bar{o}^v$, $v \in V$ such that each $(X, \sim)$-invariant distribution admits $(x^v, \bar{o}^v)$-decomposition.

The first part of the Theorem is relatively straightforward, and its (elementary) proof can be found in appendix G. We describe the ideas lying behind the second part in Section 7 below. The formal proof can be found in Section 7.3.3.

Next, we discuss how the systems of orientations and symmetric functions fit into examples from Sections 5.4 and 5.5.

6.5. **Example: Bundles of goods.** Consider Example 5.4. Assume that $\mathcal{U}$ consists of shocks η_X, η_p, and η_x for $p \in P$ and $x \in X$. There is only one orientation of each shock $\eta \in \mathcal{U}$, $\mathcal{O}_\eta = \{o_\eta\} = \{(\eta, \mathrm{id})\}$.

Let $(X \cup \mathcal{O}, \sim)$ be the relational system induced by a unary relation $R_{\mathcal{O}}$ and binary relation R_P defined so that for each x, x'

- $R_{\mathcal{O}} x$ if and only if x is an orientation, and
- $x R_P x'$ if and only if x and x' are associated with the same good (we say that $x \in X \cup \mathcal{O}$ is associated with good $p \in P$ if and only if either $x = (p, p')$ for some $p' \in P$, or x is an orientation of shock $\eta = \eta_p$ and or $\eta = \eta_{p,p'}$ for some $p' \neq p$. for each $x, x' \in X \cup \mathcal{O}$).

Take any $x = (p_1, p_2)$ and let $\bar{o} = \left(o_X, o_{\eta_{p_1}}, o_{\eta_{p_2}}, o_{\eta_{p_1,p_2}}\right)$. If tuple $\bar{o}$ is analogous to tuple $\bar{o}'$ relative to x, then either $\bar{o} = \bar{o}'$, or $\bar{o}' = \left(o_X, o_{\eta_{p_2}}, o_{\eta_{p_1}}, o_{\eta_{p_1,p_2}}\right)$. Therefore, a measurable $f : [0,1)^4 \to Y$ is $(x, \bar{o})$-symmetric if and only if condition (5.3) holds for all realizations $\eta_X, \eta_{p_1}, \eta_{p_2}, \eta_{(p_1,p_2)} \in [0,1)$.

6.6. **Example: Multiple customers and two goods.** Consider Example 5.5. Let $\mathcal{U} = \{\eta_X\} \cup \{\eta_c, c \in C\}$. Let $\{\mathrm{id}, q_0\}$ be the set of orientations from Example 1. Let $\mathcal{O} = \mathcal{U} \times \{\mathrm{id}, q_0\}$. In other words, each shock has two orientations. We associate orientation id with good p_1 and orientation q_0 with good p_2. The association between orientations and goods is arbitrary, and the opposite association would not change the analysis.

Define unary $R_{\mathcal{O}}$ and binary R_P, R_C relations on set $X \cup \mathcal{O}$: for each x, x',

- $R_{\mathcal{O}} x$ if and only if x is an orientation,

- xR_Px' if and only if x and x' are associated with the same good,
- xR_Cx' if and only if $x, x' \in X$ and x and x' are associated with the same customer,

Then, the relational system induced by the three relations is a system of orientations.

Fix $x = (c, p)$, and a tuple of orientations $\bar{o} = ((\eta_X, \mathrm{id}), (\eta_c, \mathrm{id}))$. Tuple $x\hat{\ }\bar{o}'$ is analogous to tuple $x\hat{\ }\bar{o}$ if and only if $\bar{o} = \bar{o}'$. Thus, any measurable function $f : [0,1)^2 \to Y$ is $(x, \bar{o})$-symmetric, and for each $x' = (c', p')$, each tuple $x'\hat{\ }\bar{o}'$ that is analogous to $x\hat{\ }\bar{o}$, each realization of shocks u,

$$f(\bar{o}'(u)) = \begin{cases} f(u(\eta_X), u(\eta_c)), & \text{if } p' = p, \\ f(1 - u(\eta_X), 1 - u(\eta_c)), & \text{if } p' \neq p. \end{cases}$$

7. Ideas behind the proof of Theorem 3

Here, we describe the main ideas behind the proof of Theorem 3. This section should be treated as a guide toward reading Appendices B-G.

The proof has two essentially different parts: probabilistic and algebraic. In the first part of the section, we describe the main tools that are important for the probabilistic part of the argument. The tools are illustrated in the examples discussed earlier in this paper. Section 7.2 discusses the algebraic part: The goal is to find a convenient representation of $\frac{1}{20}$-compact relational systems. Section 7.3 discusses how the various parts come together in the proof of the general case.

7.1. Main tools.

7.1.1. *Borel decomposition.* We describe a technique to replace an arbitrary probability measure by the uniform distribution on the interval $[0,1)$. Below, we always assume that $\eta_0, \eta, \eta_a \in \mathcal{U}$ for $a \in A$ are distinct, independent, uniformly distributed on the interval $[0,1)$ random shocks. Let Y and Y_0 be standard Borel spaces. The results mentioned here are standard, and their proofs can be found, for example, in Kallenberg (2005).

The key observation is that if $\omega \in \Delta Y$ is a distribution of Y-valued variable θ, then θ can be represented as a transform of a random shock η: there exists a measurable function $f : [0,1) \to R$ such that ω is equal to the distribution of $f(\eta)$.

The key observation has multiple extensions. For example, a conditional version of the above result holds. Suppose that $\omega \in \Delta(Y_0 \times Y)$ is a joint distribution of a pair of variables (θ_0, θ). Then, there is a measurable function $f : Y_0 \times [0,1) \to Y$ such that the ω-conditional distribution of θ given θ_0 is equal to $f(\theta_0, \eta)$.

The conditional version can be further compounded with the Borel decomposition result for variable θ_0 leading to a measurable $f_0 : [0,1) \to Y_0$ such that ω is equal to the joint distribution of $(f_0(\eta_0), f(f_0(\eta_0), \eta))$.

Finally, the Borel decomposition adapts in a natural way when some of the variables are independent or, more generally, conditionally independent. Suppose that $\omega \in \Delta\left(Y_0 \times Y^A\right)$ is a distribution over variables θ_0 and $\theta(a), a \in A$ such that variables $\theta(a)$ is conditionally independent from variables $\theta(a'), a' \neq a$ given θ_0. Additionally, assume that the conditional distributions of θ and θ' are equal. Then, there are measurable functions $f_0 : [0,1) \to Y_0$ and $f : [0,1)^2 \to Y$ such that ω is equal to the joint distribution of $f_0(\eta_0)$ and $f(\eta_0, \eta_a)$ for $a \in A$.

Appendix F.1 presents specific versions of the Borel decomposition results used in this paper.

7.1.2. *Conditional independence.* Suppose that $(X, \sim)$ is a relational system. Let $A, B, C \subseteq X$ be subsets of X such that A and C are disjoint. If A is finite, say that sets A *conditionally independent from* C *given* B, if for each enumeration $\bar{a}$ of set A, for each finite tuple $\bar{c} \subseteq B \cup C$, there exists tuple $\bar{b} \subseteq B$ such that $\bar{a}\hat{}\bar{c}$ and $\bar{a}\hat{}\bar{b}$ analogous. If A is infinite, then say that A is conditionally independent from C given B if each finite subset of A is conditionally independent from C given B.

The definition is motivated by the following observation: Under invariant distributions, the conditional independence of sets implies the conditional independence of random variables.

Lemma 1. *Let $A, B, C \subseteq X$ be disjoint sets such that A is conditionally independent from C given B. For any invariant distribution ω, the joint realization of random variables $\{\theta(x), x \in A\}$ is conditionally independent from $\{\theta(x), x \in C\}$ given $\{\theta(x), x \in B\}$:*

$$\omega(\theta(x), x \in A | \theta(x), x \in B \cup C) = \omega(\theta(x), x \in A | \theta(x), x \in B).$$

Proof. Suppose that A is finite and let $\bar{a}$ be an enumeration of A. For any function $q : Y^A \to R$, for any set $U \subseteq B \cup C$, define

$$V_U(q) = E_\omega(q - E_\omega(q | \theta(x), x \in U))^2.$$

Take any $U \subseteq W \subseteq B \cup C$. By Jensen's inequality, $V_U(q) \geq V_W(q)$. Moreover, $E_\omega(q|\theta(x), x \in U) = E_\omega(q|\theta(x), x \in W)$ if and only if $V_U(q) = V_W(q)$ for each $q : Y^A \to R$.

Suppose that for two sets C' and B', and their enumerations $\bar{c}$ and $\bar{b}$, tuples $\bar{a}\hat{}\bar{c}$ and $\bar{a}\hat{}\bar{b}$ are analogous. Then, the invariance of ω implies that $V_{B'}(q) = V_{C'}(q)$.

We show that $V_{B \cup C}(q) = V_B(q)$ for each $q : Y^A \to R$. On the contrary, suppose that $V_{B \cup C}(q) < V_B(q)$ for some q. There exists a finite $C' \subseteq B \cup C$ such that $V_{C'} < V_B(q)$. Take any enumeration $\bar{c}$ of C' and let $B' \subseteq B$ be a subset and $\bar{b}$ its enumeration such that $\bar{a}\hat{}\bar{c}$ and $\bar{a}\hat{}\bar{b}$ are analogous. By invariance,

$$V_B(q) > V_{C'}(q) = V_{B'}(q) \geq V_B(q).$$

The contradiction establishes the result for finite A. The infinite case follows from the standard probability arguments. $\square$

7.1.3. *Isomorphic subsets.* It is sometimes convenient to describe the decomposition of uncertainty on a properly chosen subset $X_0 \subseteq X$ instead on the original space X. Set X_0 should be sufficiently large so that the decomposition on X_0 implies the decomposition on X.

Say that mapping $\alpha : X \to X$ *preserves analogies,* if each tuple $\bar{x} = (x_1, ..., x_k)$ is analogous to tuple $\alpha(\bar{x}) = (\alpha(x_1), ..., \alpha(x_n))$. Say that X_0 is *isomorphic* to X if there exists a bijection $\alpha : X \to X_0$ such that α and α^{-1} preserve analogies. Using bijection α, we can go back and forth between sets X_0 and X. On the one hand, an invariant distribution on X can be mapped to an invariant distribution on X_0. On the other, if we can find the decomposition on X_0, then the inverse α^{-1} maps the decomposition back into the original space X.

7.1.4. *Example: Exchangeability.* We illustrate the above techniques in the case of the original de Finetti Theorem. Suppose that and that the distribution ω of variables $\theta(x), x \in X$ is exchangeable, i.e., the joint distribution over any (the same length) tuple of variables is the same. We assume without the loss of generality that $X = \mathbf{Z}$ is a set of integers.

Let $X_0 = \{x : x \geq 0\}$ and $E = \{x : x < 0\}$. Then, sets X_0 and X are isomorphic. Moreover, notice that for each $x \in X_0$ is conditionally independent from all $x' \neq x$ given E.

Let $\theta(X_0)$ and $\theta(E)$ denote the collections of all variables indexed with, respectively, non-negative integers X_0 and set E. Because sets X_0 and X are isomorphic, the joint distribution of $\theta(X_0)$ is equivalent to the joint distribution of $\theta(X_0) \cup \theta(E)$. So, it is enough to show that the marginal distribution over variables $\theta(X_0)$ has representation (1.1). Because of Lemma 1, variable $\theta(x)$ is (probabilistically) conditionally independent from any set of variables $\Theta \subseteq \theta(X_0) \setminus \{\theta(x)\}$. Additionally, exchangeability implies that the conditional distributions of variables in $\theta(X_0)$ given $\theta(E)$ are identical.

Let $\bar{e}$ be an arbitrary enumeration of set E. Let $\theta(\bar{e}) = (\theta(e_1), \theta(e_2), ...) \in Y^{|E|}$ be a random $|E|$-tuple of elements of the outcome space Y. By the Borel decomposition, there exists a measurable function $f_X : [0,1) \to Y^{|E|}$ such that the ω-distribution of $\theta(\bar{e})$ is equal to $f_0(\eta_X)$. By another application of the Borel decomposition, there exists measurable functions $f_x : Y^{|E|} \times [0,1) \to Y$ such that the conditional distribution of $\theta(x)$ given $\theta(\bar{e})$ is equal to $f_x(\theta(\bar{e}), \eta_x)$. Invariance implies that we can choose the functions so that $f_x = f_{x'}$ for each $x, x' \in X_0$. Because of (probabilistic) conditional independence, the conditional distribution of $\theta(X)$ is equal to the joint distribution of variables $f(\theta(\bar{e}), \eta_x)$. Finally, we can compound the two functions to obtain

$$f(\eta_X, \eta_x) := f_x(f_X(\eta_X), \eta_x) \text{ for each } x \in X.$$

7.1.5. *Hierarchy of conditional independent sets.* In many cases, we are able to identify an entire hierarchy of conditionally independent subsets. Precisely, suppose that $\mathcal{S}_0$ is a collection of subsets of X partially ordered by inclusion. For each element $S \in \mathcal{S}_0$, define

$$\begin{aligned}
\sqcup S &= \left\{S' \in \mathcal{S}_0 : S' \supseteq S, S' \neq S\right\}, \\
\sqcap S &= \left\{S' \in \mathcal{S}_0 : S' \subseteq S\right\}.
\end{aligned}$$

Here, $\sqcup S$ consists of the elements of $\mathcal{S}_0$ that strictly include S; $\sqcap S$ consists of the elements of $\mathcal{S}_0$ that are either equal or strictly included in S.

Consider a collection $\{E(S) : S \in \mathcal{S}_0\}$ of sets of X. For each $S \in \mathcal{S}_0$, let

$$E(\sqcup S) = \bigcup_{S' \in \sqcup S} E(S'),$$

and similarly define $E(\sqcap S)$.

We say that collection $E(.)$ is a (S_0-)*hierarchy of conditionally independent sets*, if for each $S \in \mathcal{S}_0$, $E(S)$ is conditionally independent from $X \backslash E(\sqcap S)$ given $E(\sqcup S)$.

In general, a hierarchy of independent sets may involve a series of conditional independence statements, whose length depends on the length of chains in collection $\mathcal{S}_0$. In the case analyzed in the above Section 7.1.4, a hierarchy is almost trivial as it involves only one level of conditional independence. (Notice that we can take $\mathcal{S}_0 = \{X\} \cup \{x \in X_0\}$ and let $E(X) = E$, and $E(x) = \{x\}$ for each $x \in X_0$. Then, the hierarchy implies that $E(x)$ is independent from $X_0 \backslash E(x)$ given $E(X)$.) Next, we present a less trivial example of a hierarchy.

7.1.6. *Example: Multiple customers and goods.* Next, suppose that $X = C \times P$ and that distribution ω of variables $\theta(x), x \in X$ is invariant with respect to the analogy relation from Section 2.1. W.l.o.g. assume that countable sets C and P are disjoint copies of the set of integers $\mathbf{Z}$. Define

$$\begin{aligned}
&C_0 = \{c \in C : c \geq 0\}, \; P_0 = \{p \in P : p \geq 0\}, \\
&X_0 = C_0 \times P_0.
\end{aligned}$$

Then, set X_0 is isomorphic to X.

Define a collection of sets

$$\mathcal{S}_0 = \{X\} \cup \{S_c : c \in C_0\} \cup \{S_p : p \in P_0\} \cup \{\{x\} : x \in X_0\}.$$

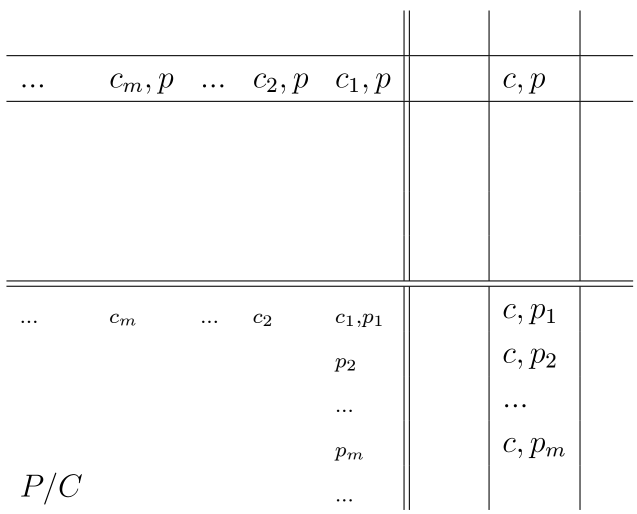
Define collection $\{E(S) : S \in \mathcal{S}_0\}$:

$$\begin{aligned} E(X) &= \{(c,p) : c, p < 0\}, \\ E(S_p) &= (C \backslash C_0) \times \{p\} \text{ for each } p \geq 0, \\ E(S_c) &= \{c\} \times (P \backslash P_0) \text{ for each } c \geq 0, \\ E(x) &= x \text{ for each } x \in X_0. \end{aligned} \tag{7.1}$$

Then, collection $E(.)$ is a hierarchy of conditionally independent sets. Specifically,

- set $E(S_c)$ is conditionally independent from $X \backslash (\{c\} \times P)$ given $E(X)$,
- set $E(S_p)$ is conditionally independent from $X \backslash (C \times \{p\})$ given $E(X)$,
- $x = (c,p) \in X_0$ is conditionally independent from any $X_0 \backslash \{x\}$ given $E \cup E(S_c) \cup E(S_p)$.

For each set A, let $\theta(A) = \{\theta(a), a \in A\}$. Then, because X_0 and X are isomorphic, the distribution of $\theta(X_0)$ is equivalent to the distribution of all variables $\theta(X)$. A hierarchy of conditional independencies leads to a hierarchy of Borel decompositions:



(1) Let $\bar{e}^X$ be an enumeration of set $E(X)$. Let $\theta(\bar{e}^X) = (\theta(\bar{e}^X_1), \ldots)$. Then, there exists a measurable function $f_X : [0,1) \to Y^{|E(X)|}$ such that the distribution of $\theta(\bar{e}^X)$[6] is equal to $f_X(\eta_X)$.

(2) For each $c \in C_0$, let $\bar{e}^c$ be an enumeration of set $E(S_c)$. Then, there exist measurable functions $f_c : Y^{|E(X)|} \times [0,1) \to Y^{|E(S_c)|}$ such that the conditional distribution of $\theta(\bar{e}^c)$ given $\theta(\bar{e}^X)$ is equal to $f_c(\theta(\bar{e}^X), \eta_{S_c})$.

(3) For each $p \in P_0$, let $\bar{e}^p$ be an enumeration of set $E(S_p)$. Then, there exist measurable functions $f_p : Y^{|E(X)|} \times [0,1) \to Y^{|E(S_p)|}$ such that the conditional distribution of $\theta(\bar{e}^p)$ given $\theta(\bar{e}^X)$ is equal to $f_p(\theta(\bar{e}^X), \eta_{S_p})$.

[6] Here, and below we use notation $\theta(\bar{e}) = (\theta(e_1), \theta(e_2), \ldots)$ for any finite or infinite tuple $\bar{e} = (e_1, e_2, \ldots)$.

(4) For each $x = (c, p) \in X_0$, there exist a measurable function $f_x : Y^{|E(X)|} \times Y^{|E(S_c)|} \times Y^{|E(S_p)|} \times [0, 1) \to Y$ such that the conditional distribution of $\theta(x)$ given $\theta\left(\bar{e}^X\right), \theta\left(\bar{e}^c\right)$, and $\theta\left(\bar{e}^p\right)$ is equal to $f_x\left(\theta\left(\bar{e}^X\right), \theta\left(\bar{e}^c\right), \theta\left(\bar{e}^p\right), \eta_x\right)$.

In general, functions f_c (or f_p, or f_x) do not have to be equal for different values of c. In order to make sure that the functions in fact are equal, we need to choose the enumerations $\bar{e}^c$ more carefully. Let $\bar{p} = (p_1, p_2, ...)$ be an enumeration of set $P \backslash P_0$. For each $c \in C_0$, let $\bar{e}^c = ((c, p_1), (c, p_2), ...)$. Then, invariance implies that $f_c = f_{c'}$ for each $c, c' \in C_0$. Similarly, an appropriate choice of $\bar{e}^p$ ensures that $f_p = f_{p'}$ and $f_x = f_{x'}$ for each $p, p' \in P_0$ and $x, x' \in X_0$.

Finally, we can compose the functions thus obtained to get

$$f\left(\eta_X, \eta_{S_c}, \eta_{S_p}, \eta_x\right) = f_x\left(f_X\left(\eta_X\right), f_c\left(f_X\left(\eta_X\right), \eta_{S_c}\right), f_p\left(f_X\left(\eta_X\right), \eta_{S_p}\right), \eta_x\right).$$

7.1.7. *Example: Multiple customers and two goods.* In the above two examples, the application of the Borel decompositions always follows a careful choice of the enumeration of the sets in the conditionally independent hierarchy. The enumerations have to be "consistent" in a certain sense. The precise notion of "consistency" is difficult to explain with elementary definitions and we postpone it till Section 7.2. Here, we only observe that sometimes, we may need to enumerate sets in more than one way. Going ahead of the formal statements, we observe that different enumeration of set $E(S)$ will correspond to different orientations of shock η_S.

We illustrate the role of multiple enumerations in the example from Section 5.5. Suppose that $X = C \times \{p_1, p_2\}$, C is equal to the set of integers, and that the distribution ω is an invariant distribution.

Define sets $E = \{(c, p) \in X : c < 0\}$ and $X_0 = X \backslash E$. As in the exchangeability case, X_0 and X are isomorphic, and finite sets $S_c = \{(c, p_1), (c, p_2)\}$ are conditionally independent from $X_0 \backslash S_c$ given E. Additionally, the joint distribution of pairs of variables $\theta(S_c)$ and $\theta(S_{c'})$ given $\theta(E)$ are equal.

Let $h : X \to X$ be an analogy-preserving bijection that exchanges the names of the goods: for each (c, p_i), $h(c, p_i) = (c, p_{-i})$, where $-i \in \{1, 2\}$ and $-i \neq i$. Notice that

$$h^2 = \mathrm{id}. \tag{7.2}$$

Let $\bar{e}$ be an enumeration of set E. Then, $h(\bar{e}) = (h(e_1), h(e_2), ...)$ be another enumeration of set E. Let $\theta(\bar{e})$ and $\theta(h(\bar{e}))$ be random $|E|$-tuples of elements of the outcome space Y. Define a mapping $\hat{h} : Y^{|E|} \to Y^{|E|}$ so that for each realization θ,

$$\hat{h}^{-1}(\theta(\bar{e})) = \theta(h(\bar{e})). \tag{7.3}$$

(Compare with equation (F.3) from appendix F.1.) The above equation characterizes mapping $\hat{h}$ uniquely; moreover, $\hat{h}$ is a bijection on $Y^{|E|}$.

Let $P = \{\mathrm{id}, q_0\}$ be the regular set of orientations from Section 6.6. Let $I = \left[0, \frac{1}{2}\right)$ and $I_0 = \left[0, \frac{1}{2}\right)$; then, interval I is partitioned into intervals I_0 and $q_0\left(I_0\right)$. Recall that

$$q_0^2 = \mathrm{id}\,. \tag{7.4}$$

Finally, fix customer c and an enumeration $\bar{s} = (s_1, s_2)$ of set S_c.

By the Borel decomposition, there exists a function $\delta_X : I_0 \to Y^{|E|}$ such that the (unconditional) distribution of $\theta\left(\bar{e}\right)$ is equal to $\delta_E\left(\eta'_X\right)$, where η'_X is chosen from the uniform distribution on I_0. Define function $f_X : I \to Y^{|E|}$ by

$$f_E\left(\eta_X\right) = \begin{cases} \delta_E\left(\eta_X\right), & \text{if } \eta_X \in I_0, \\ \left(\hat{h}^{-1} \circ f_E\right)\left(q_0^{-1}\left(\eta_X\right)\right), & \text{if } \eta_X \in q_0\left(I_0\right). \end{cases}$$

Then, by the invariance of distribution ω, the fact that h preserves analogies and that q_0 preserves the measure, the (unconditional) distribution of $\theta\left(\bar{e}\right)$ is equal to the distribution of $f_E\left(\eta_X\right)$. Moreover,

$$f_X \circ q_0 = \hat{h} \circ f_X.$$

By another application of the Borel decomposition, for each c, there exists a function $\delta_c : Y^{|E|} \times I_0 \to Y^2$ such that the conditional distribution of the ordered pair of variables $\theta\left(\bar{s}\right) = \left(\theta\left(s_1\right), \theta\left(s_2\right)\right)$ given $\theta\left(\bar{e}\right)$ is equal to the distribution of $\delta_c\left(\theta\left(\bar{e}\right), \eta'_c\right)$, where η'_c is chosen from the uniform distribution on I_0. Let $\delta_{c,1}$ and $\delta_{c,2}$ denote, respectively, the first and the second coordinate of function δ_c. Define function $f_c : Y^{|E|} \times I \to Y$ so that for each η_c

$$f_c\left(\theta\left(\bar{e}\right), \eta_c\right) = \begin{cases} \delta_{c,1}\left(\theta\left(\bar{e}\right), \eta_c\right) & \text{if } \eta_c \in I_0, \\ \delta_{c,2}\left(\hat{h}^{-1} \circ \theta\left(\bar{e}\right), q_0^{-1}\left(\eta_c\right)\right), & \text{if } \eta_c \in q_0\left(I_0\right). \end{cases}$$

Consider a pair of random variables (more precisely, functions of random variables $\theta\left(\bar{e}\right)$ and η_c):

$$\left(f_0\left(\theta\left(\bar{e}\right), \eta_c\right), f_0\left(\hat{h}\left(\theta\left(\bar{e}\right)\right), q_0\left(\eta_c\right)\right)\right)$$

By the definition of function f_c, definition (7.3), and equations (7.2) and (7.4), the pair is equal to

$$\begin{aligned} &\left(f_c\left(\theta\left(\bar{e}\right), \eta_c\right), f_c\left(\hat{h}\left(\theta\left(\bar{e}\right)\right), q_0\left(\eta_c\right)\right)\right) \\ &= \begin{cases} \left(\delta_{c,1}\left(\theta\left(\bar{e}\right), \eta_c\right), \delta_{c,2}\left(\theta\left(\bar{e}\right), \eta_c\right)\right), & \text{if } \eta_c \in I_0, \\ \left(\delta_{c,2}\left(\hat{h} \circ \theta\left(\bar{e}\right), q_0\left(\eta_c\right)\right), \delta_{c,1}\left(\hat{h} \circ \theta\left(\bar{e}\right), q_0\left(\eta_c\right)\right)\right), & \text{if } \eta_c \in q_0\left(I_0\right). \end{cases} \end{aligned} \tag{7.5}$$

The joint distribution of the variables (7.5) is equal to the conditional distribution of pair $\theta\left(\bar{s}\right)$ given $\theta\left(\bar{e}\right)$. Indeed, if $\eta_c \in I_0$, the claim is immediate; if $\eta_c \in q_0\left(I_0\right)$, then the claim is implied by the invariance of distribution ω, the fact that h preserves analogies and that q_0 preserves measure. Notice that invariance implies that $f_c = f_{c'}$ for each $c, c' \geq 0$.

Finally, we can "glue" functions f_c and f_X together. Define function $f : I^2 \to Y$ so that

$$f\left(\eta_X, \eta_c\right) = f_c\left(f_X\left(\eta_X\right), \eta_c\right).$$

The above remarks imply that the conditional distribution of variables

$$\theta\left(\bar{s}\right) = \left(f\left(\eta_X, \eta_c\right), f\left(q_0\left(\eta_X\right), q_0\left(\eta_c\right)\right)\right),$$

given η_X is equal to the conditional distribution of $\theta\left(\bar{s}\right)$ given $\theta\left(\bar{e}\right) = f_E\left(\eta_X\right)$. This follows immediately from the definitions of functions f_0 and f_E. Together with conditional independencies, the last step leads to the demanded decomposition.

7.2. **Representation of relational systems.** It turns out that each relational system can be equivalently described through an algebraic group. This is an important observation that allows us to use the language and tools of group theory.

Recall that bijection $g : X \to X$ preserves analogies (see Section 7.1.3), if each tuple $\bar{x} = (x_1, ..., x_k)$ is analogous to tuple

$$g \cdot \bar{x} = \left(g\left(x_1\right), ..., g\left(x_k\right)\right). \tag{7.6}$$

Let G be the set of all bijections that preserve analogies. It is easy to check that G contains identity mapping and G is closed with respect to taking inverses and compositions. In other words, G is an *algebraic group.* We say that G *acts on set* X and we write $G \longmapsto X$.[7]

It turns out that the descriptions through analogy relations and through groups of bijections are equivalent.

Lemma 2. *Any two tuples $\bar{x}$ and $\bar{x}'$ are analogous* if and only if *there is a analogy-preserving bijection g such that $g \cdot \bar{x} = \bar{x}'$.*

Proof. We show that if X is countable, and $(X, \sim)$ is a relational system, then for any two finite and analogous tuples $\bar{x}, \bar{x}' \in X^k$, there exists a relation-preserving bijection g such that $g \cdot \bar{x} = \bar{x}'$. The proof is a simple exercise in a back-and-forth method (Poizat (2000)). It is enough to construct enumerations $\bar{z}$ and $\bar{z}'$ of X such that $\bar{z}^{(k)} = \bar{x}$ and $\bar{z}'^{(k)} = \bar{x}'$. Fix a bijection $\iota : X \to \mathbf{N}$. For each $l \leq k$, let $z_l = x_l$, and $z'_l = x'_l$. For each $l > k$, suppose that $z_{l'}$ and $z'_{l'}$ for $l' \leq k$ are constructed.

- If l is odd, choose $z_{l+1} = \arg\min\left\{i\left(z\right) : z \in X \backslash \left\{z_1, ..., z_l\right\}\right\}$. By the extension axioms, there exists z'_{l+1} such that tuples $z^{(l)} {}^\frown z_{l+1}$ and $z'^{(l)} {}^\frown z'_{l+1}$ are analogous,

[7]For a more thorough introduction to group theory, see, for example, Lang (2002) or Dixon and Mortimer (1996).

- If l is even, choose $z'_{l+1} = \arg\min \{i(z) : z \in X \backslash \{z'_1, ..., z'_l\}\}$. By the extension axioms, there exists z_{l+1} such that tuples $z^{(l)} \hat{} z_{l+1}$ and $z'^{(l)} \hat{} z'_{l+1}$ are analogous.

□

The following table describes groups of permutations associated with examples of relational systems used in this paper: For each set A, let Π_A denote the group of all bijections of set A. Following the group-theoretic terminology, we refer to Π_A as the *symmetric group* on A.

Relational system	Section	Group	Spanning family $\mathcal{C}^*$
Trivial	5.1	Π_X	$\{x, x \in X\}$
M. customers, goods	2.1	$\Pi_C \times \Pi_P$	$\{S_p, p \in P\}, \{S_c, c \in C\}$
M. goods, discon. customers	5.2	$\Pi_P \rtimes (\Pi_C)^P$	$\{S_p, p \in P\}, S_p$ for $p \in P$
Cust., goods, incomes, prices	5.3	$\Pi_C \times \Pi_P$	$\{S_p, p \in P\}, \{S_c, c \in C\}$
Bundles of goods	5.4	Π_P	$\{S_p, p \in P\}$
M. customers, two goods	6.6	$\Pi_C \times \Pi_{\{0,1\}}$	$\{S_c, c \in C\}$

Table 1. Examples of group actions.

Here, symbol "$\times$" denotes the direct ("Cartesian") product of two groups, "$\rtimes$" denotes a semidirect product, and $(\Pi_C)^P$ is a direct product of P copies of group Π_C.[8]

Notice that notation (7.6) extends the action of group G onto finite tuples of X. Similarly, if we define $g \cdot U = \{g \cdot x : x \in U\}$ for some $U \subseteq X$, we can extend the group action onto subsets of U. In the same vein, we can extend the group action to infinite tuples, tuples of sets, sets of sets, sets of tuples, etc.

Suppose that X_0 is a subset of X and H is a subgroup of G (i.e., $H \subseteq G$ and $H.is$ a group). Say that the action group H on X_0 is *isomorphic* to the action of group G on X if there exists an analogy-preserving bijection $\alpha : X \to X_0$ such that for each $g \in G$ and $h \in H$, $\alpha \circ g \circ \alpha^{-1} \in H$ and $\alpha^{-1} \circ h \circ \alpha \in G$. In particular, set X_0 is isomorphic to X.

We are ready to state the representation theorem of the compact relational systems.

Theorem 4. *Suppose that X is a countably infinite, relational system $(X, \sim)$ finitely many types of 1-tuples and it is $\frac{1}{20}$-compact. Let G be the group of all analogy-preserving bijections. Then, there exists set $X_0 \subseteq X$, a collection of concepts $\mathcal{S}_0$, a partition $\{E(S), S \in \mathcal{S}_0\}$ of set X, and a subgroup $H \subseteq G$ such that*

(1) *The action of group H on X_0 is isomorphic to the action of group G on X.*

[8] Group $\Pi_P \rtimes (\Pi_C)^P$ consists of all pairs (g, h) of bjections $g \in \Pi_P$ and functions $h : P \to \Pi_C$. The group action on $X = C \times P$ is defined through the following formula:

$$(g, h) \cdot (c, p) = (h(g \cdot p) \cdot c, g \cdot p).$$

(2) *Collection* $\{E(S), S \in \mathcal{S}_0\}$ *is a* $\mathcal{S}_0$*-hierarchy of conditionally independent sets.*
(3) *For each* $S \in \mathcal{S}_0$*, each enumeration* $\bar{e}$ *of set* $E(S)$*, set* $\{h \cdot \bar{e} : h \in H, h \cdot S = S\}$ *is finite.*

The motivation behind the statements of the theorem follows from the examples and discussion from the preceding section. The first claim ensures that set X_0 is isomorphic to X. Second, the Theorem finds an appropriate hierarchy of conditional sets. Finally, there is a consistent (with respect to the action of group H) way of choosing *finitely* many enumerations of the elements of the hierarchy.

The goal of this section is to develop ideas used in the proof of Theorem 4. In the rest of this section, we always assume that

$$\psi \leq \frac{1}{20}.$$

7.2.1. *Multiply transitive group actions.* Notice that all groups from Table 1 are either equal to the symmetric groups on some infinite set or the groups are (possibly, different kinds of) products of such groups together with some finite groups. As it turns out, this is not an accident. In fact, our results show that the groups associated with $\frac{1}{20}$-compact relational systems can be represented as (kinds of) products of infinite groups of all permutations and (possibly) some finite groups.

It is useful to formalize some properties of symmetric groups. Say that a group action is *transitive*, if for any $x, x' \in X$, there is $g \in G$ so that $g \cdot x = x'$. The group action is n*-transitive* if each n-tuple of distinct elements of X can be mapped (via some $g \in G$) into any other n-tuple of distinct elements of X. The group action is *highly transitive* if it is n-transitive for each n. Notice that each symmetric group is highly transitive.

It is convenient to define slightly weaker versions of multiple transitivity. Say that $B \subseteq X$ is a *block* if for each $g \in G$, ether $g \cdot B = B$ or $g \cdot B \cap B = \varnothing$. An infinite group action is *block* n*- (or highly) transitive*, if there exists finite block $B \subseteq X$ such that the group action on blocks $G \longmapsto [B]$ is n- (or highly) transitive.

It turns out that finite multiply transitive group actions are relatively rare. A collection of results known together as the Classification of Finite Simple Groups (CFSG) implies that all finite 2-transitive group actions belong to either one of eight well-understood infinite families and or to one of finitely many special (so called sporadic) cases. The first two families are $|X|$-, or $(|X|-2)$-transitive, and the remaining 6 families are at most 3-transitive. All the infinite classes of finite 2-transitive groups are listed in appendix B.1.2. Additionally, all 6-transitive finite groups are either $|X|$-, or $(|X|-2)$-transitive.

7.2.2. *Types.* All definitions stated in the language of analogy relations have simple counterparts in the language of groups. As an example, we restate and expand the definition of a type. Let e and f be two arbitrary X-based objects: elements of X, finite or infinite tuples of elements, subsets, tuples of subsets, etc. Let $G_f = \{g \in G : g \cdot f = f\}$ be the set of analogy preserving bijections that keep object f fixed. Then, G_f is a subgroup of G (i.e., a group that is a subset of G). A *relative type of* e *given* f consists of all objects that can be obtained from e by bijections $g \in G_f$,

$$[e; f] = \{g \cdot e : g \in G_f\}$$

For example, the (unconditional) type of e is equal to the set of all all objects that are obtained from e by bijections $g \in G$, $[e] = [e; \varnothing]$. If $f_1, ..., f_n$ is a list of objects, we often write $[e; f_1, ..., f_n]$ to denote the relative type of e with respect to the tuple $f_1\hat{}...\hat{}f_n$.

It is useful to distinguish two types of relations between objects e and f : Say that e is f*-definable*, if $[e; f] = \{e\}$; e is f*-algebraic*, if $|[e; f]| < \infty$. Of course, any f-definable object is also f-algebraic. Moreover, if e is f-algebraic, and f is h-algebraic, then e is h-algebraic.

7.2.3. *Concepts.* Recall that a *concept* is a set $S \subseteq X$ such that $\sup_{x \in S} |[S; x]| < \infty$. In particular, S is x-algebraic for each $x \in S$, and there exists an uniform bound on the size of the relative type $[S; x]$. A *block* is a concept S such that for each $x \in S$, S- is x-definable.

Concept S is *coinfinite*, if for each other concept $S' \in [S]$, S' is not S-algebraic, i.e., $|[S; S']| = \infty$. It turns out that each concept S is contained in a coinfinite concept S' in such a way that S is S'-algebraic (Lemma 36). The last property means that, for many purposes, it is enough to work with coinfinite concepts and keep track of the associated (standard) concepts.

Lemma 34 shows that each concept has a *code*: There is a tuple $\bar{x}$ such that concept S is $\bar{x}$-definable. Codes are useful whenever it is easier to analyze the group action on finite tuples rather than (possibly, infinite) concepts. It is often important to control the length of the code, i.e., the number of elements in tuple $\bar{x}$. In general, the length may depend on the index of the concept. However, it turns out that each coinfinite concept S has a two-element code: there is a tuple $\bar{x} \in X^2$ such that S is $\bar{x}$-definable (Lemma 34).

Let $\mathcal{S}$ be the set of all coinfinite concepts. Consider the action of group G on the set of elements and coinfinite concepts, $G \longmapsto X \cup \mathcal{S}$. A concept $C \subseteq X \cup \mathcal{S}$ under such group action is defined in exactly the same way as the concept $S \subseteq X$ under the group action $G \longmapsto X$. Because C may contain elements of collection $\mathcal{S}$, C is sometimes referred to as a *concept of concepts.*

7.2.4. *Compact group actions.* Next, we restate and refine the definitions that underlie the assumptions of our results. Take any group action $G \longmapsto X$. Finite set $U \subseteq X$ is *local* if for all tuples $\bar{x}, \bar{x}' \subseteq U$ such that the tuples have the same type, $[\bar{x}] = [\bar{x}']$, the two have the same relative type given U, $[\bar{x}'; U] = [\bar{x}; U]$. Also, say that set U is k-local if the relative analogy is required to hold only for k-tuples $\bar{x}, \bar{x}' \in U^k$.

The group action is *ψ-compact,* if it has finitely many types of 1-tuples, and there exists local U_0 such that for each local $U \supseteq U_0$ and $x \in X$, there exists local $U' \supseteq U, x$ such that

$$\log |U'| \leq \psi + \log |U|. \tag{7.7}$$

Lemma 37 shows that, if the group action $G \longmapsto X$ is ψ-compact, then the group action $G \longmapsto X \cup \mathcal{S}$ satisfies two quasi-compact properties: for each k,

- there exists a constant c_k such that for each finite V, there exists k-local $U \supseteq V$ such that
$$\log |U| \leq \psi \log |V| + c_k, \text{ and} \tag{7.8}$$
- for each $\varepsilon > 0$, $x \in X \cup \mathcal{S}$, and finite set $V \supseteq [x]$, there exists k-local $U \subseteq [x]$ such that $V \subseteq U$ and for each $x' \in [x]$, there exists a k-local $U' \supseteq U, x'$ so that
$$\log |U'| \leq \psi + \varepsilon + \log |U|.$$

  The two properties are related but logically independent.

7.2.5. *Finitely many tuple types.* Say that the group action has *finitely many tuple types,* if for each k, the set of types of k tuples of elements is finite, $\left|\left\{[\bar{x}] : \bar{x} \in X^k\right\}\right| < \infty$. For example, any highly transitive group action has finitely many tuple types.

It turns out that ψ-compact group actions have finitely many tuple types. The formal argument is presented in Appendix B.2. Here, we give an intuition and illustrate the role of compactness. For simplicity, suppose that the group action $G \longmapsto X$ is transitive, i.e., there *is* only one type of 1-tuples, $X = [x]$. On the other hand, suppose that there are infinitely many types of 2-tuples, $|\{[x \hat{} x'] : x, x' \in X\}| = \infty$.

Take any local set U. There exist $x_0 \in U$ and $x_1 \notin U$ such that the type of tuple $x_0 \hat{} x_1$ is not represented in U : for each $x, x' \in U$, tuples $x \hat{} x'$ and $x_1 \hat{} x_1$ are not analogous. Take any local set $V \supseteq U, x_1$. Consider a graph with nodes V and such that there exists a directed edge from node x to node x' if and only if $x \hat{} x'$ s analogous to $x_0 \hat{} x_1$. Let k denote the out-degree of node $x \in V$ (the number of edges going out of x) and l denote the in-degree of x. By transitivity and because V is local, the out- and in-degrees do not depend on the choice of x. By the choice of x_0 and x_1, there is no edge that goes out of a node in U into a (possibly, different) node in U. Thus, the number of edges that go out of nodes in U and the number of edges that go into the nodes of U can be bounded by

$$|U| \, k \leq (|V \backslash U|) \, l \text{ and } |U| \, l \leq (|V \backslash U|) \, k.$$

The two inequalities put together imply that $|U| \leq |V \backslash U|$ and $|V| \geq 2|U|$. Because the argument does not depend on the choice of local U, the latter inequality leads to a direct contradiction with bound (7.7).

Additionally, we show that the group action $G \longmapsto X \cup \mathcal{S}$ has finitely many tuple types (Lemma 35). The idea is to represent each cofinite concept by its two-element code (see Section 7.2.3). Then, the number of types of n-tuples of coinfinite concepts can be bounded by the number of types of $2n$-tuples of elements of X. Because of the finitely many tuple types, it is easy to show that $\mathcal{S}$ must be countable.

7.2.6. *Robustly exchangeable concepts.* Let $C \subseteq X \cup \mathcal{S}$ be an infinite set of elements and/or coinfinite concepts. Say that C is *exchangeable* if C is infinite and the group action $G_C \longmapsto C$ is highly transitive. Say that C is *robustly exchangeable*, if C is exchangeable, and for each finite tuple $\bar{u} \subseteq X \cup \mathcal{S}$, there exists finite set $C_0 \subseteq C$ such that for each permutation $g \in G_{C,\bar{u}}$, $g \cdot C_0 = C_0$ and the group action $G_{C,\bar{u}} \longmapsto C \backslash C_0$ is exchangeable. Here, robustness means that the highly transitive group action $G_C \longmapsto C$ cannot be "broken" by a finite tuple $\bar{u}$, except for, possibly, some finite set C_0. We often refer to set C_0 as the *exceptional set.*

It turns out that robustly exchangeable concepts can be found, in some sense, everywhere.

Lemma 3. *Consider the group action $G \longmapsto X \cup \mathcal{S}$. For each tuple $\bar{x} \subseteq X \cup \mathcal{S}$ and each $x \in \mathcal{S}$ such that the relative type $[x; \bar{x}]$ is infinite, there exists a robustly exchangeable concept C such that $C \backslash [x; \bar{x}]$ is finite.*

Below, we sketch an argument behind Lemma 3 and illustrate the role of compactness. The Lemma is formally proved in Appendix D. The subsequent sketch can be omitted in the first reading. From now on, we assume for simplicity that $x, \bar{x} \subseteq X$.

Splitting sequence. Here, we argue that there exists a tuple $\bar{w} \supseteq \bar{x}$ and $w \in [x; \bar{x}]$ such that the relative type $[w; \bar{w}]$ has infinite cardinality and the group action $G_{\bar{w}} \longmapsto [w; \bar{w}]$ is block highly transitive (Lemma 41). We need a few preliminary observations. Because of the CFSG, it is enough to show that the group action $G_{\bar{w}} \longmapsto [w; \bar{w}]$ is block 2-transitive. Second, Lemma 25 shows that 2-transitivity follows if we show that the relative type $[w; \bar{w}]$ *cannot be split:* for all $w' \in [w; \bar{w}]$, set $[w; \bar{w}] \backslash [w'; w\hat{\ }\bar{w}]$ is finite. Finally, suppose that for each tuple $\bar{w} \supseteq \bar{x}$ and $w \in [x; \bar{x}]$, the relative type $[w; \bar{w}]$ is infinite and it can be split, i.e., there exists $w' \in [w; \bar{w}]$ such that set $[w; \bar{w}] \backslash [w'; w\hat{\ }\bar{w}]$ is infinite. In such a case, we show that there exists a *splitting sequence* (Lemmas 26 and 27): a sequence $s_0, t_0, s_1, ..., \in [x]$ such that for all m, if $\bar{s}_m = (s_0, t_0, ..., s_m, t_m)$, then for all $m, k \geq 0$ (a) $s_{m+k+1}, t_{m+k+1} \in [s_{m+1}; \bar{s}_m]$ and (b) for any t such that $t_m\hat{\ }t \in [s_m\hat{\ }s_{m+1}; \bar{s}_{m-1}]$, $t \notin [s_{m+1}; \bar{s}_m]$. See Figure 1.

We show that the existence of the splitting sequence contradicts the ψ-compactness of the group action $G \longmapsto X$ (Lemma 28). Indeed, find a sequence of local sets U_n such that

My work/A pisanina/learning/decomposition/splitting.wmf

Figure 1. Splitting sequence

$U_n \supseteq s_0, ..., t_{n-1}$. Because of the compactness, and the fact that $|\{s_0, ..., t_n\}| \leq 2(n+1)$, we can pick local sets so that

$$\limsup_{n\to\infty} \frac{1}{n} \log |U_n| \leq 2\psi. \tag{7.9}$$

On the other hand, for each n, and each $m \leq k-1$, tuples $\bar{s}_{m-1}\hat{\ }s_m$ and $\bar{s}_{m-1}\hat{\ }t_m$ are analogous. Thus, there exists a permutation $g_{n,m}$ such that $g_{n,m} \cdot \bar{s}_{m-1}\hat{\ }s_m = \bar{s}_{m-1}\hat{\ }t_m$. Because U_n is local, we can assume that permutation $g_{n,m}$ keeps set U_n fixed, $g_{n,m} \in G_{U_n}$. Let $A_{n,n} = \{s_n, t_n\}$, and by backward induction on m, define sets

$$A_{n,m} = \{s_m\} \cup A_{n,m+1} \cup g_{n,m} \cdot A_{n,m+1} \subseteq U_n.$$

The definition of the splitting sequence, together with the backward induction on m, shows that $A_{n,m+1} \subseteq [s_{m+1}, \bar{s}_m]$ and sets $A_{n,m+1}$ and $g_{n,m} \cdot A_{n,m+1}$ are disjoint. Thus,

$$|U_n| \geq |A_{n,0}| \geq 2|A_{n,1}| \geq ... \geq 2^{n+1},$$

which contracts (7.9).

Robust exchangeability. From now on, assume for simplicity that the group action $G_{\bar{w}} \longmapsto [w; \bar{w}]$ is highly transitive (rather than, more generally, block highly transitive). We show that $[w; \bar{w}]$ is robustly exchangeable.

It turns out that it is enough to show that there is no tuple $\bar{u} \supseteq \bar{w}$ and element $w' \in [w; \bar{w}]$ such that the relative type $[w'; \bar{u}]$ and the set $[w; \bar{w}] \setminus [w'; \bar{u}]$ are infinite (see Lemma 41). The claim is based on the following *counting argument*. On the contrary, suppose that $\bar{u}$ and w' with such properties exists. Let $V = [w'; \bar{u}]$. Let V_n be an increasing sequence of subsets of $[w; \bar{w}]$ such that $|V_n| = 2n$, and $|V_n \cap V| = |V_n \backslash V| = n$. Consider a sequence of local sets $U_n \supseteq V_n$ such that (7.9) holds.

By definition, set V is $\bar{u}$-definable. Similarly, set $V \cap U_n$ is $(\bar{u}\hat{\ }U_n)$-definable. These two observations yield a lower bound on the cardinality of the relative type of tuple $\bar{u}$ given $\bar{w}\hat{\ }U$:

$$|[\bar{u}; \bar{w}, U_n]| \geq |[V \cap U_n; \bar{w}; U_n]|.$$

Because the action $G_{\bar{w}} \longmapsto [w; \bar{w}]$ is highly transitive, and set U_n is local, the group action $G_{\bar{w}\hat{\ }U} \longmapsto [w; \bar{w}] \cap U_n$ is also highly transitive. Therefore, the cardinality of the relative type of set $V \cap U_n$ given $\bar{w}\hat{\ }U_n$ is equal to the number of ways a set of cardinality $|V \cap U_n|$ can be chosen from a cardinality $|[w;\bar{w}] \cap U_n|$. Hence,

$$|[\bar{u};\bar{w},U_n]| \geq \begin{pmatrix} |[w;\bar{w}] \cap U_n| \\ V \cap U_n \end{pmatrix} \geq \begin{pmatrix} 2n \\ n \end{pmatrix} \geq e^{n \log n},$$

where the last inequality comes from Stirling's approximation.

Finally, suppose that the length of tuple $\bar{u}$ is k. Then,

$$\frac{1}{n}\log|U_n| \geq \frac{1}{k}\frac{1}{n}\log||[\bar{u};U_n]|| \geq \frac{1}{nk}\log|[\bar{u};\bar{w},U_n]| \geq \frac{\log n}{k} \to \infty \text{ when } n \to \infty,$$

which contradicts (7.8).

Robustly exchangeable concepts. In the last part of this subsection, we construct a robustly exchangeable concept C such that $C\backslash[x;\bar{x}]$ is finite.

Let $D = [w;\bar{w}]$. The above argument implies that, for any set D' that has the same type as D, either the sets $D'\backslash D$ and $D\backslash D'$ are finite or the intersection between D and D' is finite. Indeed, notice that $D' = [w';\bar{w}']$ for some w' and tuple $\bar{w}'$. Then, if $D\backslash D'$ and $D \cap D'$ is infinite, then there exists $z \in D\backslash D'$ such that the sets $[z;\bar{w},\bar{w}'] \subseteq D\backslash D'$ and $D\backslash[z;\bar{w},\bar{w}']$ are infinite. But this contradicts the robust exchangeability of $[w;\bar{w}]$.

Say that set C is *complete*, if for each C' of the same type as C but $C' \neq C$, the intersection of C and C' is finite. Lemma 42 shows that there exists a complete and robustly exchangeable set $C \supseteq D$ such that $|C\backslash D| < \infty$. The idea is to define C as the union of all D's such that $D'\backslash D$ and $D\backslash D'$ are finite. An elementary argument based on Lemma 24 shows that set $C\backslash D$ must be finite.

Finally, suppose that $C \subseteq X$ is a complete and robustly exchangeable set C. Lemma 43 shows that set C must be a concept.

7.2.7. *Correlation and independence.* Consider now two robustly exchangeable concepts C and C' and the action of group $G_{C,C'}$ that fixes both concepts. Then, the action of $G_{C,C'}$ on each of the two concepts is highly transitive except for possibly finite exceptional set. (Indeed, because concept C' has a code $\bar{x}'$, C' is $\bar{x}$'-definable, which implies that $G_{\bar{x}} \subseteq G_{C'}$. Because C is robustly exchangeable, there exists a finite set $C_0 \subseteq C$ such that the action of $G_{\bar{x}'} \cap G_C \subseteq G_{C,C'}$ is highly transitive on $C\backslash C_0$.) For simplicity, we assume that the exceptional sets are empty and that group actions $G_{C,C'} \longmapsto C$ and $G_{C,C'} \longmapsto C'$ are highly transitive. In particular, for any two tuples $\bar{x},\bar{y} \in C$ of equal length, there exists a permutation $g \in G_{C,C'}$ such that $g \cdot \bar{x} = \bar{y}$.

We are interested in the joint action of $G_{C,C'}$on the union of the two concepts. The difficulty is that permutations over one of the concepts may depend on the permutations over the other concept. There are two extreme cases: On the one hand, permutations on the two concepts might be independent of each other. Precisely, for each tuple $\bar{x} \subseteq X \cup \mathcal{S}$, say that the joint action of $G_{C,C'}$ is $\bar{x}$*-independent* if there are exceptional sets finite sets $C_0 \subseteq C$ and $C'_0 \subseteq C'$ such that for each two tuples $\bar{u}, \bar{v} \subseteq C \backslash C_0$ of equal length, and any two tuples $\bar{u}', \bar{v}' \subseteq C \backslash C'_0$ of equal length, there exists a permutation $g \in G_{\bar{x}} \cap G_{C,C'}$ such that simultaneously $g \cdot \bar{u} = \bar{v}$ and $g \cdot \bar{u} = \bar{v}$. In other words, apart from the exceptional sets, the permutations on C and C' can be chosen independently. The two concepts are *independent*, if they are $\bar{x}$-independent for all tuples $\bar{x}$.

On the other hand, the two actions can be perfectly correlated. Formally, for each tuple $\bar{x} \subseteq X \cup \mathcal{S}$, the joint action of $G_{C,C'}$ is $\bar{x}$*-correlated*, if there are finite exceptional sets $C_0 \subseteq C$ and $C'_0 \subseteq C'$ and a *correlating function* $j : C \backslash C_0 \to C' \backslash C'_0$ such that for each permutation $g \in G_{\bar{x}} \cap G_{C,C'}$ and each tuple $\bar{u} \in C$, and each $g \in G_{C,C'}$, $g \cdot j(\bar{u}) = j(g \cdot \bar{u})$. That means that the permutation on concept C uniquely determines the permutation of C'. The two concepts are $\bar{x}$*-correlated*, if they are $\bar{x}$-correlated for all tuples $\bar{x}$.

At first sight, one can imagine that there is a range of imperfect correlations in which permutations on one concept limit, but not determine, permutations of the other concept. However, it turns out that, for each tuple $\bar{x}$, any two robustly exchangeable concepts can be either $\bar{x}$-independent or $\bar{x}$-correlated (Lemma 39). The argument is elementary, i.e., it does not rely on any compactness assumptions.

Additionally, if the group action is ψ-compact for any $\psi < \infty$, then any two robust exchangeable concepts are either independent or correlated. The argument relies on the counting argument from Section 7.2.6: Notice that if the two concepts are $\bar{x}$-independent and $\bar{x}\hat{\ }x$-correlated for some tuple $\bar{x}$ and x, then the correlating function is defined by x given $\bar{x}$. Because the number of the choices of the correlating functions is proportional to $n!$ where n is the number of elements in the intersection of C with some local U, the counting argument would lead to a contradiction with compactness.

7.2.8. *Coordinate system.* The discussion in Sections 7.2.6 and 7.2.7 focuses on either individual robustly exchangeable concepts or the relationships between pairs of such concepts. Next, we take a macroscopic view to describe the properties of the collection of all robustly exchangeable concepts. Observe that the correlation of robustly exchangeable concepts is transitive: if C is correlated with C' and C' is correlated with C'', then C and C'' are correlated. Thus, we can divide all robustly exchangeable concepts into correlation classes, i.e., collections of concepts such that all concepts from the same collection are correlated and all concepts from two different collections are independent. Because of the above discussion, any two concepts from two different classes are independent. Lemma 4 shows that each correlation class contains the "largest" member.

Lemma 4. *There exists a family $\mathcal{C}^*$ of mutually disjoint and independent concepts such that each concept C is correlated with the unique concept $C' \in \mathcal{C}^*$ and the correlating function $j : C \rightarrow C'$ is such that for each $x \in C$, $x \subseteq j(x)$.*

The proof is constructive. The idea is to use the correlating functions to construct equivalence classes on the elements of concepts in a correlation class $\mathcal{B}$. Take any $C, C' \in \mathcal{B}$. Informally, say that $x \in C$ and $x' \in C'$ are *directly connected* if $x' = j(x)$, where j is the correlating function between C and C'. Say that x and x' are *connected* if there exists a finite path of elements $x = x_0, ..., x_m = x'$ such that each consecutive elements are directly connected. We show that the relation of being connected is a relation of equivalence, that the union of all connected elements is a coinfinite concept, and that the collection of such a union forms a robustly exchangeable concept $C_{\mathcal{B}}$.

Define $\mathcal{C}^*$ as the family of all representative "largest" concepts $C_{\mathcal{B}}$ for all correlation classes $\mathcal{B}$. Let $\mathcal{S}^* = \bigcup \mathcal{C}^* \subseteq X \cup \mathcal{S}$ be the union of all concepts in $\mathcal{C}^*$. By construction, the concepts in $\mathcal{C}^*$ are mutually independent, and each robustly exchangeable concept is correlated with exactly one concept in $\mathcal{C}^*$. Additionally, we can show that all concepts in $\mathcal{C}^*$ are disjoint. (If not, and there are concepts C and C' with a non-empty intersection $C \cap C'$, then one shows that at least one of the concepts C or C' cannot be the largest member of its correlation class.) We refer to $\mathcal{C}^*$ as the *spanning family* of concepts.

Concepts in $\mathcal{C}^*$ can be interpreted as "dimensions" of the relational system and elements of $\mathcal{S}^*$ as "coordinates." Families $\mathcal{C}^*$ corresponding to the examples from this paper are described in Table 1. For example, in the multiple customers and goods case (Section 2.1), family $\mathcal{C}^*$ contains two elements: the concept of all concepts of customers and the concept of concepts of goods. As another example, consider the multiple goods with disconnected customers case from Section 5.2. There are infinitely many members of the spanning family: a concept of the concepts of goods and all concepts of goods. Finally, in the multiple customers and two goods case from Section 5.5, family $\mathcal{C}^*$ contains only one element: the concept of the concepts of customers.

Hierarchy of concepts. The elements of $\mathcal{S}^*$ and $\mathcal{C}^*$ can be partially ordered by inclusion. More precisely, notice that each object $e \in X \cup \mathcal{S}^* \cup \mathcal{C}^*$ can be associated with its $\mathcal{S}^*$-*cover*:

$$L(e) = \{x \in \mathcal{S}^* : e \neq x \text{ and } e \subseteq x\}.$$

For any two concepts $C, C' \in \mathcal{C}^*$ such that the intersection of $L(C)$ and C' is not empty, say that C is *included* in C', write $C \prec C'$. One shows that the relation "$\prec$" is a proper partial order on the spanning family $\mathcal{C}^*$.

In the multiple customers and goods case from Section 2.1, neither of the two elements of family $\mathcal{C}^*$ is included in the other. In the multiple goods with disconnected customers case from Section 5.2, each concept of goods is included in the concept of concepts of goods. In general, the partial order may lead to chains of concepts in $\mathcal{C}^*$ with two or more elements.

Coordinatewise description. We interpret the cover $L(e)$ as a "coordinatewise" description of the object e. The next result establishes three properties of the cover:

Lemma 5. *1. For each e, $L(e)$ is finite. 2. For each $x \in C \in \mathcal{C}^*$, $L(x) = L(C)$. 3. For each $L \subseteq \mathcal{S}^*$, sets $\{x : L(x) = L\}$ and $\{C : L(C) = L\}$ are finite.*

The first property means that the "coordinatewise" description is finite; the second, that the description of concept C is the same as its elements; and the third that the "coordinatewise" descriptions $L(.)$ can be used to almost uniquely identify elements x or concepts $C \in \mathcal{C}^*$. Here, "almost" means "up to finitely many other candidates." Together, the Lemma gives meaning to our interpretation of $\mathcal{C}^*$ as "coordinates" and $L(.)$ as the coordinate description of object e.

In the examples from Sections 2.1 and 5.2, each element $x \in X$ has two coordinates: a concept of a good and a concept of a customer. The two coordinates determine x uniquely.

We sketch the argument behind Lemma 5. First, each cover is finite. This follows from the fact that there are finitely many types of concepts and that, by a definition of a concept, each element can be contained in at most finitely many concepts of the same type.

Second, the cover of each $S \in C \in \mathcal{C}^*$ is the same as the cover of C, $L(S) = L(C)$. If not, then there is $S' \in L(S)$, but not $S' \in L(C)$. It is easy to show that S' is S-algebraic and that the relative type $[S'; C]$ is infinite. Using these two facts and the robust exchangeability of C, we show that there is a robustly exchangeable concept $C' \subseteq [S'; C]$ such that C' is correlated with C. Because C' consists of concepts S' that contain concepts $S \in C$, it must be that C' is "larger" than C in the sense defined above. However, that contradicts the choice of C as the largest member of its correlation class.

The proof of the third property relies on Lemma 3 stated above in Section 7.2.6: We need to show that if e is an element of X or $\mathcal{C}^*$, then e is $L(e)$-algebraic. Instead, if the relative type of e given $L(e)$ were infinite, then one could find a robustly exchangeable concept C that consists of elements of type e and that would be independent from concepts that contain the elements of cover $L(e)$. By taking the largest member C' of the correlation class of C, we would show that there exists $S' \in C' \in \mathcal{C}^*$ such that $e \in S'$ and $S \notin L(e)$. That would yield a contradiction with the definition of the cover $L(e)$.

Coordinate labelling β. Notice that each concept $C \in \mathcal{C}^*$ is countable (as a subset of countable set $\mathcal{S}^*$). Because all concepts in the spanning family $\mathcal{C}^*$ are disjoint, we can find a map $\beta : \mathcal{S}^* \to \mathbf{Z}$ such that for each $C \in \mathcal{C}^*$, $\beta|_C$ is a bijection between C and the set of integers $\mathbf{Z}$. We interpret β as a labeling of elements of $\mathcal{S}^*$.

We use the labeling to clarify the meaning of independence in the spanning family $\mathcal{C}^*$. Lemma 6 below shows that, for each concept $C \in \mathcal{C}^*$ and elements $s_0, s_1 \in C$, there exists a permutation g with the following properties:

- $g \cdot s_0 = s_1$, $g \cdot s_1 = s_0$,
- $g \cdot s = s$ for each $s \in \mathcal{S}^*$ such that $L(s) \cap \{s_0, s_1\} = \varnothing$. In particular, $g \cdot s = s$ for each $s \in C \setminus \{s_0, s_1\}$.
- $\beta(s) = \beta(g \cdot s)$ for each $s \in \mathcal{S}^* \setminus \{s_0, s_1\}$.

Any permutation of such form is called a *permutation of* (s_0, s_1)-type.

Lemma 6. *For each concept $C \in \mathcal{C}^*$, any two elements $s, s' \in C$, there exists a permutation of (s, s')-type.*

The idea behind Lemma 6 is to use the robust exchangeability and mutual independence of concepts in $\mathcal{C}^*$.

Positive and negative coordinates. Theorem 4 claims the existence of a subset X_0 of space X such that X_0 is isomorphic to X. Here, we show how X_0 s constructed.

The construction uses labeling β to divide coordinates into positive and negative. Say that coordinate $S \in \mathcal{S}^*$ is *positive* if S and each coordinate $S' \in L(S)$ has a non-negative label, $\beta(S), \beta(S') \geq 0$. Let $\mathcal{S}_0^* \subseteq S^*$ be the collection of all positive coordinates $S \in \mathcal{S}^*$. Define

$$X_0 = \{x \in X : L(x) \subseteq \mathcal{S}_0^*\}, \text{ and } \mathcal{S}_0 = \left\{\bigcap L : L \subseteq \mathcal{S}_0^*\right\}.$$

Here, set X_0 consists of all elements of X that have positive coordinates. Set $\mathcal{S}_0$ consists of intersections of positive coordinates; because each intersection of concepts is a concept, set $\mathcal{S}_0$ consists of concepts that are contained only in the positive coordinates. By convention, we take $\varnothing \in \mathcal{S}_0^*$, $\bigcap \varnothing = X$, which implies that $X \in \mathcal{S}_0$.

Say that permutation h *preserves negative coordinates* if $\beta(S) = \beta(h \cdot S)$ for all $S \in \mathcal{S}^* \setminus \mathcal{S}_0^*$. Let $G^{nc} \subseteq G$ be a subgroup of permutations that preserve negative coordinates. Then, $G^{nc} \subseteq G_{X_0}$.

Lemma 7. *There exists a bijection $\alpha : X \to X_0$ such that for any $h \in G^{nc}$, $\alpha^{-1} \circ h \circ \alpha \in G$, and for any $g \in G$, $\alpha \circ g \circ \alpha^{-1} \in G$. In particular, α preserves relations.*

That means that the two group actions $G \longmapsto X$ and $G^{nc} \longmapsto X_0$ are *isomorphic*. According to the Lemma, it is possible to go back and forth between isomorphic group actions without losing any information. As an important consequence, invariant distributions under each of the group actions correspond to invariant distributions under the other.

Mapping α from Lemma 7 is constructed in steps: Fix a bijection $\gamma : \mathbf{Z} \to \mathbf{N}$ that maps integers into natural numbers (we assume that $\mathbf{N}$ includes 0). Suppose that $A_0 \subseteq A_1 \subseteq \ldots$

is an increasing sequence of integers such that $\bigcup A_i \subseteq \mathbf{Z}$. Let $V_A \subseteq X \cup \mathcal{S}^* \cup \mathcal{C}^*$ be the set of objects e such that for each $\beta(L(e)) \subseteq A$. Using permutations of (s_0, s_1)-type (Lemma 6), we construct a sequence of permutations $g_0, g_1,...$ such that for each i, $g_i(V_{A_i}) \subseteq V_{\gamma(A_i)}$, and $g_i|_{V_{A_i}} = g_{i+1}|_{V_{A_i}}$. The limit of such mappings forms the required bijection α. Because each permutation g_i preserves relations, the limit α preserves relations as well. Finally, the isomorphy between $G \longmapsto X$ and $G^{nc} \longmapsto X_0$ follows from another application of Lemma 6.

Conditional independence. Recall the notion of a hierarchy of conditionally independent sets from Section 7.1.5. Here, we describe how such a hierarchy can be constructed.

Notice that collection $\mathcal{S}_0$ is partially ordered by inclusion and is closed with respect to finite intersections. For each $x \in X$, let $\min_{x\in S, S\cap\mathcal{S}_0} S$ denote the smallest element of collection $\mathcal{S}_0$ that contains x. Notice that $\min_{x\in S,S\cap\mathcal{S}_0} S$ is equal to $\min_{x\in S,S\in\mathcal{S}_0} x = \bigcap\left(L(x) \cap \mathcal{S}_0^*\right)$. For each $S \in \mathcal{S}_0$, define the set of elements x such that S is the smallest member of collection $\mathcal{S}_0$ that includes x,

$$E(S) = \left\{x \in X : \min_{x\subseteq S, S\in\mathcal{S}_0} x = S\right\}.$$

It is easy to see that $\{E(S) : S \in \mathcal{S}_0\}$ is a partition of set X. In order to shorten the notation, for any collection of sets $L \subseteq \mathcal{S}_0$, let $E(L) = \bigcup_{S\in L} E(S)$.

Using permutations of (s_0, s_1)-type (Lemma 6), we show that

Lemma 8. *$E(.)$ is a $\mathcal{S}_0$-hierarchy of conditionally independent sets.*

Finite orientations. As in the examples, we associate elements S of family $\mathcal{S}_0$, or, more precisely, sets $E(S)$ with shocks $\eta \in \mathcal{U}$. In order to construct orientations, we need more precise information about sets $E(S)$. Recall the multiple customers and two product example discussed in Section 7.1.7. There, the orientations of set $E = E(X)$ and sets $S_c = E(S_c)$ for positive S_c are constructed with the help of a certain permutation h such that $h^2 = \mathrm{id}$ and that $h \cdot (c, p_1) = (c, p_2)$. It turns out that such as construction can be generalized.

Lemma 9. *There exists subgroup $H \subseteq G^{nc}$ such that $H \longmapsto X_0$ and $G^{nc} \longmapsto X_0$ are isomorphic and for each $S \in \mathcal{S}_0$, the action of the S-fixing subgroup H_S on set $E(S)$ is finite, $\left|\left\{h|_{E(S)} : h \in H_S\right\}\right| < \infty$.*

Lemma 9 is proven in three steps. In Appendices E.6 and E.7, we develop tools that we use later to ensure two required properties of H. The construction of H is presented in Appendix E.8.

This ends the proof of Theorem 4.

7.3. **Decomposition of uncertainty.** In the last part of this section, we show how Theorem 4 together with applications of tools developed in Section 7.1 lead to the proof of the necessity part of Theorem 3.

7.3.1. *System of orientations.* We use Theorem 4 to construct a system of orientations on set X_0. We start with some notation. There are finitely many types t of concepts in collection $\mathcal{S}_0$. For each type t, fix a representative $S^t \in t \cap \mathcal{S}_0$ and fix an enumeration $\bar{e}^t$ of set $E(S^t)$. Let $H^t = \{h|_{E(S^t)} : h \in H_{S^t}\}$ be the finite set of permutation of set $E(S^t)$. For each concept $S \in t \cap \mathcal{S}_0$, fix a permutation h_S such that $h_S \cdot S^t = S$ and let $\bar{e}_S = h_S \cdot \bar{e}$.

Second, we construct orientations. Partition the interval $I = [0,1)$ into equal length subintervals $I_{\bar{e}}$, indexed with enumerations $\bar{e} \in H^t$. For each $h \in H^t$, let p_h be the measure-preserving bijection on I such that for each subinterval I_{h_0}, $p_h(I_{h_0}) = I_{h_0 h}$ and p_h is an affine monotonic shift on each of the subintervals.[9] Then, it is easy to check that for each h and h',

$$p_{hh'} = p_{h'} \circ p_h, \tag{7.10}$$

which implies that $Q^t = \{p_h, h \in H_S\}$ is a finite regular set of orientations (i.e., it contains identity, it is closed with respect to compositions, and $\{p_h(I_{\mathrm{id}}) : h \in H^t\}$ is a partition of the interval I). Notice that equation (7.10) ensures that the structure of the group H^t is replicated by the structure of compositions in Q^t.

Third, let $\mathcal{U} = \{\eta_S : S \in \mathcal{S}_0\}$ be a collection of i.i.d. random shocks associated with concepts in $\mathcal{S}_0$. Let $Q_{\eta_S} = Q^{[S]}$ be the set of orientations of shock η_S and let $\mathcal{O} = \bigcup_{S\in\mathcal{S}_0} \{\eta_S\} \times Q_{\eta_S}$ be the space of orientations of shocks in $\mathcal{U}$.

Fourth, we extend the action of group H on space $\mathcal{O}$. For each permutation $h \in H$, let $q(h,S) = h_{h\cdot S}^{-1} h h_S|_{E(S^t)} \in H^t$ be the restriction of permutation $h_{h\cdot S}^{-1} h h_S \in H_{S^t}$ to set $E(S^t)$. Then, for any two permutations h and h',

$$\begin{aligned} q(h'h,S) &= \left(h_{h'h\cdot S}^{-1} h' h h_S\right)|_{E(S^t)} = \left(\left(h_{h'h\cdot S}^{-1} h' h_{h\cdot S}\right)\left(h_{h\cdot S}^{-1} h h_S\right)\right)|_{E(S^t)} \\ &= q(h', h\cdot S)\, q(h,S) \end{aligned} \tag{7.11}$$

For each orientation (η_S, p), define

$$h \cdot (\eta_S, p) = \left(\eta_{h\cdot S}, p \circ p_{q(h,S)}\right). \tag{7.12}$$

We show that equation (7.12) defines a system of orientation. Indeed, because of (7.10) and (7.11), for any two permutations h and h',

$$h'h \cdot (\eta_S, p) = h'(h \cdot (\eta_S, p))$$

which means that equation (7.12) extends the action of group H on the set of orientations $\mathcal{O}$. By the equivalence between group actions and relational systems (Theorem 2), $X_0 \cup \mathcal{O}$ is an extension of the relational system on X_0. Also, if $\bar{o}$ is a tuple of orientations of the same concept, and tuple $\bar{o}'$ is analogous to $\bar{o}$, then all elements of tuple $\bar{o}'$ are also orientations of

[9] For each h and h_0, there exists b such that for each $x \in I_h$, then $p_h(x) = x + b$.

the same concept. The last two properties of a system of orientation follow directly from (7.12).

Finally, we show that orientations of shocks η_S can be "tied" with enumerations of set $E(S)$. Let

$$\bar{E}(S) = [\bar{e}_S; S] = \left\{h \cdot \bar{e}^t : h \in H \text{ and } h \cdot S^t = S\right\}$$

be a finite set of orientations. We define a mapping $\rho : \mathcal{O} \to \bigcup_{S \in \mathcal{S}_0} \bar{E}(S)$: for each orientation $o = (\eta_S, p_h)$, let

$$\rho(o) := h_S h \cdot \bar{e}^t. \tag{7.13}$$

Then, for each permutation $g \in H$,

$$\begin{aligned}\rho(g \cdot o) &= h_{g \cdot S} q(g, S) h \cdot \bar{e}^t = h_{g \cdot S} h_{g \cdot S}^{-1} g h_S h \cdot \bar{e}^t \\ &= g g h_S h \cdot \bar{e}^t = g \cdot \rho(o).\end{aligned}$$

In other words, the movement of orientation o under any permutation g is traced by the movement of enumeration $\rho(o)$.

Then, set $\bar{E}(S)$ is a finite set of enumerations of $E(S)$.

7.3.2. *Invariant distribution* ω^*. Let $Z = \bigcup_t Y^{|E(S^t)|}$, where t varies over types of concepts in collection $\mathcal{S}_0$. For each mapping $\theta : X \to Y$, and each (finite or infinite) tuple $\bar{e} = (e_1, e_2, ...)$ of elements of X, let $\theta(\bar{e}) = (\theta(e_1), \theta(e_2), ...)$ be a tuple of elements of Y. Define a mapping $O(\theta) : \mathcal{O} \to Z$ so that for each $o \in \mathcal{O}$,

$$O(\theta)(o) = \theta(\rho(o)),$$

where ρ is mapping that associated orientations of shocks with the enumerations of associated sets. Because the collection of sets $\{E(S) : S \in \mathcal{S}_0\}$ is a partition of X, there is a one-to-one relationship between mappings θ and $O(\theta)$.

Suppose that $\omega \in \Delta(Y)^X$ is an invariant distribution over θs. Let $\omega^* \in \Delta\left(Z^{\mathcal{O}}\right)$ be the associated distribution over mappings $O(\theta)$: for any measurable subset $E \subseteq Y^X$, let

$$\omega^*(O(E)) = \omega(E).$$

Because mapping O is one-to-one, there is a one-to-one relationship between distributions ω and ω^*. Moreover, distribution ω^* is invariant with respect to the action of group H :

Recall that the collection of sets $\{E(S) : S \in \mathcal{S}_0\}$ is a hierarchy of conditionally independent sets. Together with Lemma 1, this implies that distribution ω^* exhibits the following hierarchy of conditional independencies:

CI: For each $o \in \mathcal{O}_S$, $\theta(o)$ is conditionally independent from

$$\left\{\theta(o'), o' \in \bigcup \{\mathcal{O}_{S'} : S' \in \mathcal{S}, S' \backslash S \neq \varnothing\}\right\}$$

given

$$\left\{\theta\left(o'\right), o' \in \bigcup\left\{\mathcal{O}_{S'} : S' \in \mathcal{S}, S' \supsetneq S\right\}\right\}.$$

Let $\lambda^{\mathcal{S}} \in \Delta\left[0,1\right)^{\mathcal{S}}$ be the product of uniform distributions. A realization from $\lambda^{\mathcal{S}}$ is denoted by $\eta \in \left[0,1\right)^{\mathcal{S}}$. For each tuple of orientations $\bar{o} \in \mathcal{O}^{n}$, let $\bar{o}\left(\eta\right) = \left(q_1\left(\eta\left(S_1\right)\right), ..., q_n\left(\eta\left(S_n\right)\right)\right).$

Lemma 10. *For all $S \in \mathcal{S}$ and all orientations $o \in \mathcal{O}_S$, there exist tuples of orientations $\bar{o}^{o}$ such that if o and o' are analogous, then $o\hat{\ }\bar{o}^{o}$ and $o'\hat{\ }\bar{o}^{o'}$ are analogous, and for all H-invariant distribution $\omega^{*} \in \Delta\left(Z^{\mathcal{O}}\right)$ that satisfies CI, for all $t \in T$, some $o^{t} \in \mathcal{O}_S$ and $S \in t$, there exist $\left(o^{t}, \bar{o}^{o^{t}}\right)$-symmetric functions f^{t} such that ω is equal to the joint distribution of*

$$f^{t}\left(o\hat{\ }\bar{o}^{o}\left(u\right)\right), \text{ for } o \in \mathcal{O}_S, S \in t, \text{ and } t \in T.$$

The proof is by induction on the hierarchy of conditional independencies. At each level of the hierarchy, we apply a version of the Borel decomposition (Lemma 48 from Appendix F.1) to decompose variable $\theta\left(o\right)$ for some orientation $o = \left(\eta_S, p\right)$ into an independent shock η_S as well as the realizations of variables $\theta\left(o'\right)$ for orientations o' of shocks that are higher in the hierarchy. We combine function f together with the outcomes of the decompositions of higher-level orientations to find an symmetric function f^{t}. The proof of the Lemma can be found in Appendix F.2.

7.3.3. *Proof of necessity part of Theorem 3.* We use the results and notation from the above section. Let ω be an invariant distribution and let ω^{*} be defined as in the previous subsection. Let V be the set of types of elements of X_0. For each type v, fix a representative $x^{v} \in t \cap X_0$. Because the collection of sets $\left\{E\left(S\right) : S \in \mathcal{S}_0\right\}$ is a partition of X, there exists a unique $S^{v} \in \mathcal{S}_0$ such that $x^{v} \in E\left(S^{v}\right)$. Fix an orientation o^{v} of the shock associated with concept S^{v}. Let $\bar{o}^{v} = \bar{o}^{o^{v}}$ be the tuple of orientations from Lemma 10

Let t^{v} be the type of concept S^{v}. . Let $f_0 = f^{t^{v}} : \left[0,1\right)^{n^{t^{v}}} \to Y^{\left|E\left(S^{t^{v}}\right)\right|}$ be a function from Lemma 10.

Recall that $\rho\left(o^{v}\right)$ is an enumeration of set $E\left(S^{v}\right)$. Let m^{v} be the position in that enumeration occupied by x^{v}. Define function

$$f^{v}\left(u_1, ..., u_{n^{t^{v}}}\right) = \left(f^{t^{v}}\left(u_1, ..., u_{n^{t^{v}}}\right)\right)_{m^{v}}.$$

We show that function f^{v} is $(x^{v}, o^{v}\hat{\ }\bar{o}^{v})$-symmetric. Indeed, suppose that tuple $x^{v}\hat{\ }o\hat{\ }\bar{o}$ is analogous to $x^{v}\hat{\ }o^{v}\hat{\ }\bar{o}^{v}$. Then, the tuple of orientations $o\hat{\ }\bar{o}$ is analogous to $o^{v}\hat{\ }\bar{o}^{v}$. By Lemma 10,

$$O\left(\theta\right)\left(\bar{o}^{v}\right) = f^{t^{v}}\left(o^{v}\hat{\ }\bar{o}^{v}\left(u\right)\right) \text{ and } O\left(\theta\right)\left(\bar{o}\right) = f^{t^{v}}\left(o\hat{\ }\bar{o}\left(u\right)\right)$$

given a realization of shocks u. Because $(\rho(o))_{m^v} = (\rho(o^v))_{m^v} = x^v$ and because of the definition of operator $O(.)$, it must be that

$$\left(f^{t^v}\left(o^v\hat{}\bar{o}^v(u)\right)\right)_{m^v} = \left(f^{t^v}\left(o\hat{}\bar{o}(u)\right)\right)_{m^v}.$$

For each $x \in v \cap X_0$, find the tuple of orientations $\bar{o}^x$ that is analogous to tuple $o^v\hat{}\bar{o}^v$. It follows from Lemma 10 that the joint distribution of

$$f^v\left(\bar{o}^x(u)\right) \text{ for } x \in v \in V$$

is equal to ω.

8. Counterexample

Here, we present an example of a 1-compact relational system with finitely many types of 1-tuples that admits invariant distributions without finite decomposition. This shows that the constant $\frac{1}{20}$ in the statement of Theorem 1 cannot be increased too much.

Assume that X is a collection of finite subsets of the set of natural numbers $\mathbf{N}$ including the empty set $\varnothing$. Thus, X is countable. For each $n \in \mathbf{N}$, define a binary relation on pairs $(x, x') \in X$:

$$xR_n x' \text{ if either } n \in x \cap x', \text{ or } n \notin x \cup x'.$$

Let $\sim$ be the analogy relation induced by binary relations $\{R_n, n \in \mathbf{N}\}$. In Appendix I, we show that the relational system is 1-compact.

Lemma 11. *For each local $U \subseteq X$, and each x, there exists local $U' \supseteq U, x$ such that*

$$|U'| \leq 2|U|.$$

Suppose that $\mathcal{U} = \{\eta_n, n \in \mathbf{N}\}$ is a collection of i.i.d. random shocks uniformly distributed on the interval $[0,1]$. For each $x \in [0,1]$, define

$$\theta(x) := \sum\nolimits_{n \in x} \frac{1}{2^n}\left(\eta_n - \frac{1}{2}\right) + \sum\nolimits_{n \notin x} \frac{1}{2^n}\left(1 - \eta_n - \frac{1}{2}\right).$$

It is easy to check that the joint distribution ω of variables $\theta(x)$ is stationary. Notice that for any x, x', the correlation between variables $\theta(x)$ and $\theta(x')$ is equal to

$$E_\omega\left[\theta(x)\theta(x')\right] = \sum\nolimits_{n \notin x \triangle x'} \frac{1}{2^{n+2}} - \sum\nolimits_{n \notin x \triangle x'} \frac{1}{2^{n+2}}.$$

In particular, $E_\omega\left[\theta(x)\theta(x')\right] = E_\omega\left[\theta(y)\theta(y')\right]$ if and only if $x \triangle x' = y \triangle y'$. Because there are infinitely many sets $x \triangle x'$, there are infinitely many correlations.

We show that ω has no finite decomposition in the sense of Theorem 1. Indeed, suppose that there is such a decomposition with assignment functions $k : X \to \{1, ..., k_0\}$ and

$n : X \to \mathcal{U}^{m_0}$ for some finite k_0, m_0. For each pair of tuples (x_1, x_2) and (x_1', x_2') , write $(x_1, x_2)\, \mathcal{R}\, (x_1', x_2')$ if

$$k(x_1) = k(x_1'),\ k(x_1) = k(x_1'), \text{ and for each } m, m' \leq m_0$$
$$n_m(x_1) = n_{m'}(x_2) \text{ if and only if } n_m(x_1') = n_{m'}(x_2').$$

Then, $\mathcal{R}$ is an equivalence relation on X^2. Because all shocks are i.i.d., all the variables associated with $\mathcal{R}$-equivalent tuples (x_1, x_2) and (x_1', x_2') have the same correlations,

$$E[\theta(x_1)\theta(x_2)] = E[\theta(x_1')\theta(x_2')],$$

where E is the expectation operator.

Because $k_0, m_0 < \infty$, $\mathcal{R}$ has finitely many classes of equivalence. In particular, there are finitely many values of correlations between variables $\theta(x_1)$, and $\theta(x_2)$ for all pairs of $x_1, x_2 \in X$. Contradiction.

References

Al-Najjar, N. I. (1995): "Decomposition and Characterization of Risk with a Continuum of Random Variables," *Econometrica*, 63.

Aldous, D. (1981): "Representations for Partially Exchangeable Arrays of Random Variables," *Journal of Multivariate Analysis*, 11, 581–598.

de Finetti, B. (1980): "On the Condition of Partial Exchangeability," in *Studies in Inductive Logic and Probability II*, ed. by R. Jeffrey, chap. 9. University of California, Berkeley.

Diaconis, P. (1988): "Recent Progress on de Finetti's Notions of Exchangeability," *Bayesian Statistics*, 3, 111–125.

Diaconis, P., and D. Freedman (1980): "De Finetti's Theorem for Markov Chains," *Annals of Probability*, 8, 115–130.

Dixon, J. D., and B. Mortimer (1996): *Permutation Groups*, Graduate Texts in Mathematics. Springer, New York.

Hoover, D. (1982): *Row-Column Exchangeability and a Generalized Model for Probability*North-Holland.

Jackson, M. O., E. Kalai, and R. Smorodinsky (1999): "Bayesian Representation of Stochastic Processes under Learning: De Finetti Revisited," *Econometrica*, 67, 875–894.

Kallenberg, O. (2005): *Probabilistic Symmetries and Invariance Principles*, Probability and Its Applications. Springer, New York.

Kreps, D. M. (1988): *Notes on the Theory of Choice*. Westview Press, Boulder.

Lang, S. (2002): *Algebra*, Graduate Texts in Mathematics. Springer, New York.

Poizat, B. (2000): *A Course in Model Theory: An Introduction to Contemporary Mathematical Logic*, Universitext. Springer, New York.

Appendix A. Concepts in the multiple customers and goods case

Recall the example from Section 2.1. There is an alternative way of characterizing the relational system $(X,\sim)$, where $X = C \times P$. Let Π_C be the set of all permutations (i.e., bijections) of C. Similarly, let Π_P be the set of all permutations of P. Let $G = \Pi_C \times \Pi_P$. The set G is a group of permutations of set X. As it is discussed in the beginning of Secton 7.2, there is a natural extension of the action of group G on the tuples of elements of X. Then, two tuples $\bar{x}$ and $\bar{x}'$ are analogous if and only if there is $g \in G$ such that $g \cdot \bar{x} = \bar{x}'$.

Using the group of permutations G, we can restate the definition of the concept. For each set $S \subseteq X$ and each permutation $g \in G$, let $g \cdot S = \{g \cdot x : x \in S\}$ be the permutation of set S. Then, sets S and S' are analogous relative to x if and only if there exists permutation $g \in G$ so that $g \cdot x = x$ and $g \cdot S = S'$. In particular, set S is a concept if there exists $i_S < \infty$ so that for each $x \in S$, $|\{g \cdot S : g \in G \text{ and } g \cdot x = x\}| \leq i_S$.

The rest of the proof is divided into four steps. Suppose that S is a concept.

1. If there are $(c,p), (c',p) \in S$ such that $c \neq c'$, then $S_p \subseteq S$. Indeed, let $C_S = \{c' : (c',p) \in S\}$. Then, $C_S \backslash \{c\}$ is not empty. We show that if $C \backslash C_S$ is not empty, then S cannot be a concept. Then, either $C_S \backslash \{c\}$ or $C \backslash C_S$ has infinitely many elements. We consider only the former case (the latter is similar). Fix $c'' \in C \backslash C_S$. For each $c' \in C_S \backslash \{c\}$, find a permutation $\pi_{c'} : C \to C$ such that $\pi(c') = c''$, $\pi(c'') = \pi(c')$ and such that $\pi|_{C\backslash\{c',c''\}} = \mathrm{id}\,|_{C\backslash\{c',c''\}}$. Let $\mathrm{id}_P \in \Pi_P$ be the identity permutation of the set of goods, and let $g_{c'} = (\pi_{c'}, \mathrm{id}_P)$ for each $c' \in C_S \backslash \{c\}$. Then, for each $c'_0, c'_1 \in C_S \backslash \{c\}$ st. $c'_0 \neq c'_1$, $\pi_{c'_0}(C_S) \neq \pi_{c'_1}(C_S)$, and $g_{c'_0}(S) \neq g_{c'_1}(S)$. Because set $C_S \backslash \{c\}$ has infinitely many elements, there are infinitely many different sets $g \cdot S$ such that $(c,p) \in g \cdot S$. Thus, S is not a concept.
2. It follows that if $S_c, S_{c'} \subseteq S$ for some $c \neq c'$, then for each p, $S_p \subseteq S$, which implies that $S = X$.
3. In a similar way, we show that if there are $(c,p), (c,p') \in S$ such that $p \neq p'$, then $S_p \subseteq S$, and if if $S_p, S_{p'} \subseteq S$ for some $p \neq p'$, then $S = X$.
4. Suppose that $(c,p), (c',p') \in S$ for some $c \neq c'$ and $p \neq p'$. We show that $S = X$. If not, then by the above steps, either $S_c \cap S = \{(c,p)\}$, or $S_{c'} \cap S = \{(c',p')\}$. W.l.o.g. suppose that the latter. For each $c'' \neq c, c'$, find permutation $g_{c''} = (\pi_{c''}, \mathrm{id}_P) \in G$ such that $\pi_{c''}(c') = \pi_{c''}(c'')$, $\pi_{c''}(c'') = \pi_{c''}(c')$, and

The proof of the second part of the Lemma is similar.

Appendix B. Group actions

B.1. **Group theory.** We review some basic notation, definitions, and results from group theory (for details, see Lang (2002) and Dixon and Mortimer (1996)). Suppose that $G \longmapsto X$ is a group action. The cardinality of group G is called the *order* of G, and the cardinality of X is called the *degree* of $G \longmapsto X$.

For each set X, the set $\Pi(X)$ of all permutations on X is a group, and it is called a *symmetric group* of X. If X is finite and $|X| > 1$, there exists a unique subgroup $A(X) \subseteq \Pi(X)$ with index $|[\Pi(X) : A(X)]| = 2$. Group $A(X)$ is called an *alternating group of* X. (Alternating groups can also be defined for infinite sets X.) When $X = \{1, ..., n\}$, then we write Π_n and A_n, instead of, respectively, $\Pi(X)$ and $A(X)$.

Group action $G \longmapsto X$ is *transitive*, if for any $x, x' \in X$, there is $g \in G$ such that $g \cdot x = x'$. Group action $G \longmapsto X$ is k*-transitive*, if for any $U \subseteq X$, if $|U| \leq k-1$, then $G_{x:x\in U} \longmapsto X \backslash U$ is transitive. Group action $G \longmapsto X$ is *highly transitive*, if it is k-transitive for each k. The symmetric group is highly transitive, and the alternating group is $(|X| - 2)$-transitive.

B.1.1. *Index.* For each subgroup $H \subseteq G$, for each $g \in G$, set $gH := \{gh : h \in H\}$ is called a coset of H. Different cosets are disjoint, and the (possibly infinite) cardinality of the collection of cosets is called an *index* of H in G: $[G : H] := |\{gH : g \in G\}|$. The next result presents some bounds on indices.

Lemma 12. *If $J \subseteq H \subseteq G$ are groups, then $[G : J] = [G : H][H : J]$. If $H_1, H_2 \subseteq G$ are groups, then $[G : H_1 \cap H_2] \leq [G : H_1][G : H_2]$, and $[H_1 : H_1 \cap H_2] \leq [G : H_2]$.*

Lemma 13 (Dixon and Mortimer (1996))**.** *If X is finite, $G \longmapsto X$ is alternating or symmetric, and $H \subseteq G$ is a subgroup such that $[G : H] < |X|$, then H is alternating or symmetric. If X is infinite, $G \longmapsto X$ is highly transitive, and $H \subseteq G$ is a subgroup with a finite index, $[G : H] < \infty$, then H is highly transitive.*

B.1.2. *Classification of finite simple groups.* The entire list of finite and 2-transitive groups can be derived from the powerful result known as the Classification of Finite Simple Groups (see Dixon and Mortimer (1996)). There are eight infinite families of such groups:

1. symmetric group Π_n for each n;
2. alternating group A_n for each n;
3. *affine group* $A\Gamma L_d(b)$ and some of its subgroups, where $d \in \mathbf{N}$, and $b = p^n$ is an nth power of a prime number p, and the degree is equal to p^{nd};
4. *projective groups* $PSL_d(b)$, where $d \in \mathbf{N}$, and $b = p^n$ is an nth power of a prime p, and the degree is equal to p^{nd};
5. *unitary groups* $PSU_3(b)$, where $b = p^n$ is an nth power of a prime p, and the degree is equal to $b^3 + 1$;

(6) *symplectic groups* $SP_{2m}(2)$, where $m \in \mathbf{N}$. The symplectic group has two actions with degrees equal to $2^{m-1}(2^m+1)$, and $2^{m-1}(2^m-1)$;
(7) *Suzuki groups* $Sz(b)$, with $b = 2^{2m+1}$ and the degree equal to b^2+1,
(8) *Ree groups* $R(b)$ with $b = 3^{2m+1}$ and the degree equal to b^3+1.

Families 3-6 are also called classical Lie groups. Additionally, there are finitely many of the so-called *sporadic groups* that do not belong to any of the infinite families. Only the alternating and the symmetric group are 6-transitive.

Lemma 14. *Suppose that $X_0 \subsetneq X_1$ and G is a group such that $G_{X_0} \longmapsto X_0$ and $G \longmapsto X_1$ are 2-transitive, and they belong to the same family 3-8. Then, $|X_1| \geq 2|X_0|$.*

Proof. The result directly follows from the characterization of degree in cases 7-8. In cases 3-5, the result follows from the fact that if G has degree p^n and H is a subgroup of G that belongs to the same family 3-5, then H's degree is equal to $p^{n'}$ for some $n' < n$ (see **?**). Finally, in case 6, the result follows from the fact that the group with degree equal to $2^{m-1}(2^m-1)$ is not a subgroup of the group with the degree equal to $2^{m-1}(2^m+1)$. □

Lemma 15. *Suppose that a sequence $X_1 \subsetneq X_2 \subsetneq ..$ is such that*

$$\lim_{n\to\infty} \frac{\log|X_n|}{n} < \frac{1}{10},$$

and $G_n \longmapsto X_n$ is 2-transitive for each n[10]*. Then, $G \longmapsto X$ is highly transitive.*

Proof. It is enough to show that for each n, there exists $n' > n$ such that $G_{X_{n'}} \longmapsto X_{n'}$ is symmetric or alternating. Suppose not, and that there exists n^* such that for each $n > n^*$, $G_{X_n} \longmapsto X_n$ is 2-transitive, but not symmetric nor alternating. We can assume that n is large enough so that $G_{X_n} \longmapsto X_n$ does not belong to the sporadic cases.

Because $\frac{1}{10} < \frac{1}{9}$, there exist $n > n^*$ such that $|X_{n+9}| < 2|X_n|$. On the other hand, there exist $n \leq n_0 < n_1 \leq n+9$ such that $G_{X_{n_o}} \longmapsto X_{n_0}$ and $G_{X_{n_1}} \longmapsto X_{n_1}$ belong to the same infinite class of 2-transitive actions. By Lemma 14, $2|X_n| \leq 2|X_{n_0}| \leq |X_{n_1}| \leq |X_{n+9}|$. Contradiction. □

Lemma 16. *Suppose that $X = X_0 \cup X_1$ is a union of disjoint finite sets X_0, X_1 and $G \longmapsto X$ is a group action such that for each i, $|X_i| \geq 8$, $G \cdot X_i = X_i$ and for each $\bar{x}_i \in (X_i)^6$, $G_{\bar{x}_i} \longmapsto X_{-i}$ is 6-transitive. Then, for each i, each enumeration $\bar{x}_i^*$ of X_i, $G_{\bar{x}_i^*} \longmapsto X_{-i}$ is alternating or symmetric.*

Proof. For each i, find permutation $\pi_i \in \Pi(X_i) \setminus A(X_i)$ such that $(\pi_i)^2 = \mathrm{id}_{X_i}$. For each i, and for each $\bar{x}_{-i} \in X_{-i}^6$, the Classification of Finite Simple Groups implies that $G_{\bar{x}_{-i}} \longmapsto X_i$

[10]Here, and elsewhere, log has always basis 2.

is either alternating or symmetric. Denote

$$G_i^0 = \{g : g|_{X_i} \in \{\mathrm{id}_{X_i}, \pi_i\}\},$$
$$G_i^{00} = \{g : g|_{X_i} = \mathrm{id}_{X_i}\}.$$

Because of the choice of π_i, $G_i^{00} \subseteq G_i^0 \subseteq G$ are subgroups of G and $[G_i^0 : G_i^{00}] \leq 2$.

We show that $G_i^0 \longmapsto X_{-i}$ is 6-transitive. Indeed, by the hypothesis, $G \longmapsto X_{-i}$ is 6-transitive. Take any two tuples $\bar{x}, \bar{x}' \in (X_{-i})^6$ and any $g \in G$ such that $g \cdot \bar{x} = \bar{x}'$. Because $G_{\bar{x}'} \longmapsto X_i$ is alternating or symmetric, there is $g' \in G_{\bar{x}'}$ such that $(g'g)\,|x_i \in \{\mathrm{id}_{X_i}, \pi_i\}$. Then, $g'g \cdot \bar{x} = \bar{x}'$ and $g'g \in G_i^0$.

By the Classification of Finite Simple Groups, $G_i^0 \longmapsto X_{-i}$ is alternating or symmetric. Lemma 13 shows that $G_i^{00} \longmapsto X_{-i}$ is alternating or symmetric. □

B.2. **Finitely many tuple types.** Group action $G \longmapsto X$ has *finitely many tuple types,* if for each k, $\left|\left\{[\bar{x}] : \bar{x} \in X^k\right\}\right| < \infty$. In this section, we show that compact group actions have finitely many tuple types. The next simple observation is used without mention throughout the rest of the paper.

Lemma 17. *If $G \longmapsto X$ has finitely many tuple types, then for each finite tuple $\bar{u} \subseteq X$, $G_{\bar{u}} \longmapsto X$ has finitely many tuple types. In particular, for each k, $\left|\left\{[\bar{x};\bar{u}] : \ddot{x} \in X^k\right\}\right| < \infty$.*

Proof. Notice that for each k, $\left|\left\{[\bar{x};\bar{u}]\,; \bar{x} \in X^k\right\}\right| = \left|\left\{[\bar{x},\bar{u}]\,; \bar{x} \in X^k\right\}\right| < \infty$. □

Lemma 18. *For any local U and finite $\bar{x}^* \subseteq U$ with length l, set U is a local set of group action $G_{\bar{x}^*} \longmapsto X$.*

Proof. Take any $\bar{x}, \bar{x}' \in U^k$ and assume that $[\bar{x};\bar{x}^*] = [\bar{x}';\bar{x}^*]$. By the definition of the relative type, there is a permutation g such that $g \cdot \bar{x}^* = \bar{x}^*$ and $g \cdot \bar{x} = \bar{x}'$. If set U is local, then there is a g' such that $g' \in G_{\bar{x}^*}$, $g' \cdot U = U$ and $g' \cdot \bar{x} = \bar{x}'$. Thus, U is local under $G_{\bar{x}^*} \longmapsto X$. □

Lemma 19. *Suppose $U \subseteq X$ is local under group action $G \longmapsto X$, and that there are $x_0 \in U$ and $x_1 \notin U, [x_1] \cap U \neq \varnothing$ such that $x_0\hat{\ }x_1$ is not analogous to $x\hat{\ }x'$ for any $x, x' \in U$. Then, for any local $V \supseteq U, x_1$,*

$$\textit{either } \left|V \cap [x_0]\right| \geq 2\left|U \cap [x_0]\right|, \textit{ or } \left|V \cap [x_1]\right| \geq 2\left|U \cap [x_1]\right|. \tag{B.1}$$

Proof. We need to show that either $m_0^* \leq m_1^*$, or $m_0 \leq m_1$. Recall the argument described in Section 7.2.2. Consider a graph with nodes V and such that there exists a directed edge from node x to node x' if and only if $x\hat{\ }x'$ s analogous to $x_0\hat{\ }x_1$. Let k denote the out-degree of node x. Because V is 1-local, the out-degree does not depend on the choice of $x \in V \cap [x_0]$. Similarly, let l denote the in-degree of $x \in V \cap [x_1]$. By the choice of x_0 and x_1, there is no edge that goes out of a node in $U \cap [x_0]$ into a (possibly, different) node in $U \cap [x_1]$. Thus, the number of edges that go out of $U \cap [x_0]$ can be bounded by

$$|U \cap [x_0]|\, k \leq |(V \backslash U) \cap [x_1]|\, l.$$

Similarly, the number of edges that go into $U \cap [x_1]$ can be bounded by

$$|U \cap [x_1]|\, l \leq |(V \backslash U) \cap [x_0]|\, k.$$

The two inequalities put together imply that

$$k \leq \frac{|V \cap [x_1]|}{|U \cap [x_0]|} l \leq \frac{|(V \backslash U) \cap [x_1]|}{|U \cap [x_0]|} \frac{|(V \backslash U) \cap [x_0]|}{|U \cap [x_1]|} k,$$

which implies that at least one of the inequalities (B.1) holds. □

Lemma 20. *Suppose that group action $G \longmapsto X$ is ψ-compact group action for some $\psi < \frac{1}{2}$. Then, it has finitely many tuple types.*

Proof. Let $G \longmapsto X$ be a ψ-compact group action for any $\psi < \frac{1}{2}$ group action. By assumption, any compact group action has finitely many tuple types of 1-tuples. Suppose that it has finitely many k types for some $k \geq 1$, but infinitely many $(k+1)$-types. Then, there are $x_0, x^* \in X$ and $\bar{x}^* \in X^{k-1}$ such that $|\{[x; \bar{x}^*, x^*] : x \in [x_0; \bar{x}^*]\}| = \infty$.

Find $(k+1)$-local $U \supseteq \bar{x}^*, x^*$ and that for each $x \in X$, there is $x' \in U \cap [x]$. Assume that U is large enough so that for each local U', each $x \notin U$, there is local $U'' \supseteq U' \cup \{x\}$ so that $\log |U'| \leq \psi + \log |U|$. Let $U_0 = U$ and find an increasing sequence of local sets $U_0 \subseteq U_1 \subseteq ...$ and elements $x_1, x_2, ... \in [x_0; \bar{x}^*]$ such that

- $|U_{n+1}| \leq \Psi |U_n|$,
- $U_{n+1} \supseteq U_n, x_n$, and
- for each $x \in U_n$, $x_n \notin [x; \bar{x}^*, x^*]$.

By Lemma 18, sets U_n are local under group action $G_{\bar{x}^*}$. By Lemma 19, for each n, either

$$\text{either } \frac{|U_{n+1} \cap [x^*; \bar{x}^*]|}{|U_n \cap [x^*; \bar{x}^*]|} \geq 2, \text{ or } \frac{|U_{n+1} \cap [x_0; \bar{x}^*]|}{|U_n \cap [x_0; \bar{x}^*]|} \geq 2.$$

Thus, for either $x = x^*$ or $x = x_0$,

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n} \log \frac{|U_n|}{|U_0|} &\geq \lim_{n\to\infty} \frac{1}{n} \log \frac{|U_0 \cap [x; \bar{x}^*]|}{|U_0|} \prod\nolimits_{m=1}^{n} \frac{|U_m \cap [x; \bar{x}^*]|}{|U_{m-1} \cap [x; \bar{x}^*]|} \\ &\geq \log \sqrt{2} > \psi.\end{aligned}$$

Contradiction. □

B.3. **Transitivity.** A set $B \subseteq X$ is a *block* under group action $G \longmapsto X$ if for each $g \in G$, either $g \cdot B = B$, or $g \cdot B \cap B = \varnothing$. Group action $G \longmapsto X$ is *block* (k-, *highly*) *transitive (if X is finite, alternating) with block $B \subseteq X$*, if group action $G \longmapsto [B]$ is (k-, highly) transitive (alternating).

Lemma 21. *Take any group action $G \longmapsto X$ such that all types of 1-tuples have infinite cardinality, $|[x]| = \infty$ for each x. For any finite U and V, there is $g \in G$ such that $g \cdot U \cap V = \varnothing$.*

Proof. The proof proceeds by induction on $|U|$. If $|U| = 1$, then the claim follows from the assumption about infinite cardinality of types of 1-tuples. Suppose that the claim holds for all finite V and all U such that $|U| \leq k-1$. Take any $U, |U| \leq k-1$ and $u \notin U$. By the repeated application of the inductive claim, we can find infinitely many $g_1, g_2, \ldots$ such that $g_n \cdot U \cap \left(V \cup \bigcup_{m<n} g_m \cdot U\right) = \varnothing$. Suppose that there is u such that $g_n \cdot u \in V$ for each n. Because V is finite, there exists v_0 such that set $N_0 = \{n : g_n \cdot u = v_0\}$ has infinite cardinality. Because of the assumption of infinite cardinality of types of 1-tuples, there exists $g \in G$ such that $g \cdot u \notin V$. Because V is finite and sets $g_n \cdot U$ are disjoint, there is $n \in N_0$ such that $gg_n \cdot U \cap V = \varnothing$. This ends the proof of the inductive claim. $\square$

Lemma 22. *If $G \longmapsto X$ is block highly transitive with block B such that $|B| < |X|$, then for each $x \in B$, there exists block $B' \in [B]$ such that $G_x \longmapsto X \backslash B$ is block highly transitive with block B'.*

Proof. Notice that $G_B \longmapsto X \backslash B$ is block highly transitive with block $B' \in [B] \setminus \{B\}$, and $[G_B : G_x] \leq |B|$. The result follows from Lemma 13. $\square$

Lemma 23. *Suppose that $G \longmapsto X$ is transitive and for each x, there exists finite $U_x \subseteq X$ such that $G_{x,U_x} \longmapsto X \backslash U_x$ is transitive. Then, $G \longmapsto X$ is block 2-transitive.*

Proof. For any x, define $B_x = \bigcup \{x' : x' \text{ is } x\text{-algebraic}\}$. Because of the transitivity of the group action $G_{x,U_x} \longmapsto X \backslash U_x$, it must be that $B_x \subseteq U_x$, and B_x is finite. Because $G_{B_x} \supseteq G_x \supseteq G_{x,U_x}$, and $U_x \backslash B_x$ is finite, it must be that $G_{B_x} \longmapsto X \backslash B_x$ is transitive.

For any $x, x', x''X$, $[x''; x] = \bigcup_{x_1' \in [x';x]} [x''; x_1']$. Then, for each $x' \in B_x$, each $x'' \in B_{x''}$, x'' is x-algebraic, and $x'' \in B_x$. This implies that $B_x = B_{x'}$ for each $x' \in B_x$. Because for any $g \in G$, $g \cdot B_x = B_{g \cdot x}$, it must be that B is a block. Because $G_B \longmapsto X \backslash B$ is transitive, it must be that $G \longmapsto X$ is block 2-transitive with block B. $\square$

Lemma 24. *Suppose that the group action $G \longmapsto X$ is transitive and it has finitely many tuple types. Fix x and tuple $\bar{x}$ and for each $\bar{x}'$ such that $[x; \bar{x}]$ is infinite and for each $g \in G$, $(g \cdot [x; \bar{x}]) \setminus [x; \bar{x}]$ is finite. Then, $X \setminus [x; \bar{x}]$ is finite.*

Proof. For any two sets A, B, recall that $A \triangle B = (A \backslash B) \cup (B \backslash A)$ s the symmetric difference between A and B. The symmetric difference is symmetric ($A \triangle B = B \triangle A$), and for any triple of sets, $(A \triangle B) \triangle (A \triangle C) = B \triangle C$.

For each $\bar{x}' \in [\bar{x}]$, define $S(\bar{x}') = g \cdot [x; \bar{x}]$ for some g such that $g \cdot \bar{x} = \bar{x}'$ (of course, the definition does not depend on the choice of g). Suppose that $S(\bar{x})$ is infinite, and for each $\bar{x}' \in [\bar{x}]$, the set $S(\bar{x}') \backslash S(\bar{x})$ is finite. Suppose that $X \backslash S(\bar{x})$ is infinite.

By definition, $S(\bar{x}) = [x; \bar{x}]$ is infinite. Because of the finitely many tuple types, there exists a finite set $X_0(\bar{x}) \subseteq X \backslash S(\bar{x})$ such that $G_{\bar{x}} \cdot X_0(\bar{x}) = X_0(\bar{x})$ and that for each $x' \in X \backslash (S(\bar{x}) \cup X_0(\bar{x}))$, the relative type $[x'; \bar{x}]$ is infinite.

Because of the finitely many tuple types, there exists $N < \infty$ such that for all $\bar{x}' \in [\bar{x}]$, the cardinality of set

$$W(\bar{x}, \bar{x}') = (S(\bar{x}) \triangle S(\bar{x}')) \backslash (X_0(\bar{x}) \cup X_0(\bar{x}'))$$

is bounded by N. Let N be the smallest constant with such a property.

We show that $N > 0$. Indeed, it is enough to show that there exists $\bar{x}'$ such that $S(\bar{x}') \backslash (S(\bar{x}) \cup X_0(\bar{x}))$ is not empty. But this follows from transitivity of $G \longmapsto X$.

Find $\bar{x}'$ so that $|W(\bar{x}, \bar{x}')| = N$. By Lemma 21, there exists $g \in G_{\bar{x}}$ such that

$$\begin{gathered} g \cdot ((S(\bar{x}) \triangle S(\bar{x}') \cup X_0(\bar{x}')) \backslash X_0(\bar{x})) \\ \cap (S(\bar{x}) \triangle S(\bar{x}') \cup X_0(\bar{x}')) = \varnothing. \end{gathered}$$

Let $\bar{x}'' = g \cdot \bar{x}'$. Then,

$$\begin{aligned} W(\bar{x}', \bar{x}'') &= (S(\bar{x}') \triangle S(\bar{x}'')) \backslash (X_0(\bar{x}') \cup X_0(\bar{x}'')) \\ &= (S(\bar{x}) \triangle S(\bar{x}'') \cup S(\bar{x}) \triangle S(\bar{x}')) \backslash (X_0(\bar{x}') \cup X_0(\bar{x}'')) \\ &= ((S(\bar{x}) \triangle S(\bar{x}'')) \backslash (X_0(\bar{x}') \cup X_0(\bar{x}''))) \\ &\cup ((S(\bar{x}) \triangle S(\bar{x}')) \backslash (X_0(\bar{x}') \cup X_0(\bar{x}''))). \end{aligned}$$

The two sets in the union above are disjoint. Moreover, because $S(\bar{x}) \triangle S(\bar{x}'')$ is disjoint from $X_0(\bar{x}') \backslash X_0(\bar{x})$, it must be that

$$((S(\bar{x}) \triangle S(\bar{x}'')) \backslash (X_0(\bar{x}') \cup X_0(\bar{x}''))) \supseteq ((S(\bar{x}) \triangle S(\bar{x}'')) \backslash (X_0(\bar{x}) \cup X_0(\bar{x}''))) = W(\bar{x}, \bar{x}'')$$

and, because $S(\bar{x}) \triangle S(\bar{x}')$ is disjoint from $X_0(\bar{x}'') \backslash X_0(\bar{x})$, it must be that

$$(S(\bar{x}) \triangle S(\bar{x}')) \backslash (X_0(\bar{x}') \cup X_0(\bar{x}'')) \supseteq (S(\bar{x}) \triangle S(\bar{x}')) \backslash (X_0(\bar{x}') \cup X_0(\bar{x})) = W(\bar{x}, \bar{x}').$$

Thus, $|W(\bar{x}', \bar{x}')| \geq |W(\bar{x}, \bar{x}')| + |W(\bar{x}, \bar{x}'')| = 2N$, which yields a contradiction with the choice of constant N. $\square$

B.4. **Splitting.** This section establishes a useful technical property of group actions with finitely many tuple types. Let $G \longmapsto X$ be a group action with finitely many types. Suppose that the type $[x]$ is infinite, $|[x]| = \infty$. The type $[x]$ *can be split*, if there is $z \in [x]$ such that sets $[z;x]$ and $[x] \setminus [z;x]$ have infinite cardinality. The type $[x]$ *can be robustly split*, if for each tuple $\bar{u} \subseteq X$ and $x' \in [x]$ such that $|[x';\bar{u}]| = \infty$, the relative type $[x';\bar{u}]$ can be split.

A *splitting sequence* of elements of type $[x]$ is a sequence $s_0, t_0, s_1, ..., \in [x]$ such that for all m, if $\bar{s}_m = (s_0, t_0, ..., s_m, t_m)$, then for all $m, k \geq 0$ (a) $s_{m+k+1}, t_{m+k+1} \in [s_{m+1}; \bar{s}_m]$ and (b) for any t such that $t_m\hat{\ }t \in [s_m\hat{\ }s_{m+1}; \bar{s}_{m-1}]$, $t \notin [s_{m+1}; \bar{s}_m]$.

Lemma 25. *Suppose that the group action $G \longmapsto X$ is transitive. If it cannot be split, then, $G \longmapsto X$ is block 2-transitive.*

Proof. It follows from Lemma 23. □

Lemma 26. *Suppose that the group action $G \longmapsto X$ has finitely many tuple types. If the type $[x]$ can be split, then there exist $s_0, t_0, s \in X$ such that $|[s; s_0, t_0]| = \infty$ and for any t so that $t_0\hat{\ }t$ is analogous to $s_0\hat{\ }s$, $t \notin [s; s_0, t_0]$.*

Proof. If the group action $G \longmapsto X$ can be split, then there exists $z \in [x]$ such that sets $[z;x]$ and $[x] \setminus [z;x]$ have infinite cardinality. By Lemma 24, it must be that there exists a permutation g such that $[z;x] \setminus g \cdot [z;x]$ is infinite. Take $s_0 = x$ and $t_0 = g \cdot x$. Because of the finitely many tuple types, there exists $s \in [z;x] \setminus g \cdot [z;x]$ such that the relative type $[s; s_0, t_0]$ is infinite. The Lemma follows. □

Lemma 27. *Suppose that the group action $G \longmapsto X$ has finitely many tuple types. Let $x, u \in X$ be such that $|[x;u]| = 1$. Suppose that the type $[x]$ can be robustly split. Then, there exist a splitting sequence of type $[u]$.*

Proof. We construct a splitting sequence $s_0, t_0, s_1, ...$ of elements of type $[u]$ and a sequence $x_0, x_1, x_2, ... \in [x]$ such that for each m, the relative type $[x_m; \bar{s}_{m-1}]$ is infinite, and $x^s_m, x^t_m, x_{m+1} \in [x_m; \bar{s}_{m-1}]$, where x^s and x^t are the unique elements such that $x^s_m\hat{\ }s_m$ and $x^t_m\hat{\ }t_m$ is analogous to $x\hat{\ }u$. The construction follows from a repeated application of Lemma 26. □

Lemma 28. *Suppose that the group action $G \longmapsto X$ is ψ-compact for some $\psi < \frac{1}{2}$. Then, there is no splitting sequence.*

Proof. Find U_0 from the definition of ψ-compactness. Find a splitting sequence $s_0, t_0, s_1, ...$ Find a collection of local sets $U_n \supseteq U_0, s_n, t_n$. Because of ψ-compactness, we can ensure that $|U_n| \leq 2^{\psi(|U_0|+2n)}$.

On the other hand, for each $0 < m \leq n$, find $g_{m,n} \in G_{U_n}$ such that $g_{m,n} \cdot \bar{s}_{m-1} = \bar{s}_{m-1}$ and $g_{m,n} \cdot s_m = t_m$. Such bijections exist because U_n is local. Let $A_{n,n} = \{s_n, t_n\}$, and for each

$m < n$,

$$A_{m,n} = A_{m+1,n} \cup g_{m,n} \cdot A_{m+1,n}.$$

Then, by induction and the choice of the sequence, $A_{m+1,n} \subseteq [s_{m+1}; \bar{s}_m]$. Moreover, for each $t \in g_{m,n} \cdot A_{m+1,n}$, $t_m\hat{\ }t \in [s_m\hat{\ }s_{m+1}; \bar{s}_{m-1}]$, and $t \notin A_{m,n}$. Thus, sets $A_{m,n}$ and $g_{m,n} \cdot A_{m+1,n}$ are disjoint. Because they have equal cardinality, $|A_{m,n}| = 2\,|A_{m+1,n}|$, and $A_{0,n} = 2^n$. Because $A_{0,n} \subseteq U_n$, it must be that $|U_n| \geq 2^n$. Contradiction. $\square$

B.5. **Generation and small orbits.** Group action $G \longmapsto X$ *has uniformly bounded* 1-*types*, if there exists a constant $m < \infty$ such that $|[x]| \leq m$ for each $x \in X$. The next Lemma shows that, for group actions with uniformly bounded 1-types, any finite set of permutations extends to a finite subgroup of permutations.

Lemma 29. *Suppose that $G \longmapsto X$ is a group action with uniformly bounded 1-types. For any finite set $G_0 \subseteq G$, there exists a finite subgroup $G' \subseteq G$ so that $G_0 \subseteq G'$.*

Proof. Let $m = \sup_{x \in X} |[x]|$. For each $x \in X$, choose a bijection $i_{[x]} : [x] \to \{1, ..., |[x]|\}$. For each $g \in G$, define permutation bijection $g_{[x]} = i_{[x]} \circ g \circ i_{[x]}^{-1}$ of set $\{1, ..., k_x\}$.

For each $[x]$, let $U[x] = \left\{[x'] : g_{[x']} = g_{[x]} \text{ for each } g \in G'\right\}$. Then, $\{U[x]\}_{[x] \subseteq X}$ is a finite partition of the infinite set $\{[x] : x \in X\}$ of all 1-types, and $\left|\{U[x]\}_{[x] \subseteq X}\right| \leq (m!)^{|G_0| m} < \infty$.

For each $[x]$, let $G_{[x]} \supseteq \left\{g_{[x]} : g \in G\right\}$ be the smallest group generated by permutations $\left\{g_{[x]} : g \in G_0\right\}$. Because $G_{[x]}$ is a subset of symmetric group $G_{[x]} \subseteq \Pi\{1, ..., |[x]|\}$, $\left|G_{[x]}\right| \leq m!$.

Without loss of generality, assume that $\mathrm{id}_X \in G_0$ and that $g^{-1} \in G_0$ for each $g \in G_0$. Let $G' = \{g_1 ... g_n : g_i \in G, n < \infty\}$ be the set of all finite products of permutations in G_0. Clearly, G' is a group. Moreover, for each $g \in G'$, $g_{[x]} \in G_{[x]}$. Hence, $|G'| \leq \left(\sup_x \left|G_{[x]}\right|\right)^{\left|\{U[x]\}_{[x] \subseteq X}\right|} < \infty$. $\square$

B.6. **Countable extensions.** The last result establishes a simple extension property of permutations on a countable set.

Lemma 30. *Suppose that X is countable. Suppose that $g_0, g_1, ... \in G$ is a sequence of permutations such that there exists a partition of set X into finite sets $\mathcal{P} = \{V \subseteq X\}$ such that for each $V \in \mathcal{P}$, $g_n \cdot V \in V$ for each n, and there exists n_V so that for each $m > n_V$,*

$$g_m \cdot V = V \text{ and } g_m \cdot (g_{n_V} .... g_1 \cdot V) = g_{n_x} .... g_1 \cdot V.$$

Then, there exists g that preserves analogies and such that $g \cdot V = g_{n_x} .... g_1 \cdot V$ for each $V \in \mathcal{P}$.

Proof. For each $V \in \mathcal{P}$, define $g \cdot V = g_{n_V} .... g_1 \cdot V$. Similarly, for each $V \in \mathcal{P}$, define $g^{-1} \cdot V = (g_{n_V} .... g_1)^{-1} \cdot V$. Then, g is a bijection on $\mathcal{P}$, and g^{-1} is its inverse.

Let $V_0, V_1, ...$ be an enumeration of partition $\mathcal{P}$. By taking subsequences, we can find a sequence of permutations $g^0, g^1, ...$ such that $g^i \cdot V = g \cdot V$ for each $V \in \mathcal{P}$, and $g^{i+1}|_{V_0 \cup ... \cup V_i} = g^i|_{V_0 \cup ... \cup V_i}$. Then, the pointwise limit $g = \lim_{i \to \infty} g^i$ is a well -defined bijection. Moreover, for each finite $\bar{x}$, there is i such that $g^i \cdot \bar{x} = g \cdot \bar{x}$, and $\bar{x}$ and $g \cdot \bar{x}$ are analogous. $\square$

Appendix C. Concepts

This section deals with concepts and their properties. Throughout the section, we assume that $G \longmapsto X$ is a group action, and we list additional properties only when they are needed for extra results.

We list all definitions used in this appendix. A concept is a subset $S \subseteq X$ such that $i_S := \sup_{x \in S} |[S; x]| < \infty$. For each concept S, let $i(S) = \sup_{x \in S} |[S; x]| < \infty$.

Concept S is a *block,* if for each $g \in G$, either $g \cdot S = S$, or $g \cdot S \cap S = \varnothing$. In other words, S is a block if $i(S) = 1$.

A tuple of variables $\bar{x}$ is a *code* of concept S, if $[S; \bar{x}] = \{S\}$. For example, if S is a block, then any $x \in S$ is a code of S.

Concept $S \subseteq X$ is *coinfinite,* if for each concept $S' \in [S]$ either $S' = S$ or $|[S'; S]| = \infty$. Let $\mathcal{S}$ be the set of all coinfinite concepts.

It is useful to study the action $G \longmapsto X \cup \mathcal{S}$ of group G on the elements of space X and coinfinite concepts in $\mathcal{S}$. In order to distinguish concepts under the group actions $G \longmapsto X$ and $G \longmapsto X \cup \mathcal{S}$, we reserve letters $S, S', S'', s \subseteq X$ for the former, and $C, C', C'', c \subseteq X \cup \mathcal{S}$. Of course, any concept under the former group action is also a concept under the latter. For each $C \subseteq X \cup \mathcal{S}$, define the union of elements of C as

$$pC = \bigcup C \subseteq X.$$

Here, we abuse slightly the notation, and we treat elements of C as subsets of X; this is immediate when $x \in C \cap \mathcal{S}$, and if $x \in C \cap X$, then we interpret x as one-element set $\{x\}$.

Say that subset (not necessarily a concept) $C \subseteq X \cup \mathcal{S}$ is *robustly block exchangeable*, if C is infinite, $C_C \to C$ is transitive, block highly transitive with finite block $B \subseteq C$, and for each tuple $\bar{x}$, there exists $x' \in C$ such that $|C \backslash [x'; C, \bar{x}]| < \infty$ and $G_C, \bar{x} \longmapsto [x'; C, \bar{x}]$ is block highly transitive with block $B' \in [B]$. If $|B| = 1$, we drop the word "block."

Two robustly exchangeable concepts C^1 and C^2 are $\bar{x}$-*independent* for some tuple $\bar{x} \subseteq X \cup \mathcal{S}$, if for each i, there are $x_0^i \in C^i$ such that $|C^i \backslash [x_0^i; C^1, C^2, \bar{x}]| < \infty$ and, for any finite tuple of concepts $\bar{x}^{-i} \subseteq C^{-i}$, $[x_0^i; C^1, C^2, \bar{x}] = [x_0^i; C^1, C^2, \bar{x}, \bar{x}^{-i}]$. Together with robust exchangeability, the latter implies that the group actions $G_{C^1, C^2, \bar{x}} \cap G_{x^{-i}} \longmapsto [x_0^i; C^1, C^2, \bar{x}, \bar{x}^{-i}]$

are highly transitive. If C^1 and C^2 are $\bar{x}$-independent for each tuple $\bar{x}$, then we say that they are *independent*.

Two robustly exchangeable concepts C^1, C^2 are $\bar{x}$-*correlated*, if for each i, there are $x_0^i \in [C^i]$ such that $|C^i \setminus [x_0^i; C^1, C^2, \bar{x}]| < \infty$ and a bijection $j : [x_0^1; C^1, C^2, \bar{x}] \to [x_0^2; C^1, C^2, \bar{x}]$ such that $(g \circ j)(x') = (j \circ g)(x')$ for each $g \in G_{C^1,C^2,\bar{x}}$ and each $x' \in [x_0^1; C^1, C^2]$. We refer to j as the *correlating function*. Robust exchangeability implies that the group actions $G_{C^1,C^2,\bar{x}} \longmapsto [x_0^i; C^1, C^2, H]$ are highly transitive. It is easy to check that if two robustly exchangeable concepts of concepts are $\bar{x}$-correlated, then they are $\bar{x}'$-correlated for each $\bar{x}' \supseteq \bar{x}$.

C.1. **Basic properties.**

Lemma 31. *If S is a concept, $S' \subseteq S$, and $S' \neq S$, then $S' \notin [S]$.*

Proof. On the contrary, suppose that $S' \sim S, S' \neq S$, and $S' \supseteq S$. Find $g \in G$ such that $g \cdot S' = S$. Consider a decreasing sequence of sets $S_0 = S', S_n = g \cdot S_{n-1}$. Then, $S_{n-1} \supseteq S_n$ and $S_{n-1} \neq S_n$. Find $n > i(S)$ and $x \in S_n$. Then, $i(S) \geq |\{S'' \in [S] : x \in S\}| \geq |\{S_0, ..., S_n\}| \geq n + 1 > i(S)$. Contradiction. ☐

Lemma 32. *Suppose that there exists a finite collection of concepts $\mathcal{S}$ such that $X = \bigcup_{S \in \mathcal{S}} S$. Then, there exists $S \in \mathcal{S}$ such that $[S] < \infty$.*

Proof. On the contrary, suppose that $|[S]| = \infty$ for each $S \in \mathcal{S}$. By Lemma 21, there exists a sequence of permutations $g_1, g_2, ...$ such that $g_n \cdot \mathcal{S} \neq g_m \cdot \mathcal{S}$ for $m \neq n$. Because $X = \bigcup_{S \in \mathcal{S}} S = \bigcup_{S \in g_n \cdot \mathcal{S}} S$ for each n, and $\mathcal{S}$ is finite, it means that for each $x \in X$, there exists $S \in \mathcal{S}$, such that x belongs to $g_n \cdot S$ for infinitely many n. This yields a contradiction with the fact that S is a concept. ☐

Lemma 33. *If C is a concept under the group action $G \longmapsto X \cup \mathcal{S}$, then pC is a concept under the group action $G \longmapsto X$ and C is pC-algebraic.*

Proof. For each $x \in X$,

$$\begin{aligned}
|\{P : x \in P, P \in [pC]\}| &= |\{pC' : x \in pC', C' \in [C]\}| \\
&\leq |\{s : x \in s, s \in \mathcal{S}\}| \sup_{s \in \mathcal{S}} |\{C' : s \in C', C' \in [C]\}| \\
&\leq i(S)\, i(C) < \infty.
\end{aligned}$$

This shows the first part of the claim. For the second, notice that for each $x \in X$,

$$\{C' \in [C] : x \in pC'\} = \bigcup_{S' \in [S] : x_0 \in S'} \{C' \in [C] : S' \in C'\}.$$

Hence, for each $x \in pC$,

$$\{C' \in [C] : pC' = pC\} \leq |\{C' \in [C] : x \in pC'\}| < \infty.$$

$\square$

C.2. **Coinfinite concepts.**

Lemma 34. *Suppose that $G \longmapsto X$ has finitely many tuple types. Any concept has a code. Any coinfinite concept has a two-element code.*

Proof. Suppose that S is a concept. If S is finite, then any enumeration $\bar{x}$ of S is a code of S. Suppose that S is infinite. By Lemma 31, for each $S' \in [S] \setminus \{S\}$, there is always $x' \in S \backslash S'$. Thus, we can find a set $V \subseteq S$, $|V| \leq i(S)$ such that if $S' \in S$ and $S' \supseteq V$, then $S' = S$. Then, any enumeration $\bar{x}$ of set V is a code of S.

Suppose that S is a coinfinite concept. Consider two cases. Suppose that for each $x \in S$, there exists $x' \in S$ so that $\{S' \in [S] : x \in S\} \cap \{S' \in [S] : x' \in S\} = \{S\}$. Then, $[S; x, x'] = \{S\}$.

Alternatively, suppose that there exists $x \in S$, such that for each $x' \in S$, there exists $S' \in [S] \setminus \{S\}$ such that $x, x' \in S'$. In other words, $S = \bigcup_{S' \in \mathcal{S}} S'$, where $\mathcal{S} = \{S' \in [S] \setminus \{S\} : x \in S'\}$. By Lemma 32, there exists $S_0 \in \mathcal{S}$ such that $S_0 \cap S$ is S-algebraic. Because S_0 is a concept, it must be that S_0 is S-algebraic. That contradicts the fact that S is coinfinite. $\square$

Lemma 35. *Suppose that $G \longmapsto X$ has finitely many tuple types. If S is a concept, then $G \longmapsto X \cup [S]$ has finitely many tuple types. Moreover, $G \longmapsto X \cup \mathcal{S}$ has finitely many tuple types.*

Proof. Suppose that S is a concept. By the first part of Lemma 34, concept S has a code $\bar{x}$. Let M be the length of tuple $\bar{x}$. Then, the number of n-tuples of the group action $G \longmapsto X \cup [S]$ is not higher then the number of Mn-tuples of the group action $G \longmapsto X$.

Suppose that S is a coinfinite concept. First, we show that there are finitely many concepts S such that pair of elements $\bar{x}^* \in X^2$ is a code of concept S. Notice that $\bar{x}^*$ partitions the space X into relative types $\Pi = \{[x; \bar{x}^*] : x \in X\}$. Let $M = |\Pi|$ be the size of the partition. Then, M is not higher than the number of types of 3-tuples. Because $G \longmapsto X$ has finitely many tuple types, $M < \infty$.

One easily checks that if concept S is coded by $\bar{x}^*$, then S must be measurable with respect to partition Π. In particular, there are at most 2^M concepts encoded by tuple $\bar{x}^*$.

Take any n. The number of types of n-tuples $\bar{s} \in (X \cup \mathcal{S})^n$ of the group action $G \longmapsto X \cup \mathcal{S}$ is bounded by $\left(2^M\right)^n$ times the number of types of $2n$-tuples of the group action $G \longmapsto X$. $\square$

Lemma 36. *Suppose that $G \longmapsto X$ has finitely many tuple types. For each concept S_0, there exists the largest coinfinite concept $T_0 \supseteq S_0$ such that S_0 is T_0-algebraic. If $C_0 \subseteq X \cup \mathcal{S}$ is a concept under the group action $G \longmapsto X \cup \mathcal{S}$, then there exists the largest coinfinite concept $T_0 \supseteq pC_0$ such that C_0 is T_0-algebraic.*

Proof. By Lemma 35, the group action $G \longmapsto [S_0]$ has finitely many tuple types.

We show that for each concept $S \in [S_0]$, there exists a coinfinite concept T such that $S \subseteq T$ and S is T-algebraic. For each $S \in [S_0]$, define $B(S) = \{S' \in [S] : S' \text{ is } S \text{ algebraic}\}$. We show that $B(S)$ is a finite block. The claim follows from the following observations. First, because of finitely many types of 2-tuples, $B(S)$ is a union of finitely many finite sets $[S'; S]$, hence $B(S)$ is finite. Second, for each permutation g, $g \cdot B(S) = B(g \cdot S)$. Third, $B(S) = B(S')$ for each $S' \in B(S)$. Indeed, because S and S' have the same type, it must be that $|B(S')| = B(S)$. Moreover, if $S'' \in B(S')$, then S'' is S'-algebraic, and hence, also S-algebraic. Thus, $B(S') \subseteq B(S)$.

Define $T = \bigcup B(S)$. Then,

$$|[S;T]| \leq |[S; B(S)]| \, |[B(S); T]| \leq |B(S)| \, i(S) < \infty.$$

Next, we show that T is coinfinite. Suppose not. Then, there exists T' such that $T' \neq T$ and T' is T-algebraic. Suppose that $T' = B(S')$. Then, $S' \notin B(S)$. On the other hand,

$$|[S'; S]| \leq |[S'; B(S')]| \, |[B(S'); T']| \, |[T'; T]| \, |[T'; S]|.$$

Because each of the terms on the right-hand side is finite ($|[S'; B(S')]| \leq |B(S)|$, $|[B(S'); T']| \leq i(S)$, $|[T'; T]| < \infty$, and $|[T; S]| = 1$) the left-hand side is finite as well. This contradicts the fact that $S' \notin B(S)$.

Finally, we show that, for each coinfinite concept S, there exists the largest coinfinite concept T such that $S \subseteq T$ and S is T-algebraic. Let $\mathcal{S} = \{T : T \subseteq X \text{ is a coinfinite concept}\}$. For each S, define $B(S) = \{T \in \mathcal{S} : S \subseteq T \text{ and } S \text{ is } T\text{-algebraic}\}$. Then, because of the finitely many tuple types, one shows that $T = \bigcup B(S) \in B(S)$. Such T is the largest coinfinite concept that contains S.

The Lemma follows from the above observations and Lemma 33. □

C.3. **Compactness properties of $G \longmapsto X \cup \mathcal{S}$.**

Lemma 37. *Suppose that the group action $G \longmapsto X$ has finitely many tuple types and it is ψ-compact for some $\psi < \infty$. Consider the group action $G \longmapsto X \cup \mathcal{S}$. Then, for each k,*

(1) *there exists a constant c_k such that for each finite V, there exists k-local $U \supseteq V$ such that $\log |U| \leq \psi \log |V| + c_k$, and*

(2) *for each* $\varepsilon > 0$, $x \in X \cup \mathcal{S}$, *and finite set* $V \supseteq [x]$, *there exists* k*-local* $U \subseteq [x]$ *such that* $V \subseteq U$ *and for each* $x' \in [x]$, *there exists a* k*-local* $U' \supseteq U, x'$ *so that* $\log |U'| \leq \psi + \varepsilon + \log |U|$.

Proof. By definition, the group action $G \longmapsto X$ has finitely many tuple types. By Lemma 35, the group action $G \longmapsto \mathcal{S}$ has finitely many 1-tuples. Fix a finite set $\mathcal{T} \subseteq \mathcal{S}$ such that $\mathcal{T}$ contains exactly one representative for each type of concepts $[S] \subseteq \mathcal{S}$. For each $S \in \mathcal{T}$, find a two-element code $\bar{x}^S \in X^2$. Also, fix a finite set set $X_0 \subseteq X$ that contains exactly one representative for each type of elements of X. Let $\bar{X}_0$ be a finite set of 2-tuples that contains a two element code $\bar{x}$ for each concept $s \in \mathcal{S}$ such that $s \cap X_0 \neq \varnothing$ and such that $\bar{x}\hat{\ }s$ is analogous to $\bar{x}^S\hat{\ }S$ if $s \in [S]$.

Fix $k \geq 2$. Because $G \longmapsto X$ is ψ-compact, there exists a $2k$-local U_0 such that for each $2k$-local $U \supseteq U_0$, each $x \in X$, there exists local set $U' \supseteq U, x$ such that

$$\log |U'| \leq \psi + \log |U|. \tag{C.1}$$

Assume that $U_0 \supseteq X_0, \bar{X}_0$, and that, additionally, U_0 is large enough so that each type of $4k$-tuples is represented in U : for each tuple $\bar{x} \in X^{4k}$, there exists $\bar{x}' \subseteq U_0$ that is analogous to $\bar{x}$.

For each $U \subseteq X$ such that $U \supseteq U_0$ and U is $2k$–local under the group action $G \longmapsto X$, define

$$\begin{aligned} U^{\mathcal{S}} &= U \cup \bigcup\nolimits_{S \in \mathcal{T}} \left\{ s \in [S] : s\hat{\ }\bar{x} \in S\hat{\ }\bar{x}^S \text{ for some } \bar{x} \subseteq U \right\} \\ &= U \cup \{ s \in \mathcal{S} : s \cap U \neq \varnothing \} \end{aligned}$$

The second equality follows from the fact that U is 1-local, and that $U \supseteq U_0 \supseteq X_0, \bar{X}_0$. Notice that there exists a constant $M = \sup_{x \in X} |\{s \in \mathcal{S} : x \in s\}|$ such that

$$\left| U^{\mathcal{S}} \right| \leq M |U|. \tag{C.2}$$

For each $x \in U^{\mathcal{S}}$, define $c^U(x) = x$ if $x \in X$ and $c^U(x) = \bar{x} \subseteq U$ if $x\hat{\ }\bar{x}$ is analogous to $S\hat{\ }\bar{x}^S$ for some $S \in \mathcal{T}$. The choice of the mapping c^U is not unique.

We show that $U^{\mathcal{S}} \subseteq X \cup \mathcal{S}$ is k-local under the group action $G \longmapsto X \cup \mathcal{S}$. Indeed, take any two k-tuples $\bar{x}, \bar{x}' \subseteq U^{\mathcal{S}}$ such that $\bar{x}$ and $\bar{x}'$ are analogous. Let $\bar{c} = c^U(x_1)\hat{\ }...\hat{\ }c^U(x_k)$ and similarly define $\bar{c}'$. Because tuples $\bar{x}$ and $\bar{x}'$ are analogous, there exists a tuple $\bar{d}$ such that $\bar{x}\hat{\ }\bar{d}$ and $\bar{x}'\hat{\ }\bar{c}'$ are analogous. Because U contains the representatives of all types of $4k$-tuples, there exists a tuple $\bar{z}\hat{\ }\bar{w} \subseteq U$ such that $\bar{c}\hat{\ }\bar{d} \in [\bar{z}\hat{\ }\bar{w}; U]$. Find $g \in G_U$ such that $g \cdot \bar{c}\hat{\ }\bar{d} = \bar{z}\hat{\ }\bar{w}$ and notice that $g \in G_{U^{\mathcal{S}}}$ (this follows from the fact that a code uniquely defines the associated concept, and because of the construction of set $U^{\mathcal{S}}$.). Let $\bar{x}'' = g \cdot \bar{x}$. Then, $\bar{x}\hat{\ }\bar{c} \in \left[\bar{x}''\hat{\ }\bar{z}; U^{\mathcal{S}}\right]$. Using a similar argument, we can show that $\bar{x}'\hat{\ }\bar{c}' \in \left[\bar{x}''\hat{\ }\bar{w}; U^{\mathcal{S}}\right]$. Thus, $\bar{x}, \bar{x}' \in \left[\bar{x}''; U^{\mathcal{S}}\right]$.

The rest of the proof is concluded in two steps.

(1) For each set $V \subseteq X \cup \mathcal{S}$, find set V^X such that $V \subseteq \left(V^X\right)^{\mathcal{S}}$. We can choose V^X so that $\left|V^X\right| \leq |V|$. Because of (C.1) and (C.2), we can find $2k$-local $U \supseteq V^X \cup U_0$, $U \subseteq X$ such that

$$\log\left|U^{\mathcal{S}}\right| \leq \log|U| + \log M \leq \psi\log|V| + \psi\log|U_0| + \log M.$$

(2) Take any $x \in X \cup \mathcal{S}$ and finite set $V \subseteq X \cup \mathcal{S}$. Suppose that there is $\varepsilon > 0$ such that for each k-local $U \subseteq [x]$ such that $V \subseteq U$, there is $x^U \in [x]$ so that for each k-local $U' \supseteq U, x$, $|U'| > 2^{\psi+\varepsilon}|U|$. Let $z^U \in x^U \cap X$ (if $x^U \in X$, let $z^U = x^U$).
Construct a sequence $W_0 \subseteq W_1 \subseteq \ldots \subseteq X$ of k-local sets under the group action $G \longmapsto X$. Let $W_0 \supseteq U_0$ be a k-local set that is large enough so that $W^{\mathcal{S}} \supseteq V$. For each k, find local $W_k \supseteq W_{k+1}, z^{W^{\mathcal{S}}_{k-1}\cap[x]}$. Because of ψ-compactness, we can find the sequence so that $|W_k| \leq 2^{\psi}|W_{k-1}|$ for each k. On the other hand, for each k, $W^{\mathcal{S}}_k \cap [x] \supseteq V, x^{W^{\mathcal{S}}_x\cap[x]}$ and

$$\left|W^{\mathcal{S}}_k \cap [x]\right| \geq 2^{\psi+\varepsilon}\left|W^{\mathcal{S}}_{k-1} \cap [x]\right|.$$

Thus,

$$\lim_{k\to\infty}\frac{\left|W^{\mathcal{S}}_k\right|}{|W_k|} \geq \lim_{k\to\infty}\frac{2^{(\psi+\varepsilon)k}\left|W^{\mathcal{S}}_0 \cap [x]\right|}{2^{\psi k}|W_0|} = \infty,$$

which contradicts (C.2).

□

C.4. **Robust exchangeability.**

Lemma 38. *If C is robustly exchangeable concept, and C' is a concept, then either $C \supseteq C'$, $|C' \cap C| = 1$, or C and C' are disjoint.*

Proof. It is easy to see that $C \cap C'$ is a concept. Thus, it is enough to show that for any robustly exchangeable concept C, any concept $C' \subseteq C$, $C' \neq C$, it must be that $|C'| = 1$. Suppose not. Because C' is a concept, then $i(C') < \infty$. Find a subset $A \subseteq C$ such that $i(C') + 2 \leq |A| < \infty$, $|A \cap C'| \geq 2$, and $|A \backslash C'| \geq 1$. Fix $x_0 \in A \cap C'$. Then,

$$i(C') \geq |\{g \cdot C' : x_0 \in g \cdot C'\}| \geq |\{g \cdot (A \cap C') : g \in G_{x_0}\}| \geq i(C') + 1,$$

which yields a contradiction with the fact that S' is a concept. □

C.5. **Independence and correlation.**

Lemma 39. *For any tuple $\bar{x}$, any two robustly exchangeable concepts C^1 and C^2 are either $\bar{x}$-correlated or $\bar{x}$-independent.*

Proof. Suppose that C^1 and C^2 are robustly exchangeable. For each i, find $x_0^i \in [C^i]$ such that $|C^i \backslash [x_0^i; C^1, C^2, \bar{x}]| < \infty$. By robust exchangeability, the group actions $G_{C^1,C^2,\bar{x}} \longmapsto [x_0^i; C^1, C^2, \bar{x}]$ are highly transitive for each $i = 1, 2$.

Find an infinite sequence of distinct elements $x_1^i, x_2^i, ... \in C^i$ and let $E_n^1 = \{x_1^1, ..., x_n^1\}$. Let $G_n = G_{C^1,C^2,\bar{x}} \cap \bigcap_{x \in E_n^1} G_x$. Because of robust exchangeability, for each n, there is finite $E_n^2 \subseteq [x_0^2; C^1, C^2, \bar{x}]$ so that $G_n \longmapsto [x_0^2; C^1, C^2, \bar{x}] \backslash E_n^2$ is highly transitive and $G_n \cdot E_n^2 = E_n^2$. Of course, the sequence of finite sets E_n^2 is (weakly) increasing in the set order, $E_n^2 \subseteq E_{n+1}^2$ for each n. To shorten the subsequent notation, take $E_0^1 = E_0^2 = \varnothing$ and define $C_n^i = [x_0^i; C^1, C^2, \bar{x}] \backslash E_n^i$.

If $|E_1^2| = 1$, then let $j(x_1^1) = x_1^2$. Because of the high transitivity of $G_1 \longmapsto C^2 \backslash E_1^2$, j can be extended to a bijection $j : C^1 \rightarrow C^2$ such that $(g \circ j)(x') = (j \circ g)(x')$ for each $g \in G_{C^1,C^2,\bar{x}}$ and each $x' \in [x_0^1; C^1, C^2]$ .for any $g' \in G_{C^1,C^2}$. Hence, C^1 and C^2 are $\bar{x}$-correlated.

If $E_n^2 = \varnothing$ for each n, then, C^1 and C^2 are $\bar{x}$-independent.

We show that there is no other possibility.

On the contrary, suppose that $E_n^2 = \varnothing$ and $\left|E_{n+1}^2\right| \geq 2$ for some $n \geq 0$. Then, group action $G_n \longmapsto C_n^i$ is highly transitive. Because $\left(G_n \cap G_{x_{n+1}^1}\right) \cdot E_{n+1}^2 = E_{n+1}^2$, this implies that $G_n \longmapsto \left[E_{n+1}^2; C^1, C^2\right]$ is highly transitive and E_{n+1}^2 is a finite and non-trivial block of group action $G_n \longmapsto C_n^2$. Because highly transitive group action does not have non-trivial blocks, we get a contradiction.

Alternatively, suppose that $|E_n^2| = 0$ and $\left|E_{n+1}^2\right| = 1$ for some $n \geq 1$. Let $E_{n+1}^2 = \left\{x_{n+1}^2\right\}$. For any $g, g' \in G_{n-1}$ such that $g \cdot x_n^1 = g' \cdot x_n^1$ and $g \cdot x_{n+1}^1 = g' \cdot x_{n+1}^1$, we have $g \cdot x_{n+1}^2 = g' \cdot x_{n+1}^2$. Indeed, if not, then $g^{-1}g' \in G_{n+1}$, but $g^{-1}g' \notin G_{x_{n+1}^2}$, which contradicts the choice of x_{n+1}^2. For each $x_a, x_b \in C_{n-1}^1$, define $j_{x_a}(x_b) = g \cdot x_{n+1}^2$ for some g so that $g \cdot x_n^1 = x_a$ and $g \cdot x_{n+1}^1 = x_b$. The definition does not depend on the choice of g and for any $g \in G^{n-1}$, $g \cdot j_{x_a}(x_b) = j_{g \cdot x_a}(g \cdot x_b)$.

Because $G_n \longmapsto C_n^2$ is highly transitive, it must be that $j_{x_a}\left(C_{n-1}^1 \backslash \{x_a\}\right) = C_n^2$ (otherwise $j_{x_n^1}\left(C_{n-1}^1 \backslash \{x_n^1\}\right) \neq C_n^2$, and there is $g \in G_{n-1}$ such that $g \cdot j_{x_n^1}\left(C_{n-1}^1 \backslash \{x_n^1\}\right) \neq j_{x_n^1}\left(C_{n-1}^1 \backslash \{x_n^1\}\right)$, which implies that $g \cdot C_{n-1}^1 \backslash \{x_n^1\} \neq C_{n-1}^1 \backslash \{x_n^1\}$ and $g \notin G_n$). Similarly, one shows that $j_{C_{n-1}^1 \backslash \{x_b\}}(x_b) = C_n^2$. Because $G_n \longmapsto C_n^2$ is highly transitive, we have $j_{x_a}(x_b) \neq j_{x_a}(x_b')$ for $x_b \neq x_b'$ (otherwise $j_{x_n^1}^{-1}(x')$ would be a block of group action $G_n \longmapsto C_{n-1}^2$.)

We show that for each $x_a, x_a', x_b \in C_{n-1}^1$, if $x_b \neq x_a, x_a'$, then $j_{x_a}(x_b) = j_{x_a'}(x_b)$. Suppose not and find $x_b' = j_{x_a}^{-1}\left(j_{x_a'}(x_b)\right) \neq x_b$.

- If $x'_b \neq x_a, x'_a$, then there is$g \in G_{x_a, x'_a, x_b}$ such that $g \cdot x'_b \neq x'_b$. But then, $g \cdot j^{-1}_{x_a}\left(j_{x'_a}(x_b)\right) = j^{-1}_{g\cdot x_a}\left(j_{g\cdot x'_a}(g \cdot x_b)\right) = j^{-1}_{x_a}\left(j_{x'_a}(x_b)\right) = x'_b$.
- If $x'_b = x_a, x'_a$, then there is $g \in G_{x_a, x'_a}$ such that $g \cdot x_b \neq x_b$. But then, $g \cdot x_b = g \cdot j^{-1}_{x_a}\left(j_{x'_a}(x'_b)\right) = j^{-1}_{x_a}\left(j_{x'_a}(x'_b)\right) = x_b$.

Define $j^* : C^1_{n-1} \to C^2_{n-1}$ by $j^*(x_b) = j_{x_a}(x_b)$ for some $x_a \neq x_b$. Then, for any $g \in G_{n-1}$, $g \circ j^*|_{C^1_{n-1}} = j^* \circ g|_{C^1_{n-1}}$, which shows that $G_n \cdot \left(E^2_{n-1} \cup j^*(x^1_n)\right) = \left(G_{n-1} \cap G_{x^1_n}\right) \cdot \left(E^2_{n-1} \cup j^*(x^1_n)\right) = E^2_{n-1} \cup j^*(x^1_n)$, which contradicts the initial claim that $G_n \longmapsto C^2 \backslash E^2_{n-1}$ is highly transitive.

This ends the proof of the Lemma. □

Lemma 40. *Suppose that the group action $G \longmapsto X$ has finitely many tuple types and it is ψ-compact for some $\psi < \infty$. Consider the group action $G \mapsto X \cup \mathcal{S}$. Any two robustly exchangeable concepts C^1 and C^2 are either independent or correlated.*

Proof. We assume that the thesis of Lemma 37 holds. We show that, if two robustly exchangeable concepts C^1 and C^2 are $\bar{v}$-independent for some tuple $\bar{v}$, then they are $\bar{v}\hat{\ }x$-independent for any $x \in X$.

Take any tuple $\bar{v}$ and element $x \in X$. Let k_0 be the length of tuple $\bar{v}$. Let $k = k_0 + 13$. Suppose that C^1 and C^2 are robustly exchangeable and $G_{\bar{v}}$-independent but $G_{\bar{v}\hat{\ }x}$-correlated. There are $x^i_0 \in C^i$ such that if $C^i_0 = [x^i_0; \bar{v}\hat{\ }x, C^1, C^2]$, then $|C^i \backslash C^i_0| < \infty$ and the group action $G_{\bar{v}\hat{\ }x, C^1, C^2} \longmapsto C^i_0$ is highly transitive. Let $j : C^1_0 \to C^2_0$ be the correlating function of $G_{\bar{v}\hat{\ }x}$-correlation. Take any finite set $V_0 \supseteq \bar{v}, x, x^1_0, x^2_0$. Let $V^i_m \subseteq [x^i_0; \bar{v}, C^1, C^2]$ be finite subsets such that $|V^i_m| = m$ and $V^2_m = j(V^1_m)$.

Suppose that $m \geq 8$. Take any k-local set $U \supseteq V_0, V^1_m, V^2_m$. Because of $G_{\bar{v}}$-independence, and by Lemma 16

$$\left|\left[\bar{V}^2_m; \bar{V}^1_m, \bar{v}, U\right]\right| \geq (m-2)!$$

Because of $G_{\bar{v}\hat{\ }x}$-correlation, any enumeration $\bar{V}^2_m$ is $\left(\bar{v}\hat{\ }x\hat{\ }\bar{V}^1_m\right)$-definable, where, for each i, $\bar{V}^i_m$ are enumerations of set V^i_m. By a version of the counting argument from Section (7.2.6),

$$|U| \geq \left|\left[x; \bar{V}^1_m, \bar{V}^2_m, \bar{v}, U\right]\right| \geq \left|\left[\bar{V}^2_m; \bar{V}^1_m, \bar{v}, U\right]\right| \geq (m-2)!$$

Thus, by Stirling's approximation,

$$\liminf_{m\to\infty} \inf_{U \supseteq V_0, V^1_m, V^2_m} \frac{\log|U|}{|V_0| + 2m} \geq \lim_{m\to\infty} \frac{(m-2)\log(m-2)}{|V_0| + 2m} = \infty,$$

which contradicts ψ-compactness. □

Appendix D. Robust exchangeability under compact group actions

The goal of this part of the appendix is to prove Lemma 3. Assume that $G \longmapsto X$ is $\frac{1}{20}$-compact. Then, by Lemmas 35 and 37, the group action $G \longmapsto X \cup \mathcal{S}$ has finitely many tuple types, and it satisfies two quasi-compact properties: for each k, there exists a constant c_k such that for each finite V, there exists k-local $U \supseteq V$ such that

$$\log |U| \leq \frac{1}{10} \log |V| + c_k, \text{ and} \tag{D.1}$$

for each $x \in X \cup \mathcal{S}$, each finite set $V \supseteq [x]$, there exists k-local $U \subseteq [x]$ such that $V \subseteq U$ and for each $x' \in [x]$, there exists a k-local $U' \supseteq U, x'$ so that

$$|U'| < \frac{3}{2} |U|. \tag{D.2}$$

Additionally, the results about splitting from Appendix B.4 apply.

D.1. Proof of Lemma 3.

Set $C \subseteq X$ is *complete*, if $|C| = \infty$, and there exists a constant $M < \infty$ such that for each $C' \in [C]$, if $C' \neq C$, then $|C' \cap C| \leq M$. We show the following partial results.

Lemma 41. *For each tuple $\bar{x} \subseteq X \cup \mathcal{S}$ and each $x \in \mathcal{S}$ such that the relative type $[x; \bar{x}]$ is infinite, there exists a tuple $\bar{w} \supseteq \bar{x}$ and $w \in [x; \bar{x}]$ such that the relative type $[w; \bar{w}]$ is robustly block exchangeable.*

Lemma 42. *For each tuple $\bar{x} \subseteq X \cup \mathcal{S}$ and each $x \in \mathcal{S}$ such that the relative type $[x; \bar{x}]$ is robustly block exchangeable, there exists a complete and robustly block exchangeable set $C \subseteq [x]$ such that $C \setminus [x; \bar{x}]$ is finite.*

Lemma 43. *If C is complete and robustly block exchangeable, then C is a concept.*

Suppose that B is a block of a robustly block exchangeable concept $C \subseteq \mathcal{S}$. Then, B is a concept. Because each element of C is coinfinite, it must be that $|B| = 1$, and C is robustly exchangeable. Lemma 3 follows from the above results.

D.2. Proof of Lemma 41.

Take any tuple $\bar{x}$ and element x such that the relative type $[x; \bar{x}]$ is infinite.

(1) *There exists a tuple $\bar{z} \supseteq \bar{x}$ and $z \in [x; \bar{x}]$ such that the relative type $[z; \bar{z}]$ is infinite and the group action $G_{\bar{z}} \longmapsto [z; \bar{z}]$ is block 2-transitive.* It is helpful to replace tuple $\bar{x} \subseteq X \cup \mathcal{S}$ by a tuple $\bar{x}^* \subseteq X$ so that each coinfinite concept is replaced by its two-element code (which existence comes from Lemma 34). So, the length of tuple $\bar{x}^*$ is at most twice the length of tuple $\bar{x}$. Because the relative type $[x; \bar{x}]$ is infinite, because of the finitely many tuple types, there exists $x^* \in [x; \bar{x}^*]$ such that the relative type

$[x^*; \bar{x}^*]$ is infinite.

Let $\bar{c}$ be a two-element code of x (if $x \in X$, then take $\bar{c} = x\hat{\ }x$.). Because $G \longmapsto X$ is ψ-compact, the induced group action $G \longmapsto X \cup [\bar{x}]$ is 2ψ-compact. By Lemma 28, the latter group action does not have a splitting sequence. By Lemma 27, the relative type $[x^*; \bar{x}^*]$ cannot be robustly split. Thus, there exists a tuple $\bar{z} \subseteq X \cup [S]$, $\bar{z} \supseteq \bar{x}^*\hat{\ }\bar{x}$, and concept $z \in [x^*; \bar{x}^*]$ so that the relative type $[z; \bar{z}]$ is infinite and it cannot be split. By Lemma 25, the group action $G_{\bar{z}} \longmapsto [z; \bar{z}]$ is block 2-transitive with a finite block $B \subseteq [z; \bar{z}]$.

(2) *We show that the group action $G_{\bar{z}} \longmapsto [z; \bar{z}]$ is block highly transitive.* Let k_0 be the length of tuple $\bar{z}$. Let $k = |2|\,|B| + k_0$. Find a sequence $x_1 \in B_1, x_2 \in B_2, \dots$ such that $B_i \in [B; \bar{z}]$ are disjoint blocks of the block 2-transitive group action. Find a collection of k-local sets such that $U_n \supseteq \bar{z}, x_1, \dots, x_n$. Let $B^n = \{B' \in [B, \bar{z}]\,; B' \subseteq U\}$. By compactness, we can choose sets U_n so that

$$\lim_{n\to\infty} \frac{1}{n} \log |B^n| \leq \lim_{n\to\infty} \frac{1}{n} \log |U_n| \leq \frac{1}{10}.$$

Because the sets U_n are k-local, it must be that the group action $G_{\bar{z},U^n} \to B^n$ is 2-transitive. By Lemma 15, $G_{\bar{z},U^n} \to B^n$ is alternating or symmetric. Thus, $G_{\bar{z}} \to [B; \bar{z}]$ is highly transitive.

(3) *For each tuple $\bar{u} \supseteq \bar{z}$, there exists $B_{\bar{u}} \in [B; \bar{z}]$ such that set $[B; \bar{z}] \setminus [B_{\bar{u}}; \bar{u}]$ is finite.* Notice that because set B is finite, the group action $G \longmapsto X \cup [B]$ has finitely many tuple types. Then, for each tuple $\bar{u} \supseteq \bar{z}$, there are finitely many relative types $[B'; \bar{z}]$ for $B' \in [B; \bar{z}]$.

If the claim does not hold, there exists $\bar{v} \supseteq \bar{z}$ and $B^1, B^2 \in [B; \bar{z}\hat{\ }\bar{v}]$ such that the relative types $C^1 = [B^1; \bar{z}\hat{\ }\bar{v}]$ and $C^2 = [B^2; \bar{z}\hat{\ }\bar{v}]$ are infinite and disjoint. However, that will contradict compactness by the counting argument described in Section 7.2.6.

(4) *For each tuple $\bar{u} \supseteq \bar{z}$, the group action $G_{\bar{u}} \longmapsto [B_{\bar{u}}; \bar{u}]$ is highly transitive.* We can apply the previous point to tuples $\bar{u}$ and $\bar{u}\hat{\ }u$ for each $u \in [B_{\bar{u}}; \bar{u}]$ to show that the relative type $[B_{\bar{u}}; \bar{u}]$ cannot be split. By Lemma 25, the group action $G_{\bar{u}} \longmapsto [B_{\bar{u}}; \bar{u}]$ is block 2-transitive with some finite block $C \subseteq [B_{\bar{u}}; \bar{u}]$. We can use a version of the counting argument to show that it must be that $|C| = 1$. Hence, the group action $G_{\bar{u}} \longmapsto [B_{\bar{u}}; \bar{u}]$ is block 2-transitive. Because any $\bar{u}$ could have been chosen, it must be that $G_{\bar{u}} \longmapsto [B_{\bar{u}}; \bar{u}]$ is block highly transitive.

(5) *There exists tuple $\bar{w} \supseteq \bar{z}$ and $w \in [z, \bar{z}]$ such that the set $[z; \bar{z}] \setminus [w; \bar{w}]$ is finite and the relative type $[w; \bar{w}]$ is robustly exchangeable.* For each tuple $\bar{v} \supseteq \bar{z}$, pick $x_{\bar{v}} \in B_{\bar{v}}$. If the set $[x; \bar{x}] \setminus [x_{\bar{v}}; \bar{v}]$ is finite for all $\bar{v}$, then the relative type $[z; \bar{z}]$ is robustly exchangeable. Otherwise, there exists $\bar{v}$ and $x_1, x_2 \in [z; \bar{z}]$ such that the relative types of x_1 and x_2 given $\bar{v}$ are infinite and disjoint. In particular, the group action $G_{\bar{v}} \longmapsto [x^1; \bar{v}]$ is

block highly transitive with block $B'_{\bar{v}} = B_{\bar{v}} \cap [x^1; \bar{v}]$. Notice that $|B'_{\bar{v}}| < |B_{\bar{x}}|$. We can repeat the argument for the group action $G_{\bar{v}} \longmapsto [x^1; \bar{v}]$. Because the initial block B is finite, the argument will stop at a certain moment, and we find a tuple $\bar{w} \supseteq \bar{z}$ and $w \in [z, \bar{w}]$ such that the claim holds.

D.3. **Proof of Lemma 42.** Take any tuple $\bar{x}$ and element x such that the relative type $[x; \bar{x}]$ is robustly block exchangeable. For each tuple $\bar{x}' \in [\bar{x}']$, let $S(\bar{x}') = g \cdot [x; \bar{x}]$ for some g such that $g \cdot \bar{x}' = \bar{x}$ (the definition does not depend on the choice of g).

1. *There exists* $M, N < \infty$ *such that for all* $\bar{x}', \bar{x}'' \in [\bar{x}]$, *either* $|S(\bar{x}') \cap S(\bar{x}'')| \leq M$, *or* $|S(\bar{x}') \backslash S(\bar{x}'')| \leq N$. Because of robust exchangeability, there is no $\bar{x}'$ such that the two sets $S(\bar{x}) \cap S(\bar{x}')$ and $S(\bar{x}) \backslash S(\bar{x}')$ are infinite. The claim follows from the fact that the group action $G \longmapsto X \cup S$ has finitely many tuple types. Let N be the smallest constant so that the claim holds.
2. *Let* $C = \bigcup \{S(\bar{x}') : |S(\bar{x}') \backslash S(\bar{x})| \leq N\}$. *Then,* $G_C \longmapsto C$ *is transitive.* That follows from the fact that for each permutation g such that $|S(g \cdot \bar{x}) \backslash S(\bar{x})| \leq N$, $g \in G_C$.
3. $|C \backslash S(\bar{x})| < \infty$. This follows from Lemma 24.
4. C *is complete.* Suppose that there is $C' \in [C]$ such that $C \backslash C'$ is finite. Then, there is $S(\bar{x}) \subseteq C$ and $S(\bar{x}') \subseteq C'$ such that $\mathrm{S}(\bar{x}) \backslash S(\bar{x}')$ is finite. But then, for all such $\bar{x}$ and $\bar{x}'$, $S(\bar{x}) \subseteq C'$ and $S(\bar{x}') \subseteq C$, which implies that $C = C'$.
5. C *is robustly block exchangeable.* This follows from the facts that $S(\bar{x}) \subseteq C$, $|C \backslash S(\bar{x})| < \infty$, $S(\bar{x})$ is robustly block exchangeable, and that $G_C \longmapsto C$ is transitive.

D.4. **Proof of Lemma 43.** Fix an infinite and complete set $C_0 \subseteq X$ such that $G_{C_0} \longmapsto C_0$ is robustly block exchangeable with block B_0. Fix $x_0 \in C_0$. Because C_0 is complete, there exists constant $M < \infty$ such that for each $C \in [C_0]$,

$$|\{B \in [B_0; C_0] : B \cap C \neq \varnothing\}| \leq M.$$

Assume that M is the smallest such constant.

Fix $D_0 \subseteq C_0$ such that $x_0 \in D$ and D is a union of $M + 1$ distinct blocks of C_0. Let $\bar{d}$ be an enumeration of set D_0. Then, C_0 is $\bar{d}$-definable. Let $\mathcal{T} = \left\{[x; \bar{d}] : x \in [x_0]\right\}$. Because of finitely many tuple types, $\mathcal{T}$ is finite.

Because $G_{C_0} \longmapsto C_0$ is robustly block exchangeable with block B, for each $x \in [x_0]$, there exists a finite set $C(x) \subseteq C_0$ such that $G_{C_0, x} \longmapsto C_0 \backslash C(x)$ is highly transitive. Notice that if $x, x' \in [x_0]$, and x and x' have the same relative type given $\bar{d}$, then x and x' have the same relative type given C_0, and $|C(x)| = |C(x')|$. Let $N = \max_{x \in t \in \mathcal{T}} |C(x)| < \infty$.

Let $k = 10(M + N + 1)|B|^2$. Suppose that $V \subseteq [x]$ is a finite set such that $D_0 \subseteq V$ and the intersection $V \cap C_0$ contains at least k distinct blocks of the group action $G_{C_0} \longmapsto C_0$. For each such V, there exists a k-local $U^V \subseteq [x]$, $U^V \supseteq V$ such that for each $x \in [x_0]$, there is a $U \subseteq U^V, x$ such that $|U| < \frac{3}{2}\left|U^V\right|$. Because of the definition of constant M and because U^V is k-local, for each $C \in [C_0]$, $\left|C \cap U^V\right| > M\,|B|$ if and only if $C \in \left[C_0; U^V\right]$.

For each $x \in U^V$, let $i(x)$ be the number of concepts C that are analogous to C_0 relative to U^V and that contain x,

$$i(x) = \left|\left\{C \in \left[C_0; U^V\right] : x \in C\right\}\right|.$$

Because $U^V \subseteq [x_0]$ is k-local (hence, 1-local), $i(x) = i(x_0)$ for each $x \in U^V$.

(1) *We show that* $\left|\left[C_0; U^V\right]\right| < \infty$. Indeed, for any $C, C' \in \left[C_0; U^V\right]$, it must be that $\left|C \cap U^V\right|, \left|C' \cap U^V\right| \geq (M+1)\,|B|$, and by the choice of M, if $C \cap U = C' \cap U$, then $C = C'$. A simple counting argument shows that for each $x \in U^V$

$$i(x) = \frac{1}{|U^V|}\left|C_0 \cap U^V\right|\left|\left[C_0; U^V\right]\right| \tag{D.3}$$

(2) *We show that*

$$i(x) < \frac{1}{2}\frac{1}{|B|}\left|C_0 \cap U^V\right|. \tag{D.4}$$

Indeed, there exists $B_0^* \in [B_0; C_0]$ so that $B_0^* \cap C = \varnothing$ for any $C \in \left[C_0; U^V\right] \setminus \{C_0\}$. In particular, if $B\hat{\ }C \in \left[B_0^*\hat{\ }C_0; U^V\right]$ for some $C \in \left[C_0; U^V\right]$ and $B \cap B_0^* \neq \varnothing$, then $C = C_0$. Take any $x^* \in B^*$. Take any local $U \supseteq U^V, x^*$ so that $|U| < \frac{3}{2}|U|$. Then,

$$\begin{aligned}|U| &\geq \left|U^V\right| + \left|\left\{x \in g \cdot B^* : g \in G_{U^V,U}\right\}\right| \\ &\left|U^V\right| + |B|\left|\left[C_0; U^V\right]\right|.\end{aligned}$$

The claim follows from equality (D.3).

(3) *We show that* $M \leq 1$. If not, there is C and M different blocks $B_i \in [B_0; C_0]$ such that $B_i \cap C \neq \varnothing$ for $i \leq M$. W.l.o.g. assume that $x_0 \in B_M$ and that V is large enough that $B_0, .., B_M \subseteq C_0 \cap U^V$ and that $\left|C \cap U^V\right| > M\,|B|$. Because $G_{C,U} \longmapsto [x; C, U]$ is block highly transitive with block B, and by the choice of k,

$$\begin{aligned}i(x_0) &\geq [B_1; C, U, B_2, ..., B_{M_C}] \\ &\geq \frac{1}{|B|}|C \cap U| - M \geq \frac{1}{2\,|B|}|C \cap U|.\end{aligned}$$

This yields a contradiction with inequality (D.4).

(4) *We show that for each* $x \notin C_0$, *and each* $C \in [C_0]$ *such that* $x \in C$ *and* $C \cap C_0 \subseteq C(x)$. On the contrary, suppose that there are $x \notin C$ and $C' \in [C]$ such that $x \in C'$ and $C' \cap C$ and $C \backslash C(x)$ have a non-empty intersection. Assume that V contains x and at least $M+1$ distinct blocks of C. Let $n \leq N$ be the number of blocks of $C_0 \cap U^V$ that have a non-empty intersection with $C(x)$. There is at most one block

of $C_0 \cap U^V$ that has a non-empty intersection with C. Because $G_{C,x} \longmapsto C \backslash C(x)$ is highly transitive, we get

$$\begin{aligned} i(x) &\geq \left|\left\{C \in \left[C_0; U^V\right] : x \in C \text{ and } (C \cap C_V) \backslash C(x) \neq \varnothing\right\}\right| \\ &\geq \frac{1}{|B|}(|C \cap U|) - n > \frac{1}{2|B|}|C \cap U|. \end{aligned}$$

where the last inequality holds because of the choices of k and $V \subseteq U^V$. This yields a contradiction with (D.4).

(5) *We finish the proof of the Lemma.* Suppose that C_0 is not a concept. Then, because $[C_0; U]$ is finite, there exists $C \ni x_0$ such that $C \in [C_0] \setminus \left[C_0; U^V\right]$. Let $D = C \cap U$. Because $M \leq 1$, it must be that $|D| \leq |B|$.
Because of the previous step, there exists a finite subset $W \subseteq C$ such that for each $x' \in U^V \backslash C$, if $C' \in [C_0]$ and $C' \ni x'$, then $C' \cap C \subseteq W$. Let B be a block of the group action $G_C \longmapsto C$ such that B is disjoint with U^V and W. Let $x \in B$.
Let $D = C \cap U^V$. For each $D' \in \left[D; U^V\right]$, find a $B_{D'}$ such that $D'\hat{\ } B_{D'} \in \left[D\hat{\ }B; U^V\right]$. Then, if $D' \neq D$, then the intersection of $B_{D'}$ and B is empty. (Otherwise, there would be $x' \in D' \backslash D$ and $x' \in C'$ such that the intersection of C' and B is non-empty. But that would contradict the choice of B.) It follows that $B_{D'}$ and $B_{D''}$ are disjoint for all distinct $D', D'' \in \left[D; U^V\right]$.
Take any k-local $U \supseteq U^V, x$. Because U intersects C at at least two distinct blocks and $M = 1$, it must be that $C \in \left[C_0; U^V\right]$. Thus,

$$|U'| \geq |U| + |B| \left|\left[D; U^V\right]\right| \geq |U| + |B| \frac{|U|}{|D_0|} \geq 2|U|,$$

where the last inequality follows from the fact that $|D_0| \leq |B_0|$. But that contradicts the choice of U^V.

Appendix E. Coordinate system

In this part of the Appendix, we prove the Lemmas stated in Section 7.2.8. Below, we work with the action of group G on the space of elements X and coinfinite concepts $\mathcal{S}$. We always assume that the group action $G \mapsto X$ is $\frac{1}{20}$-compact and that the thesis of Lemma 37 holds.

E.1. **Proof of Lemma 4.** Fix a correlation class $\mathcal{R}$ of robustly exchangeable concepts. For any two concepts $C, C' \in \mathcal{R}$, let $d(C, C') \subseteq C$ be the finite exceptional set omitted by the correlating function $j_{C,C'} : C \backslash d(C, C') \to C' \backslash d(C', C)$, where $j_{C',C} = (j_{C,C'})^{-1}$.

For each concept C and each $x \in C$, define set

$$T^0(x, C) = \{(j_{C,C'}(x), C') : C' \in \mathcal{R}, x \notin d(C, C')\}.$$

Then, for each $x, x' \in C$, if $x \neq x'$, then sets $T^0(x, C)$ and $T^0(x', C)$ are disjoint. For each $n > 0$, define

$$T^n(x, C) = T^{n-1}(x, C) \cup \bigcup\nolimits_{(x',C') \in T^{n-1}(x,C)} T^0(x', C'), \text{ and}$$
$$T^\infty(x, C) = \bigcup\nolimits_n T^n(x, C).$$

Notice that $T^\infty(x, C) = T^\infty(x', C')$ for each $(x', C') \in T^\infty(x, C)$. Finally, define

$$T(x, C) = \{x : (x, C) \in T^\infty(x, C) \text{ for some } C \in \mathcal{R}\}.$$

Fix $x_0 \in C_0 \in \mathcal{R}$ and $T \in [T(x_0, C_0)]$. We show that

Lemma 44. *For each $C \in \mathcal{R}$, $|T \cap C| \leq 1$.*

Proof. Suppose not and that there are $x, x' \in T \cap C$ such that $x \neq x'$. We can assume that $(x', C) \in T^\infty(x, C)$. (Indeed, if $(x', C') \in T^\infty(x, C)$ for some $C' \neq C$, then using the fact that C and C' are robustly exchangeable concepts, we can show that $d(C', C) = \{x\}$ and that $j_{C',C}(x') \in T^0(x, C')$.) By construction, there exists a finite sequence $(x, C) = (x_0, C_0), ..., (x_n, C_n) = (x', C)$ such that for each $m < n$, $j_{C_m, C_{m+1}}(x_m) = x_{m+1}$ and the group actions $G_{C_m,C_{m+1}} \longmapsto [x_m; C_m, C_{m+1}]$ and $G_{C_m,C_{m+1}} \longmapsto [x_{m+1}; C_m, C_{m+1}]$ are highly transitive.

We show that there exists $C' \in \mathcal{R}$ such that the group actions $G_{C',C_m,C_{m+1}} \mapsto [x_m; C_m, C_{m+1}, C']$ and $G_{C',C_m,C_{m+1}} \mapsto [x_{m+1}; C_m, C_{m+1}, C']$ are highly transitive for each $m < n$. Pick any $C \in \mathcal{R}$ and let $\bar{x}$ be a code of C' (its existence follows from Lemma 34). For each $\bar{x}' \in [\bar{x}]$ and each m, let $F_m(\bar{x}')$ be a set such that $G_{C_m,C_{m+1},\bar{x}} \cdot F_m(\bar{x}') = F_m(\bar{x}')$ and the group action $G_{C_m,C_{m+1},\bar{x}} \longmapsto [x_m; C_m, C_{m+1}, \bar{x}'] \backslash\backslash F_m(\bar{x}')$ is highly transitive. Let $f^* = \sup_m |F_m(\bar{x})|$. Fix $N > mf^*$ and, using Lemma 37, find 6-local set $U \supseteq \bar{x}, x_1, ..., x_n$ such that for each m,

$$|U \cap C_m| \geq N.$$

Then, the group actions $G_{C_m,C_{m+1},U} \longmapsto [x_m; C_m, C_{m+1}] \cap U$ and $G_{C_m,C_{m+1},U} \longmapsto [x_{m+1}; C_m, C_{m+1}] \cap U$ are 6-transitive and, by the CFSG, highly transitive. Let α_m be the fraction of tuples $\bar{x}' \in [\bar{x}] \cap U^k$ such that $x_m \notin F_m(\bar{x}')$ and let α be the fraction of tuples $\bar{x}' \in [\bar{x}] \cap U^k$ such that $x \notin F_m(\bar{x}')$ for each m. Then,

$$\alpha_i \geq 1 - \frac{1}{N} f^* \text{ and } \alpha(\bar{x}) \geq 1 - \sum_m (1 - \alpha_i) \geq 1 - \frac{1}{N} mf^*.$$

Thus, $\alpha > 0$, and there exists $\bar{x}'$ such that $x \notin F_i(\bar{x}')$ for each player i. Find the unique C' so that $\bar{x}\hat{\ }C$ s analogous to $\bar{x}'\hat{\ }C'$. The claim follows.

Because $j_{C_0,C'}(x) \neq j_{C_0,C'}(x')$, there exists $m < n$ such that $j_{C_m,C'}(x_m) \neq j_{C_{m+1},C'}(x_{m+1})$. But this contradicts the fact that pairs of elements (x_m, x_{m+1}), $(x_m, j_{C_m,C'}(x_m))$, and $(x_{m+1}, j_{C_m,C'}(x_{m+1}))$ are correlated under the action of group $G_{C_m,C_{m+1},C'} \subseteq G_{C_m,C_{m+1}}$. □

We show that T is a concept. By construction, for each $x \in T$, there exists C such that $T = T(x, C)$. Because there are finitely many tuple types of concepts, $|[T; x]| \leq |[C; x]| < \infty$, and the bound is uniform across x.

We assume w.l.o.g. that T is coinfinite. (If not, replace T with the largest coinfinite concept $T' \supseteq T$ st. T is T'-algebraic. Such T' exists, and it is unique by Lemma 36). Thus, $T \in \mathcal{S}$.

Take any $C \in \mathcal{R}$ and consider the group action $G_C \mapsto \{T(x, C) : x \in C\}$. By Lemma 3, there exists robustly exchangeable concept $V \subseteq [T]$ such that $|V \backslash \{T(x, C) : x \in C\}| < \infty$. Because $G_C \mapsto \{T(x, C) : x \in C\}$ is robustly exchangeable, it must be that $\{T(x, C) : x \in C\} \subseteq V$ and $j_{C,V}(x) = T(x, C)$ is the correlating function.

Let $\mathcal{C}^*$ be the collection of concepts obtained in such a way for all equivalence classes $\mathcal{R}$. By construction, all concepts $C' \in \mathcal{C}_0$ are mutually independent, each concept $C \in \mathcal{C}$ is correlated with exactly one concept $C' \in \mathcal{C}^*$ such that $d(C, C')$ is empty, and that for each $x \in C'$, $x \in j_{C,C'}(x)$.

Finally, we show that the concepts in $\mathcal{C}^*$ are mutually disjoint. Suppose not and that $C, C' \in \mathcal{C}^*$ are two distinct concepts with non-empty intersection. By Lemma 38, the intersection must consist of exactly one element $\{x\} = C \cap C'$. That implies that $|[C'; C]| = \infty$. Find the largest coinfinite concept $u \supseteq pC'$ and such that C' is u-algebraic. Such a concept exists and it is unique by Lemma 36. Then, $x \subsetneq u$. Notice that $\{x' \in C : x' \subseteq u\} \subseteq C$ is a concept. By Lemma 38, there are two possibilities:

- x is the unique element of C such that $x \subsetneq x'$. In such a case, by Lemma 3, there exists a robustly exchangeable concept of concepts $C'' \subseteq \{u' : u' \hat{\ } x' \in [u \hat{\ } x; C]\}$ and such that $x' \in C''$. Clearly C'' must be correlated with C. But then, $j_{C'',C}(x') \subseteq x$. This contradicts the fact that $x \subsetneq x'$,
- for all $u \in C$, $u \subsetneq x'$. Then, x' is is C-algebraic (because x' is a concept), and, because x' is C'-algebraic, it must be that C' is C-algebraic. This yields a contradiction with the fact that $|[C'; C]| = \infty$.

E.2. **Proof of Lemma 5.**

(1) This follows from the fact that there are finitely many concepts over each $x \in X$.

(2) Suppose that $x \in C$ and $x\prime \in \mathcal{S}^*$ such that $x \subseteq x', x \neq x'$. An argument from the end of the proof of Lemma 4 shows that it must be that $u \subseteq x'$ for each $u \in C$.

(3) Let $L \subseteq \mathcal{S}^*$ be a finite set and let $\bar{L}$ be an enumeration of L. Suppose that $|\{C : L(C) = L\}| = \infty$. Because of the finitely many types of concepts, there exists C such that $\left|\left[C; \bar{L}\right]\right| = \infty$. Let x be the unique coinfinite concept such that C is x-algebraic (such a concept exists and is unique by Lemma 36). Then, $\left|\left[x; \bar{L}\right]\right| = \infty$. By Lemma 3, there exists a robustly exchangeable concept C' such that $C' \backslash \left[x; \bar{L}\right]$ is finite. Find concept $C'' \in \mathcal{C}^*$

that is correlated with C'. Then, for all $x' \in C' \cap \left[x; \bar{L}\right]$, x' is contained in a different element of concept C''. Because there are infinitely many of such x', and because each such x' is associated with a disjoint and finite set of concepts $c \in \left|\{C : L(C) = L\}\right|$, it must be that set L contains infinitely many elements of C'. But this contradicts the fact that L is finite.

E.3. **Proof of Lemma 6.** Say that set $L \subseteq \mathcal{S}^*$ is *upper*, if for each $S \in L$, $L(S) \subseteq L$.

We show that for each concept $C \in C^*$, each *finite* upper set L, each enumeration $\bar{L}$ of L, $G_{\bar{L}} \cap G_C \longmapsto C \backslash L$ is highly transitive. First, suppose that $L = L(s)$ for some (hence, by Lemma 5, all) $s \in C$. Then, $G_L \supseteq G_C$, and by Lemma 5, $[G_L : G_C] < \infty$. Because set L is finite, the index $[G_L : G_{\bar{L}}] = \left[\bar{L}, L\right]$ is finite, and

$$[G_C : G_{\bar{L}} \cap G_C] \leq [G_L : G_{\bar{L}} \cap G_C] \leq [G_L : G_{\bar{L}}]\,[G_L : G_C] < \infty.$$

Because finite index subgroups of highly transitive group actions are highly transitive (Lemma 13), $G_{\bar{L}} \cap G_C \longmapsto C$ is highly transitive.[11] More generally, assume that $L \supseteq L(s)$ for some (hence, all) $s \in C$. The proof follows induction on the size of set $\left|L \backslash L(s)\right|$. Suppose that L is an upper set, $L' = L \cup \{S'\}$ is an upper set for some $s' \in C' \in \mathcal{C}^*$, and $G_{\bar{L}} \cap G_C \longmapsto C \backslash L$ is highly transitive for some enumeration $\bar{L}$ of L. If $S' \in C$ (and $C' = C$), then the claim is trivial. So, we assume that $S' \notin C$. Notice that

$$[G_{\bar{L}} \cap G_C : G_{\bar{L}} \cap G_C \cap G_{C'}] \leq [G_L : G_{\bar{L}}]\,[G_L : G_{C'}] < \infty.$$

Thus, the group action $G_{\bar{L}} \cap G_C \cap G_{C'} \longmapsto C \backslash L$ is highly transitive. If $G_{\bar{L}^\frown S} \cap G_C \cap G_{C'} \longmapsto C \backslash L$ is not highly transitive, then C and C' are $G_{\bar{L}}$-correlated, which contradicts the fact that C and C' are independent.

Next, suppose that $s, s' \in C \in \mathcal{C}^*$. We show that for each finite set $Z \subseteq X \cup \mathcal{S}^*$, there exists a permutation $g_Z \in G$ such that $g_Z \cdot (s, s') = (s', s)$, $g_Z \cdot x = x$ and $\beta(g_Z \cdot S) = \beta(S)$ for each $S \in L(x)$ for each $x \in Z \backslash (s \cup s')$. Indeed, the existence of such a permutation follows from a repeated application of the above observation.

For each finite $A \subseteq \mathbf{Z}$, let

$$V_A = \{x \in X \cup \mathcal{S}^* : \{\beta(S) : S \in L(x)\} = A\}.$$

We show below that set V_A is necessarily finite. Then, $\mathcal{P} = \{V_A : A \subseteq \mathbf{Z}, A \text{ is finite}\}$ is a partition of $X \cup \mathcal{S}^*$.

Finally, let $Z_1 \subseteq Z_2 \subseteq ...$ be an increasing sequence of finite sets with union equal to $X \cup \mathcal{S}^*$. Let $g_1 = g_{X_1}$ and $g_i = g_{Z_{i+1}} \cdot g_{Z_i}^{-1}$. The sequence $\{g_i\}$ satisfies the assumption of Lemma 30 with partition $\mathcal{P}$. The result follows.

[11] Recall that the definition and properties of the subgroup index are stated in Section B.1.1.

Lemma 45. *V_A is finite for each finite $A \subseteq \mathbf{Z}$.*

Proof. The claim follows by induction on inclusion and Lemma 5. □

E.4. **Proof of Lemma 8.** We need to show that for each $S \in \mathcal{S}_0$, each tuple $\bar{x}^0 \subseteq E(S)$, each tuple $\bar{x} \subseteq X \backslash E(\sqcap S)$, there exists a tuple $\bar{x}' \subseteq E(\sqcup S)$ such that $\bar{x}^0 \hat{} \bar{x} \sim \bar{x}^0 \hat{} \bar{x}'$. Let $L = \bigcup_{x \in \bar{x}} L(x) \backslash \sqcup S$. Using Lemma 6, we can find a permutation $h \in G_{\bar{x}^0}$ such that for each $S \in L$, $\beta(h \cdot S) < L$. For each $x \in \bar{x}$ $x \notin E(\sqcap S)$, which implies that $L(x) \cap L(S) \subsetneq L(S)$. Thus, $h \cdot L(x) \subsetneq L(S)$, which implies that $h \cdot \bar{x} \subseteq E(\sqcup S)$.

E.5. **Proof of Lemma 7.** Recall the definition of set V_A from the proof of Lemma 6. For each finite $B \subseteq \mathbf{Z}$, define $U_B = \bigcup \{V_A : A \subseteq B\}$. Then, U_B is finite.

Fix bijection $\gamma : \mathbf{Z} \to \mathbf{N}$. Take any increasing sequence of finite sets $B_1 \subseteq B_2 \subseteq ...$ whose union is equal to $X \cup \mathcal{S}^*$. Using Lemma 6, we can find an increasing sequence of permutations g_i such that for each $A \subseteq B_i$, $g_i \cdot V_A = V_{\gamma(A)}$. Using the argument from Lemma 30, we can show that, possibly by taking subsequences, there exists a pointwise limit $\alpha = \lim_{i \to \infty} g_i$ that is a bijection $\alpha : X \to X_0$ that preserves relations.

Take any permutation $g \in G$ and consider a bijection $g_\alpha : X_0 \to X_0$ defined as

$$g_\alpha(x) = \alpha \circ g \circ \alpha^{-1}(x).$$

Because g_α preserves relations, g_α extends to $\hat{g}_\alpha \in G$ such that $\hat{g}_\alpha|_{X_0} = g_\alpha$. By choosing proper sequences of permutations from Lemma 6, together with an application of Lemma 30, we can show that there exists a permutation $g' \in G^{nc}$ such that $g'|_{X_0} = g_h$. This shows that the group actions $G \longmapsto X$ and $G^{nc} \longmapsto X_0$ are isomorphic.

E.6. **Finite family $E^{\#}(S)$.** In order to ensure that the group action $H_S \longmapsto E(S)$ is finite, it is useful to define two other families of sets. First, we expand sets $E(.)$ to include the "coordinates" $\mathcal{S}^*$: For each $S \in \mathcal{S}_0$, define the set of elements x such that S is the smallest member of collection $\mathcal{S}_0$ that includes x,

$$E^*(S) = \left\{ x \in X \cup \mathcal{S}^* : S = \bigcap \left(L(x) \cap \mathcal{S}_0^* \right) \right\}.$$

Then, $E(S) = E^*(S) \cap X$. Second, we define finite approximations of $E^*(S)$: For each m, let

$$E_m(S) = E^*(S) \cap \left(X_0 \cup \{ S \in \mathcal{S}^* : -m \leq \beta(S) < 0 \} \right).$$

Then, each set $E_m(S)$ is finite, and for each $g \in G^{nc}$, $g \cdot E_m(S) = E_m(g \cdot S)$. Let $E^*(\mathcal{L}) = \bigcup_{S \in \mathcal{L}} E^*(S)$ and $E_m(\mathcal{L}) = \bigcup_{S \in \mathcal{L}} E_m(S)$ for each set $\mathcal{L} \subseteq \mathcal{S}_0$. We show the following result.

Lemma 46. *For each $S \in \mathcal{S}_0$, the group action $G_S^{nc} \longmapsto E^*(S)$ has finite orbits and*

$$\left[\overline{E^*(S)}; G^{nc}, \overline{E^*(\sqcup S)} \right] = \left[\overline{E^*(S)}; G^{nc}, \overline{X \backslash E^*(\sqcap S)} \right]. \tag{E.1}$$

There exist constants $m_S \geq 1$ for $S \in \mathcal{S}_0$ such that if $E^{\#}(S) = E_{m_S}(S)$, then

$$\left[\overline{E^{\#}(S)}; G^{nc}, \overline{E^{\#}(\sqcup S)}\right] = \left[\overline{E^{\#}(S)}; G^{nc}, \overline{X \backslash E^{*}(\sqcap S)}\right].$$

First, fix $S \in \mathcal{S}_0$. We show that $G^{nc}_S \longmapsto E^*(S)$ has finite orbits. Take any $g \in G^{nc}_S$. Then, for each $x \in E^*(S)$, $\beta(g \cdot L(x)) = g \cdot L(x)$. (Indeed, for each $S' \in L(x)$, if S' is positive, then $S' \in L(S)$ and $\beta(g \cdot S') \in \beta(L(x))$; if S' is negative, then $\beta(g \cdot S') = \beta(S') \in \beta(L(x))$.) Additionally, if $x \in \mathcal{S}^*$, then x is negative, and $\beta(g \cdot x) = \beta(x)$. Thus, $g \cdot x \in V_{\beta(L(x))}$ if $x \in X$ and $g \cdot x \in V_{\beta(L(x) \cup \{S\})}$ if $x \in \mathcal{S}^*$. Because sets V_A are finite (Lemma 45), it must be that $G^{nc}_S \longmapsto E^*(S)$ has finite orbits.

Next, we show equality (E.1). Suppose that tuples $\bar{x}, \bar{x}' \subseteq E^*(S)$ are such that for each tuple $\bar{x}^0 \subseteq E^*(\sqcup S)$, there exists a permutation $h \in G^{nc}$ such that $h \cdot \bar{x}^0 = \bar{x}^0$ and $h \cdot \bar{x} = \bar{x}'$. We show that for each $\bar{x}^0 \subseteq X \backslash E^*(\sqcap S)$, there exists $g \in G^{nc}$ such that $h \cdot \bar{x}^0 = \bar{x}^0$ and $h \cdot \bar{x} = \bar{x}'$. Indeed, take any $\bar{x}^0 \subseteq X \backslash E^*(\sqcap S)$. By conditional independence (Lemma 7), there exists a permutation $g' \in G$ such that $g' \cdot \bar{x}^0 \subseteq E^*(\sqcup S)$ and $g' \cdot \bar{x}\,\hat{}\,\bar{x}' = \bar{x}\,\hat{}\,\bar{x}'$. Find permutation h such that $h \cdot (g' \cdot \bar{x}^0) = (g' \cdot \bar{x}^0)$. Let $g'' = g'^{-1} h g'$. Because $\beta(g'' \cdot S) = \beta(S)$ for each $S \in \bigcup \{L(x) : x \subseteq \bar{x}\,\hat{}\,\bar{x}'\,\hat{}\,\bar{x}^0\}$, one can use Lemmas 6 and 30 to find $g \in G^{nc}$ such that $g|_{\bar{x}\hat{}\bar{x}'\hat{}\bar{x}^0} = g''|\bar{x}\,\hat{}\,\bar{x}'\,\hat{}\,\bar{x}^0$.

Because of (E.1) and the fact that set $E_m(S)$ is finite, for each $S \in \mathcal{S}_0$, there exists $k^S(m)$ such that

$$\left[\overline{E_m(S)}; G^{nc}, \overline{E_{k^S(m)}(\sqcup S)}\right] = \left[\overline{E_m(S)}; G^{nc}, \overline{X \backslash E^{*}(\sqcap S)}\right].$$

For $S \in \mathcal{S}_0$ such that if $S' \subseteq S$ and $S' \in \mathcal{S}_0$, let $m_S = 1$. Recursively define

$$m_S = \max_{S' \in \sqcap S} \left(k^S(m_{S'})\right).$$

The result follows.

E.7. **Subgroup copies.** Next, we describe conditions that imply that the group actions $H \longmapsto X_0$ and $G^{nc} \longmapsto X_0$ are isomorphic.

For each subset $\mathcal{A} \subseteq X \cup \mathcal{S}^*$, two permutations h and h' are $\mathcal{A}$-*copies*, if $h|_{\mathcal{A}} = h'|_{\mathcal{A}}$. Two subgroups $H, H' \subseteq G$ are $\mathcal{A}$-*copies*, if for each $h \in H$, there exists $h' \in H'$ (and vice versa) such that h and h' are $\mathcal{A}$-*copies.* Thus, H and G^{nc} are X_0-copies, then their actions on X_0 are isomorphic.

Lemma 47. *Suppose that $S \in \mathcal{S}_0$ and sets $\mathcal{W}' = \mathcal{W} \cup [S; G^{nc}] \subseteq \mathcal{S}_0$ are such that for each $S' \in \mathcal{W}'$, $L(S') \subseteq \mathcal{W}$. Suppose that H and G^{nc} are $E^{\#}(\mathcal{W})$-copies.*

(1) *If H contains permutations of (s, s')-type for each $s, s' \in [S; G^{nc}]$, then H and G^{nc} are $\left(E^{\#}(\mathcal{W}) \cup \mathcal{W}'\right)$-copies.*

(2) *If, additionally,* $\left[\overline{E^{\#}(S)}; G^{nc}, \overline{E^{\#}(\sqcup S)}\right] = \left[\overline{E^{\#}(S)}; H, \overline{X \backslash E^{*}(\sqcap S)}\right]$*, then* H *and* G^{nc} *are* $E^{\#}(\mathcal{W}')$*-copies.*

We start with the first part. Take any permutation $g \in G^{nc}$ and find its $E^{\#}(\mathcal{W})$-copy h_0. Enumerate $C_1, C_2, ...$ all concepts $C_i \in \mathcal{C}^*$ such that the intersection $C_i \cap [S; G^{nc}]$ is non-empty. Then, $C \in [S; G^{nc}]$ and $g \cdot C = h_0 \cdot C$. Because H contains permutations of (s, s')-type for each $s, s' \in C \cap [S; G^{nc}]$, we can find a sequence of permutations h_i such that h_{i-1} and h_i are $E^{\#}(\mathcal{W}') \backslash C \cap [S; G^{nc}]$-copies and h_i and g are $C \cap [S; G^{nc}]$-copies. By Lemma 30, there exists limit $h = \lim_{i \to \infty} h_i$ such that h and g are $\left(E^{\#}(\mathcal{W}) \cup \mathcal{W}'\right)$-copies

Second, assume that $\left[\overline{E^{\#}(S)}; G^{nc}, \overline{E^{\#}(\sqcup S)}\right] = \left[\overline{E^{\#}(S)}; H, \overline{X \backslash E^{*}(\sqcap S)}\right]$. Then, by Lemma 46,

$$\left[\overline{E^{\#}(S)}; G^{nc}, \overline{E^{\#}(\sqcup S)}\right] = \left[\overline{E^{\#}(S)}; H, \overline{X \backslash E^{*}(\sqcap S)}\right]$$

Take any $g \in G^{nc}$ and find its $E^{\#}(\mathcal{W}) \cup \mathcal{W}'$-copy h_0. We show that there exists h_S such that h_S and h_0 are $E^{\#}(\mathcal{W} \cup \mathcal{W}') \backslash E^{\#}(S)$ -copies, and h_S and g are $E^{\#}(S)$-copies. Then, using Lemma 30, we can show that there exists a permutation h such that h and g are $E^{\#}(\mathcal{W} \cup \mathcal{W}')$-copies.

Because $h_0, g \in G^{nc}$, there is a permutation $g' = g h_0^{-1} \in G^{nc}$ such that g' is constant on $E^{\#}(\mathcal{W}) \setminus \cup \mathcal{W}'$ and $g' h_0 = g$. By the second part of Lemma 46, there exists permutation $h' \in H$ such that h' is constant on $X \backslash E^{*}(\sqcap S)$ and h' and g' are $E^{\#}(S)$-copies. Thus, $h_S = h' h_0$ and h_0 are $E^{\#}(\mathcal{W} \cup \mathcal{W}') \backslash E^{\#}(S)$-copies, and h_S and g are $E^{\#}(S)$-copies.

E.8. **Proof of Lemma 9.** Fix an increasing sequence of subsets $\varnothing = \mathcal{S}_0^0 \subseteq \mathcal{S}_0^1 ... \subseteq \mathcal{S}_0^m = \mathcal{S}_0$ such that for each $k \geq 0$, there exists $S \in \mathcal{S}_0$ such that $\mathcal{S}_0^{k+1} = \mathcal{S}_0^k \cup [S; G^{nc}]$ and $L(S) \subseteq \mathcal{S}_0^k$.

Say that subgroup $H \subseteq G^{nc}$ has *finite orbits up to level* k, if for each $S \in \mathcal{S}_0^k$, $\left|\left[\overline{E^{*}(S)}; H\right]\right| < \infty$.

By induction on $k \geq 0$, we show that there exists $H \subseteq G^{nc}$ with finite orbits up to level k and such that H and G^{nc} are $E^{\#}\left(\mathcal{S}_0^k\right)$-copies.

The inductive claim is immediate when $k = 0$. Fix $k \geq 0$ and suppose that $H \subseteq G^{nc}$ has finite orbits up to level k and H and G^{nc} are $E^{\#}\left(\mathcal{S}_0^k\right)$-copies. Let $S \in \mathcal{S}_0$ be such that $\mathcal{S}_0^{k+1} = \mathcal{S}_0^k \cup [S; G^{nc}]$. Let $\bar{e}^{\#}$, $\bar{e}$, and $\bar{e}^{\triangle}$ be enumerations of, respectively, $E^{\#}(S), E(S)$, and $E^{*}(\sqcup S)$.

Let H^* be the set of all permutations $h' \in G^{nc}$ such that $h'|_{E^{*}\left(\mathcal{S}_0^k\right)} \in \left\{h|_{E^{*}\left(\mathcal{S}_0^k\right)} : h \in H\right\}$. Then, H^* contains permutations (x, x')-type for each $x, x' \in C$ and $L(C) \subseteq \mathcal{S}_0^k$. By Lemma 47, H^* and G^{nc} are $E^{\#}\left(\mathcal{S}_0^k\right) \cup \mathcal{S}_0^{k+1}$-copies and for each $s' \in [S; G^{nc}]$, there exists $g_{S'} \in H^*$ such that $g_{S'} \cdot S = S'$.

There exists $F^{*}\subseteq H_{S}^{*}$ such that $\left|\left[\overline{E^{*}\left(S\right)};F^{*}\right]\right|<\infty$, F^{*} and H_{S}^{*} are $E^{*}\left(\sqcup S\right)$-copies, and

$$\left[\bar{e}^{\#};G^{nc},\overline{E^{\#}\left(\sqcup S\right)}\right]=\left\{f\cdot\bar{e}:f\in F^{*}\cap H_{\overline{X\backslash E^{*}\left(\sqcap S\right)}}^{*}\right\}.$$

By Lemma 46, there exists a finite set $F_{0}\subseteq G_{S}^{nc}\cap G_{\overline{X\backslash E^{*}\left(\sqcap S\right)}}$ such that $\left[\bar{e}^{\#};G^{nc},\overline{E^{\#}\left(\sqcup S\right)}\right]=\left\{f\cdot\bar{e}^{\#}:f\in F_{0}\right\}$. Additionally, because H^{*} has finite orbits up to level k, there is a finite set F_{1} such that for each $h\in H_{S}^{*}$, $h|_{E^{*}\left(\sqcup S\right)}\in\left\{f|_{E\left(\sqcup S\right)}:f\in F_{1}\right\}$. Because $G_{S}^{nc}\longmapsto E^{*}\left(S\right)$ has finite orbits (Lemma 8), Lemma 29 implies that there exists a subgroup $F^{*}\supseteq F_{0},F_{1}$ such that $\left|\left[\overline{E^{*}\left(S\right)};F^{*}\right]\right|<\infty$.

For each permutation $h\in H_{S}^{*}$, there exists a permutation $h'\in H_{\overline{X\backslash E^{*}\left(\sqcap S\right)}}^{*}$ such that $h'h\in F^{*}$. Indeed, because F^{*} and H_{S}^{*} are $E^{*}\left(\sqcup S\right)$-copies, for each $h\in H_{S}^{*}$, there exists $f\in F^{*}$ such that $h|_{E^{*}\left(\sqcup S\right)}=f|_{E^{*}\left(\sqcup S\right)}$. Hence, $f\cdot\bar{e}^{\#}\in\left[h\cdot\bar{e}^{\#};G^{nc},h\cdot\bar{e}^{\Delta}\right]$. By Lemma 46, $f\cdot\bar{e}^{\#}\in\left[h\cdot\bar{e}^{\#};G^{nc},\overline{X\backslash E^{*}\left(\sqcap S\right)}\right]$. The claim follows.

For each $S'\in\left[S;G^{nc}\right]$, define

$$G\left(S'\right)=g_{S'}\cdot\left[\bar{e};F^{*}\right].$$

For each permutation $h\in H^{*}$, for each $S'\in\left[S;G^{nc}\right]$, there exists a permutation $p_{S'}\left(h\right)\in H_{S'}^{*}\cap H_{\overline{X\backslash E^{*}\left(\sqcap S'\right)}}^{*}$ such that $p_{S'}\left(h\right)h\cdot G\left(h^{-1}\cdot S\right)=G\left(h\cdot S\right)$. Indeed, notice that $g_{S'}^{-1}hg_{h^{-1}\cdot S'}\in H_{S}^{*}$. By the pervious observation, there exists $h''\in H_{S}^{*}\cap H_{\overline{X\backslash E^{*}\left(\sqcap S\right)}}^{*}$ such that

$$h''g_{S'}^{-1}ht_{h^{-1}\cdot S'}\left(g_{h^{-1}\cdot S'}^{-1}G\left(h^{-1}\cdot S'\right)\right)=g_{h^{-1}\cdot S'}^{-1}G\left(h^{-1}\cdot S'\right)=\left[\bar{e};F^{*}\right].$$

Let $p_{S'}\left(h\right)=g_{S'}h''g_{S'}^{-1}$.

Let $S_{1},S_{2},...$ be an enumeration of $\left[S;G^{nc}\right]$. By Lemma 30, for each permutation $h\in H^{*}$, there exists a permutation

$$p\left(h\right)=\left(...\circ p_{S_{2}}\circ p_{S_{1}}\right)\left(h\right).$$

Then, $p\left(h\right)|_{E^{\#}\left(\mathcal{S}_{0}^{k}\right)}=h|_{E^{\#}\left(\mathcal{S}_{0}^{k}\right)}$ and for each $S'\in\left[S;G^{nc}\right]$, $p\left(h\right)\cdot G\left(S'\right)=G\left(p\left(h\right)\cdot S'\right)=G\left(h\cdot S'\right)$. Moreover, if h is a permutations of (x,x')-type for some $x,x'\in C\cap\mathcal{S}_{0}$ and $L\left(C\right)\subseteq\mathcal{S}_{0}^{k}$, then $p_{S_{m}}\left(h\right)$ is a permutation of (x,x')-type for each m, and $p\left(h\right)$ also is a permutation of (x,x')-type.

Let $H'=\left\{p\left(h\right):h\in H^{*}\right\}$. Then, $\left|\left[\bar{e};H'\right]\right|=\left|G\left(S\right)\right|<\infty$ and H' has finite orbits up to level $k+1$.By Lemma 47, H' and G^{nc} are $E^{\#}\left(\mathcal{S}_{0}^{k+1}\right)$-copies.

Appendix F. Decomposition of uncertainty

F.1. **Borel decomposition of finite invariant distributions.** The first result describes a version of the Borel decomposition argument appropriate for distributions that are invariant with respect to finite group actions.

Let B be a finite regular of orientations. Then, B has a group action on itself: $B \longmapsto B$ defined so that for each $b, b' \in B$, $b \cdot b' = b \circ b'$ (the left part of the formula defines the group action, and the right side corresponds to a composition of orientations).

Let A be a finite set. Let $H \longmapsto A \cup B$ be a group action so that $H \cdot B = B$, and the group actions $B \longmapsto B$ and $H \longmapsto B$ are isomorphic. For each $h \in H$, let $b_h \in B$ be the unique orientation such that for each $b \in B$, $h \cdot b = b \circ b_h$. Notice that if $h \in H_b$, then $b_h = \mathrm{id}$ and $h \in H_{\bar{b}}$ for any enumeration $\bar{b}$ of B. Let H_{id_B} denote the subgroup of permutations that keep all elements of B fixed.

Some notation is useful. Suppose that Z is a set, A is a countable set, and $H \longmapsto A$ is a group action. Define the action $H \longmapsto Z^A$ of group H on functions $\tau : A \to Z$,

$$(h \cdot \tau)(a) = \tau\left(h^{-1} \cdot a\right) \text{ for each } a \in A.$$

Similarly, define the group action the action $H \longmapsto Z^A$ of group H on functions $\theta : B \to Z$.

For each (possibly countably infinite) tuple $\bar{a}$ of elements of A, define mapping $\bar{a} : Z^A \to Z^{|A|}$ from that assigns functions τ with sequences of elements of Z,

$$\bar{a}(\bar{\tau}) = (\tau(a_1), \tau(a_2), \ldots).$$

The inverse mapping $\bar{a}^{-1}$ takes sequences of elements of Z into functions τ. Then,

$$\bar{a}(h \cdot \tau) = \left(h^{-1} \cdot \bar{a}\right)(\tau). \tag{F.1}$$

Let $\omega : Z^A \to \Delta Z^B$ be a collection of conditional distributions over functions $\theta \in Z^B$ given functions $\tau \in Z^A$. Conditional distributions ω are *H-invariant*, if for all $\tau \in Z^A$, all measurable sets $U \subseteq Z^B$, and all permutations $h \in H$,

$$\omega(h \cdot \theta \in U | h \cdot \tau) = \omega(\theta \in U | \tau).$$

The left-hand side is a conditional distribution given that the realization of variables on set A is equal to A.

Lemma 48. *Let A be a finite set. Let B be a finite regular set of orientations. Let $H \longmapsto A \cup B$ be a group action so that $H \cdot B = B$, and the group actions $B \longmapsto B$ and $H \longmapsto B$ are isomorphic. Suppose that conditional distributions $\omega : Z^A \to \Delta Y^B$ are H-invariant. Let $\bar{a}^*$ be an enumeration of A.*
Then, there exists a measurable function $f : [0,1) \times Z^{|A|} \to Z$ such that if η is uniformly distributed random variable, then for each $\tau : A \to Z$,

(1) *for each $h \in H_{\mathrm{id}_B}$,*

$$f(\eta, \bar{a}^*(\tau)) = f(\eta, \bar{a}^*(h \cdot \tau))$$

(2) *for any selection of permutations $h_b \in H$ so that $b = h_b \cdot \mathrm{id}_B$, the joint distribution of variables*

$$f\left(b\left(\eta\right),\left(h_b \cdot \bar{a}^*\right)\left(\tau\right)\right), b \in B \tag{F.2}$$

is equal to the conditional distribution $\omega\left(.|\tau\right)$.

Because of the first property, the values of terms in (F.2) do not depend on the choices of permutations h_b. Indeed, if $b = h \cdot b^* = h' \cdot b^*$, then $h'h^{-1} \in H_{\mathrm{id}_B}$, and for each η,

$$\begin{aligned} f\left(b\left(\eta\right),\left(h \cdot \bar{a}\right)\left(\tau\right)\right) &= f\left(b\left(\eta\right),\left(h'h^{-1}h \cdot \bar{a}\right)\left(\tau\right)\right) \\ &= f\left(b\left(\eta\right),\left(h' \cdot \bar{a}\right)\left(\tau\right)\right). \end{aligned}$$

Proof. Fix interval $I \subseteq [0,1)$ such that $\{b\left(I\right) : b \in B\}$ is a partition of the interval $[0,1)$ into $|B|$ disjoint sets. For each $\eta \in [0,1)$, there exists unique b_η so that $\eta \in b_\eta\left(I\right)$ (or, alternatively, $b_\eta^{-1}\left(\eta\right) \in I$).

Define the action $H \longmapsto Z^{|A|}$, of group H on sequences of elements of $Z^{|A|}$,

$$h \cdot \bar{z} := \bar{a}^*\left(h \cdot \left(\bar{a}^*\right)^{-1}\left(\bar{z}\right)\right) = \left(h^{-1} \cdot \bar{a}^*\right)\left(\left(\bar{a}^*\right)^{-1}\left(\bar{z}\right)\right).$$

Then, for each $h \in H$, and each $\tau : A \to Z$,

$$h \cdot \left(\bar{a}^*\left(\tau\right)\right) = \bar{a}^*\left(h \cdot \left(\bar{a}^*\right)^{-1}\left(\bar{a}^*\left(\tau\right)\right)\right) = \bar{a}^*\left(h \cdot \tau\right). \tag{F.3}$$

(The second equality follows from (F.1).)Then, for each h, h', and $\bar{z}$,

$$\begin{aligned} h' \cdot \left(h \cdot \bar{z}\right) &= \bar{a}^*\left(h' \cdot \left(\bar{a}^*\right)^{-1}\left(\bar{a}^*\left(h \cdot \left(\bar{a}^*\right)^{-1}\left(\bar{z}\right)\right)\right)\right) \\ &= \bar{a}^*\left(h' \cdot \left(h \cdot \left(\bar{a}^*\right)^{-1}\left(\bar{z}\right)\right)\right) \\ &= \bar{a}^*\left(h'h \cdot \left(\bar{a}^*\right)^{-1}\left(\bar{z}\right)\right) \\ &= \left(h'h\right) \cdot \bar{z}. \end{aligned}$$

Consider the action $H_{\mathrm{id}_B} \longmapsto Z^{|A|}$ of the subgroup that keeps all elements of B fixed. Because H_{id_B} is finite, the type of each element $\bar{z}$ is finite, $|[\bar{z}; H_{\mathrm{id}_B}]| < \infty$. One can find a measurable subset $E \subseteq Z$ such that for each $\bar{z}$, set E contains exactly one element of the type of y,

$$E \cap [\bar{z}; H_{\mathrm{id}_B}] = \{r\left(\bar{y}\right)\},$$

for some $r\left(\bar{y}\right) \in Z^{|A|}$.

Let λ_I be the uniform distribution on the interval I. By the standard Borel decomposition result (for example, see Kallenberg (2005)), there exists a function $\delta : I \times Z^{|A|} \to Z^B$ such that if u is chosen from λ_I, then the distribution of $\delta(u,\bar{z}) \in Z^B$ is equal to $\omega\left(.|\left(\bar{a}^*\right)^{-1}(\bar{z})\right)$. Because of H-invariance of ω, for each measurable $U \subseteq Z^B$, for each $h \in H$,

$$\begin{aligned}
\lambda_I(\delta(u,\bar{z}) \in U) &= \omega\left(\theta \in U|\left(\bar{a}^*\right)^{-1}(\bar{z})\right) \\
&= \omega\left(\theta \in h^{-1}\cdot U|h\cdot\left(\bar{a}^*\right)^{-1}(\bar{z})\right) \\
&= \omega\left(\theta \in h^{-1}\cdot U|\left(\bar{a}^*\right)^{-1}(h\cdot\bar{z})\right) \\
&= \lambda_I(h\cdot\delta(u,h\cdot\bar{z}) \in U), \qquad \text{(F.4)}
\end{aligned}$$

and the distribution of variables $h\cdot\delta(u,h\cdot\bar{z})$ is equal to $\omega\left(.|\left(\bar{a}^*\right)^{-1}(\bar{z})\right)$.

For each $\eta \in I$, each $b \in B$, any h_b so that $b = h_b\cdot \mathrm{id}_B$, and each $\bar{z} \in Z^{|A|}$, define

$$f(b(\eta),\bar{z}) = \delta\left(\eta, r\left(h_b^{-1}\cdot\bar{z}\right)\right)(b).$$

Notice that the definition does not depend on the choice of h_b. Indeed, if $b = h\cdot\mathrm{id}_B = h'\cdot\mathrm{id}_B$, then $(h')^{-1}h \in H_{\mathrm{id}_B}$, and

$$\left((h')^{-1}h\right)\cdot h^{-1}\cdot\bar{z} = \left((h')^{-1}hh^{-1}\right)\cdot\bar{z} = (h')^{-1}\cdot\bar{z}.$$

The latter implies that $h^{-1}\cdot\bar{z}$ and $(h')^{-1}\cdot\bar{z}$ have the same type under the group action with respect to the group action $H_{\mathrm{id}_B} \longmapsto Z^{|A|}$, and

$$\delta\left(\eta, r\left(h^{-1}\cdot\bar{z}\right)\right)(b) = \delta\left(\eta, r\left((h')^{-1}\cdot\bar{z}\right)\right)(b).$$

We show that function f satisfies the first property stated in the Lemma. Let $\bar{z} = \bar{a}^*(\tau)$. Fix $b \in B$ and $h_b \in H$ so that $b = h_b\cdot\mathrm{id}_B$. Suppose that $h \in H_{\mathrm{id}_B}$. Then, $h' = h_bhh_b^{-1} \in H_{\mathrm{id}_B}$, and

$$h_b^{-1}\cdot(h\cdot\bar{z}) = \left(h_b^{-1}h\right)\cdot z = \left(h'h_b^{-1}\right)\cdot z = h'\cdot\left(h_b^{-1}\cdot\bar{z}\right),$$

which implies that $h_b^{-1}\cdot(h\cdot\bar{z})$ and $h_b^{-1}\cdot\bar{z}$ have the same type with respect to the group action $H_{\mathrm{id}_B} \longmapsto Z^{|A|}$. Thus, $r\left(h_b^{-1}\cdot(h\cdot\bar{z})\right) = r\left(h_b^{-1}\cdot\bar{z}\right)$, and by (F.3),

$$\begin{aligned}
f(b(\eta),\bar{a}^*(h\cdot\tau)) &= f(b(\eta),h\cdot\bar{z}) \\
&= \delta\left(\eta, r\left(h_b^{-1}\cdot(h\cdot\bar{z})\right)\right)(b) \\
&= \delta\left(\eta, r\left(h_b^{-1}\cdot\bar{z}\right)\right)(b) \\
&= f(b(\eta),\bar{z}) = f(b(\eta),\bar{a}^*(\tau)).
\end{aligned}$$

We show that function f satisfies the second property. Let $\bar{z} = \bar{a}^*(\tau)$. For each $b_0 \in B$, each $\eta \in I$, and any selection $h_b \in H$ so that $b = h_b\cdot\mathrm{id}_B$,

$$\begin{aligned}
f(b(b_0(\eta)),h_b\cdot\bar{z}) &= \delta\left(\eta, r\left(h_{b\circ b_0}^{-1}\cdot(h_b\cdot\bar{z})\right)\right)(b\circ b_0) \\
&= \delta\left(\eta, r\left(h_{b_0}^{-1}\cdot\bar{z}\right)\right)(h_{b_0}\cdot b) \\
&= \delta(\eta,h\cdot\bar{z})\left(h^{-1}\cdot b\right) = h\cdot\delta(\eta,h\cdot\bar{z})(b) \qquad \text{(F.5)}
\end{aligned}$$

for some h so that $h \cdot b_0 = \mathrm{id}_B$. By (F.4), the joint distribution of variables (F.5) is equal to $\omega\left(.|\left(\bar{a}^*\right)^{-1}(\bar{z})\right)$. □

F.2. **Proof of Lemma 10.** We assume notation from Section 7.3. We start with some notation. Recall that $\mathcal{U}$ consist of shocks η_S identified by concepts $S \in \mathcal{S}_0$. In what follows, we identify shocks with the associated concepts. So, the joint realization u of all shocks in $\mathcal{U}$ is treated as an element of space $u \in [0,1)^{\mathcal{S}_0}$.

We define the action of group H on the joint realization of shocks,$H \longmapsto [0,1)^{\mathcal{S}_0}$. Because $X_0 \cup \mathcal{O}$ is a system with orientations, for each $S \in t \subseteq \mathcal{S}_0$, there exists an orientation $q(S) \in Q^t$ such that for each $(\eta_{S'}, q') \in \mathcal{O}$, $h \cdot (\eta_{S'}, q') = \left(\eta_{h(S')}, q \circ q(S')\right)$. For each $u \in [0,1)^{\mathcal{U}}$, let

$$(h \cdot u)(S) = q(S)(u(h \cdot S)) \text{ for each } S \in \mathcal{S}_0. \tag{F.6}$$

Then, for each tuple of orientations $\bar{o} \subseteq \mathcal{O}$,

$$(h \cdot \bar{o})(u) = \bar{o}(h \cdot u). \tag{F.7}$$

Slightly abusing notation, we write $(h \cdot u') \in [0,1)^{h(\mathcal{S}')}$ for any $u' \in [0,1)^{\mathcal{S}'}$ and some subset of concepts $\mathcal{S}' \subseteq \mathcal{S}_0$.

We move to the proof of the Lemma. The proof goes by induction on the set of types T. Take any set T_0, let $\mathcal{S}_0 = \bigcup T_0$, and $\mathcal{O}_0 = \bigcup_{S \in \mathcal{S}_0} \mathcal{O}_S$. Suppose that there exist tuples of orientations $\bar{o}^o$ for all $o \in \mathcal{O}_0$ that satisfy the thesis of the Lemma for all H-invariant distributions $\omega^* \in \Delta\left(Z^{\mathcal{O}_0}\right)$ that satisfy CI, and that additionally, each $\bar{o}^o$ contains each orientation in set $\bigcup \{\mathcal{O}_{S'} : S' \in \mathcal{S}, S' \supsetneq S\}$.

Suppose that $T = T_0 \cup \{t_0\}$ and T_0 is an *upper* subset of T, i.e., for each $S \in t \in T_0$, for each $S' \in t' \in T$, if $S \subseteq S'$, then $t' \in T_0$. Let ω^* be an H-invariant distribution that satisfies CI. For each $t \in T_0$, find $o^t{}^\frown\bar{o}^t$-symmetric functions f^t that decompose $\mathrm{marg}_{\mathcal{O}_0} \omega^*$ as in the thesis of the Lemma.

From now on, fix concept $S^* \in t_0$. Define the set of orientations

$$\mathcal{O}_{\sqcup S^*} = \bigcup \{\mathcal{O}_{S'} : S' \in \sqcup S^*\}.$$

Then, for each $o \in \mathcal{O}_{S^*}$, $\theta(o)$ is conditionally independent from $\{\theta(o'), o' \notin \mathcal{O}_{S^*}\}$, given $\{\theta(o'), o' \in \mathcal{O}_{\sqcup S^*}\}$.

Let $n^a = |\sqcup S^*|$, $n^b = |\mathcal{O}_{\sqcup S^*}|$, and $n = n^a + n^b$. Fix orientation $o^* = (S^*, \mathrm{id}) \in \mathcal{O}_{S^*}$. Find a tuple $\bar{o}^{*a}$ that contains exactly one orientation of each concept in set $\sqcup S^*$. Find an enumeration $\bar{o}^{*b}$ of set $\mathcal{O}_{\sqcup S^*}$. Let $\bar{o}^* = \bar{o}^{*a}{}^\frown\bar{o}^{*b}$. Define mapping $\left(o^*{}^\frown\bar{o}^{*a}\right)^{-1} : [0,1)^{n^a+1} \to [0,1)^{\sqcup S^* \cup \{S^*\}}$ so that

$$o^*{}^\frown\bar{o}^{*a}\left(\left(o^*{}^\frown\bar{o}^{*a}\right)^{-1}(u)\right) = \bar{u}$$

for each $\bar{u} \in [0,1)^{n^a+1}$. We have

$$h^{-1} \cdot \left(o^{*\frown}\bar{o}^{*a}\right)^{-1}\left(\left(h \cdot o^{*\frown}\bar{o}^{*a}\right)(u)\right) = h^{-1} \cdot \left(o^{*\frown}\bar{o}^{*a}\right)^{-1}\left(o^{*\frown}\bar{o}^{*a}(u)\right)$$
$$= h^{-1} \cdot (h \cdot u)|_{\sqcup S^* \cup \{S^*\}} = u|_{\sqcup S^* \cup \{S^*\}}, \tag{F.8}$$

where $h \cdot u$ is defined in (F.6).

Consider a marginal distribution $\hat{\omega} = \text{marg}_{\mathcal{O}_{S^*} \cup \mathcal{O}_{\sqcup S^*}} \omega^* \in \Delta\left(Z^{\mathcal{O}_{\sqcup S^*} \cup \mathcal{O}_{S^*}}\right)$. Then, the conditional distributions $\omega\left(.|\theta^*(o'), o' \in \mathcal{O}_{\sqcup S^*}\right)$ over functions $\theta : \mathcal{O}_{S^*} \to Z$ are H-invariant in the sense described in part F.1 of this appendix. By Lemma 48, there exists a measurable function $f : [0,1) \times Z^{n^b} \to Z$ such that for each $\tau : \mathcal{O}_{\sqcup S^*} \to Z$, for each $h \in H$ st. $h \cdot o^* = o^*$,

$$f\left(\eta_{S^*}, \bar{o}^{*b}(\tau)\right) = f\left(\eta(S^*), \bar{o}^{*b}(h \cdot \tau)\right) \tag{F.9}$$

and for any selection of permutations $h_o \in H$ so that $o = h_o \cdot o^*$, $o \in \mathcal{O}_{S^*}$, the joint distribution of random variables

$$f\left(o(\eta_{S^*}), \left(h_o \cdot \bar{o}^{*b}\right)(\tau)\right), o \in \mathcal{O}_{\sqcup S^*} \tag{F.10}$$

is equal to the conditional distribution $\hat{\omega}(.|\tau)$.

Define a measurable function $f^{t_0} : [0,1)^{n^a} \to Z$ so that for each $\bar{u} \in [0,1)^{n^a+1}$,

$$f^{t_0}(\bar{u}) = f\left(z_0, z_1, ..., z_{n^b}\right), \text{ where}$$

where $z_0 = (q^*)^{-1}(u_1)$, and for each $m \leq n^b$, $S_m^{*b} \in t_m$ and

$$z_m = f^{t_m}\left(\left(o_m^{*b\frown}\bar{o}^{o_m^{*b}}\right)\left(\left(\bar{o}^{*\frown}\bar{o}^{*a}\right)^{-1}(\bar{u})\right)\right).$$

We show that function f is $o^{*\frown}\bar{o}^*$-symmetric. Take any permutation $h \in H_{S^*}$ such that $h \cdot o^* = o^*$. We need to show that for each $u \in [0,1)^{\mathcal{S}_0}$

$$f^{t_0}\left(\left(o^{*\frown}\bar{o}^{*a}\right)(u)\right) = f^{t_0}\left(\left(h \cdot o^{*\frown}\bar{o}^{*a}\right)(u)\right).$$

Indeed, by (F.9),

$$f^{t_0}(\bar{u})$$
$$= f\begin{pmatrix} (o^*)^{-1}(u_1), f^{t_1}\left(\left(o_1^{*b\frown}\bar{o}^{o_1^{*b}}\right)\left(\left(\bar{o}^{*\frown}\bar{o}^{*a}\right)^{-1}(\bar{u})\right)\right), \\ ..., f^{t_{n^b}}\left(\left(o_{n^b}^{*b\frown}\bar{o}^{o_{n^b}^{*b}}\right)\left(\left(\bar{o}^{*\frown}\bar{o}^{*a}\right)^{-1}(\bar{u})\right)\right) \end{pmatrix}$$
$$= f\begin{pmatrix} (o^*)^{-1}(u_1), f^{t_1}\left(h^{-1} \cdot \left(o_1^{*b\frown}\bar{o}^{o_1^{*b}}\right)\left(\left(\bar{o}^{*\frown}\bar{o}^{*a}\right)^{-1}(\bar{u})\right)\right), \\ ..., f^{t_{n^b}}\left(h^{-1} \cdot \left(o_{n^b}^{*b\frown}\bar{o}^{o_{n^b}^{*b}}\right)\left(\left(\bar{o}^{*\frown}\bar{o}^{*a}\right)^{-1}(\bar{u})\right)\right) \end{pmatrix}.$$

By (F.7) and (F.8), if $\bar{u} = (h \cdot o^{*\frown}\bar{o}^{*a})(\eta)$, then the above is equal to

$$= f\left(\begin{array}{c} (o^*)^{-1}(u_1), f^{t_1}\left(\left(o_1^{*b\frown}\bar{o}^{o_1^{*b}}\right)\left(h^{-1}\cdot(\bar{o}^{*\frown}\bar{o}^{*a})^{-1}(\bar{u})\right)\right), \\ \dots, f^{t_{n^b}}\left(h^{-1}\cdot\left(o_{n^b}^{*b\frown}\bar{o}^{o_{n^b}^{*b}}\right)\left((\bar{o}^{*\frown}\bar{o}^{*a})^{-1}(\bar{u})\right)\right)\end{array}\right)$$
$$= f\left((o^*)^{-1}(u_1), f^{t_1}\left(\left(o_1^{*b\frown}\bar{o}^{o_1^{*b}}\right)(\eta|_{\sqcup S^*})\right), \dots, f^{t_{n^b}}\left(\left(o_{n^b}^{*b\frown}\bar{o}^{o_{n^b}^{*b}}\right)(\eta|_{\sqcup S^*})\right)\right)$$
$$= f^{t_0}\left((o^{*\frown}\bar{o}^{*a})(\eta)\right).$$

For each $o \in S \in t_0$, find tuple $\bar{o}^o$ such that $o^{*\frown}\bar{o}^*$ and $o^\frown\bar{o}^o$ are analogous. Suppose that $u \in [0,1)^{\mathcal{S}_0}$ is a realization of shocks, and let $\tau : \mathcal{O}_{\sqcup S} \to Z$ be such that

$$\tau(o') = f^t\left(o'^\frown\bar{o}^{o'}(u)\right) \text{ for } o' = (\eta_{S'}, p) \in \mathcal{O}_{\sqcup S}, S' \in t \cap \sqcup S, \text{ and } t \in T.$$

We show that the joint distribution of variables

$$f^{t_0}\left(o^\frown\bar{o}^o(\eta)\right), \text{ for } o \in \mathcal{O}_S \text{ and } S \in t_0$$

is equal to is $\hat{\omega}(.|\tau)$. Let $h_o \in H_S$ be such that $o = h_o \cdot o^*$. By symmetry of f^t for all $t \in T$ and by (F.7),

$$\begin{aligned}
&f^{t_0}\left(o^\frown\bar{o}^o(\eta)\right)\\
&= f^{t_0}\left((h_o \cdot o^*)(\eta), (h_o \cdot \bar{o}^*)(\eta)\right)\\
&= f^{t_0}\left(o^*(h_o\cdot\eta), \bar{o}^*(h_o\cdot\eta)\right)\\
&= f\left(o(\eta(S^*)), f^{t_1}\left(\left(o_1^{*b\frown}\bar{o}^{o_1^{*b}}\right)(h_o\cdot\eta)\right), \dots, f^{t_{n^b}}\left(\left(o_{n^b}^{*b\frown}\bar{o}^{o_{n^b}^{*b}}\right)(h_o\cdot\eta)\right)\right)\\
&= f\left(o(\eta(S^*)), f^{t_1}\left(\left(h_o\cdot o_1^{*b\frown}\bar{o}^{o_1^{*b}}\right)(\eta)\right), \dots, f^{t_{n^b}}\left(\left(h_o\cdot o_{n^b}^{*b\frown}\bar{o}^{o_{n^b}^{*b}}\right)(\eta)\right)\right)\\
&= f\left(o(\eta(S^*)), \tau\left(h_o\cdot o_1^{*b}\right), \dots, \tau\left(h_o\cdot o_{n^b}^{*b}\right)\right)\\
&= f\left(o(\eta(S^*)), \left(h_o\cdot\bar{o}^{*b}\right)(\tau)\right).
\end{aligned}$$

The claim follows from the fact that the joint distribution of (F.10) is equal to $\hat{\omega}(.|\tau)$.

Appendix G. Proof of sufficiency part of Theorem 3

We begin with two technical results.

Lemma 49. *For any two analogous tuples of orientations $\bar{o} \sim \bar{o}'$, there is a measurable bijection $\pi : [0,1)^{\mathcal{U}} \to [0,1)^{\mathcal{U}}$ such that π^{-1} is measurable, π preserves measure λ^{U}, i.e., for each measurable $E \subseteq [0,1)^{\mathcal{U}}$, $\lambda^{\mathrm{U}}(E) = \lambda^{\mathrm{U}}(\pi(E))$, and*

$$\bar{o}'(\pi(u)) = \bar{o}(u).$$

Proof. Let $\bar{o} = ((\eta_1, q_1), ..., (\eta_n, q_n))$, and $\bar{o}' = ((\eta'_1, q'_1), ..., (\eta'_n, q'_n))$. By the definition of a system of orientations, for any $m, m' \leq n$, $\eta_m = \eta_{m'}$ if and only if $\eta'_m = \eta'_{m'}$. Thus, there is a bijection $\iota : \mathcal{U} \to \mathcal{U}$ such that $\iota(\eta_m) = \eta'_m$ for each $m \leq n$, and $\iota(\eta) = \eta$ for each $\eta \notin \{\eta_1, ..., \eta_n\} \cup \{\eta'_1, ..., \eta'_n\}$. Also, for each $\eta \in \{\eta_1, ..., \eta_n\}$, there is $q^\eta \in Q_\eta$ such that for each $m \leq n$, $q'_m = q^{\eta_m} \circ q_m$. For each $\eta \notin \{\eta_1, ..., \eta_n\}$, let $q^\eta = \mathrm{id}_{[0,1)}$. For each $u \in [0,1)^{\mathcal{U}}$, and for each $\eta \in \mathcal{U}$, define

$$(\pi(u))_\eta = q^{i^{-1}(\eta)}\left(u_{i^{-1}(\eta)}\right)$$

It is easy to check that mapping π has the required properties. □

Lemma 50. *If $f : [0,1)^n \to Y$, is $(x^0, \bar{o}^0)$-symmetric, then for any $x \in X$, any tuples of orientations $\bar{o}$ and $\bar{o}'$ such that $x\hat{\ }\bar{o}$ and $x\hat{\ }\bar{o}'$ are analogous to $x^0\hat{\ }\bar{o}^0$, it must be that $f(\bar{o}(u)) = f(\bar{o}'(u))$ for all realizations $u \in [0,1)^{\mathcal{U}}$.*

Proof. By external consistency, there exists a tuple $\bar{o}^{0\prime}$ such that $x\hat{\ }\bar{o}\hat{\ }\bar{o}'$ is analogous to $x^0\hat{\ }\bar{o}^0\hat{\ }\bar{o}^{0\prime}$. By internal consistency, tuple $x^0\hat{\ }\bar{o}^{0\prime}$ is analogous to $x\hat{\ }\bar{o}'$, and, as a consequence, to $x^0\hat{\ }\bar{o}^0$. By symmetry, for each $u \in [0,1)^{\mathcal{U}}$, $f(\bar{\sigma}^0(u)) = f(\bar{\sigma}^{0\prime}(u))$. By Lemma 49, there exists a measurable bijection $\pi : [0,1)^{\mathcal{U}} \to [0,1)^{\mathcal{U}}$ such that for each $u \in [0,1)^{\mathcal{U}}$,

$$f(\bar{o}(u)) = f\left(\bar{\sigma}^0(\pi(u))\right) = f\left(\bar{\sigma}^{0\prime}(\pi(u))\right) = f(\bar{\sigma}'(u)).$$

□

Take any two analogous tuples $\bar{x} = (x_1, ..., x_m)$ and $\bar{x}' = (x_1, ..., x'_m)$. Find tuples $\bar{o}_1, ..., \bar{o}_m$ such that for each $k \leq m$, tuples $x\hat{\ }\bar{o}$ and $x_k\hat{\ }\bar{o}_k$. Find tuples $\bar{o}'_1, ..., \bar{o}'_m$ such that tuples $x_1\hat{\ }\bar{o}_1\hat{\ }....\hat{\ }x_m\hat{\ }\bar{o}_m$ and $x'_1\hat{\ }\bar{o}'_1\hat{\ }....\hat{\ }x'_m\hat{\ }\bar{o}'_m$ are analogous. By consistency axioms and the definition of random variables $\theta^{f,x,\bar{o}}(.;u)$, for each $k \leq m$ and each $u \in [0,1)^{\mathcal{U}}$,

$$\theta(x_k; u) = f(\bar{o}_k(u)) \text{ and } \theta(x'_k; u) = f(\bar{o}'_k(u)).$$

(We drop the superscript (f, x, σ) as it won't lead to ambiguity.) By Lemma 49, there exists a $\lambda^{\mathcal{U}}$-preserving measurable bijection $\pi : [0,1)^{\mathcal{U}} \to [0,1)^{\mathcal{U}}$ such that π^{-1} is measurable, and for each $k \leq m$ and each $u \in [0,1)^{\mathcal{U}}$,

$$f(\bar{o}'_k(u)) = f(\bar{o}_k(\pi(u))).$$

Hence, for each $k \leq m$

$$\theta(x'_k; u) = \theta(x_k; \pi(u)).$$

Because π preserves measure λ^U, the joint distribution of variables $(\theta(x_1; .), ..., \theta(x_m; .))$ is equal to the joint distribution of variables $(\theta(x'_1; .), ..., \theta(x'_m; .))$. This ends the proof of the Theorem.

Appendix H. Proofs of Theorems 1 and 2

In this section, we show that Theorems 1 and 2 are corollaries to Theorem 3.

We assume that the thesis of Theorem 3 holds: Let $V = \{[x] : x \in X\}$ be the set of types of 1-tuples. Then, there exists a relational system with orientations $(X \cup \mathcal{O}, \sim)$, elements x^v, and tuples of orientations $\bar{o}^v$, $v \in V$ such that for each each $(X, \sim)$-invariant distribution ω there exist $(x^v, \bar{o}^v)$-symmetric functions $f^v : [0,1)^{n^v} \to Y$ for $v \in V$ such that ω is equal to the joint distribution of variables

$$\theta^{f^v, x^v, \bar{o}^v}(x; u), x \in v \in V.$$

H.1. **Proof of Theorem 1.** For each $x \in v \in V$, fix a tuple $\bar{o}^x = ((\eta_1^x, q_1^x), ..., (\eta_{n^v}^x, q_{n^v}^x))$ such that $x\hat{\ }\bar{o}^x$ is analogous to $x^v\hat{\ }o^v$. Define a function

$$f_x(u_1, ..., u_{n^v}) = f^v(q_1^x(u_1), ..., q_{n^v}^x(u_{n^v})).$$

Then, the number of different functions f_x is bounded by the number of types of 1-tuples $|V|$, the number of different orientations, and the number of parameters of functions f^v. In particular, there exists finite m and n such that each invariant distribution ω admits (m, n)-decomposition.

H.2. **Proof of Theorem 2.** Suppose that ω admits $(x^v, \bar{o}^v)_{v\in V}$-decomposition with $(x^v, \bar{o}^v)$-symmetric functions f_v. Assume that assignments k and n are obtained as in the above proof of Theorem 1. Then, $x \in v \in V$ is not affected by shock η if, for almost all realizations $u \in [0,1]^{\mathcal{U}}$, $\theta^{f^v, x^v, \bar{o}^v}(x; u(-\eta), \eta)$ is an almost surely constant function of realization η.

The proof is divided into the following steps. First, because orientations are measure-preserving bijections, we can assume w.l.o.g. that for each v, tuple $\bar{o}^v = ((\eta_1^v, p_1^v), ..., (\eta_{n^v}^v, p_{n^v}^v))$ consists of orientations of distinct shocks, $\eta_m \neq \eta_{m'}$ for all $m \neq m'$ (one can always redefine tuple $\bar{o}$ and symmetric function f_v to avoid repeating orientations of the same shock).

Second, we can assume w.l.o.g. that for each $m \leq n^v$, for almost all realizations of u, $f_v(\bar{o}(u))$ is not almost surely constant function of $u(\eta_m^v)$ (otherwise, one can redefine symmetric f_v to avoid spurious parameters).

Third, for each tuple of orientations $\bar{o} = ((\eta_1, p_1), ..., (\eta_m, p_m))$, define the shock support of $\bar{o}$ as $\operatorname{supp}(\bar{o}) = \{\eta_1, ..., \eta_m\}$. Then, for any $x \in v$, and any two tuples of orientations $\bar{o}$ and $\bar{o}'$, if $x\hat{\ }\bar{o}$ and $x\hat{\ }\bar{o}'$ are analogous to $x^v\hat{\ }\bar{o}^v$, then $\operatorname{supp}(\bar{o}) = \operatorname{supp}(\bar{o}')$. (Indeed, if not, then w.l.o.g. there is $\eta \in \{\eta_1, ..., \eta_{n^v}\} \setminus \{\eta_1', ..., \eta_{n^v}'\}$, and the value of $f^v(\bar{o}(u))$ depends on the realization of $u(\eta)$, but the value of $f^v(\bar{o}'(u))$ doesn't. Because of the first two steps, $f^v(\bar{o}(u)) \neq f^v(\bar{o}'(u))$ for some realization of u, which contradicts the fact that f^v is $(x^v, \bar{o}^v)$-symmetric.) Define

$$\mathcal{U}(x) = \{\eta_1, ..., \eta_{n^v}\}.$$

Then, x is affected by shock η if and only if $\eta \in \mathcal{U}(x)$. In particular, each $x \in X$ is affected by at most finitely many shocks.

Fourth, it follows from the third step that, for each orientation o of shock $\eta \in \mathcal{U}(x_1) \backslash \mathcal{U}(x_1)$, if $x_1 \hat{} x_2 \hat{} o$ is analogous to $x_1' \hat{} x_2' \hat{} o'$, and o' is an orientation of shock η', then $\eta' \in \mathcal{U}(x_1') \backslash \mathcal{U}(x_2')$.

Fifth, suppose that $x \in D(\eta)$ for some shock η, and $D(\eta)$ is analogous to $D \subseteq X$ relative to x. We show that there is a shock η' so that $D = D(\eta')$. Indeed, there are enumerations $d_1, d_2, \ldots$, of $D(\eta)$, $x_1, x_2, \ldots$, of $X \backslash D(\eta)$, $d_1', d_2', \ldots$, of D, $x_1', x_2', \ldots$, of $X \backslash D$, such that for each m, tuples $x \hat{} d_1 \hat{} x_1 \hat{} \ldots \hat{} d_m \hat{} x_m$, and $x \hat{} d_1' \hat{} x_1' \hat{} \ldots \hat{} d_m' \hat{} x_m'$ are analogous. Let o be an orientation of shock η. By external consistency, there exist orientations $o_m = (\eta_m, p_m)$ such that tuples $o \hat{} x \hat{} d_1 \hat{} x_1 \hat{} \ldots \hat{} d_m \hat{} x_m$, and $o_m \hat{} x \hat{} d_1' \hat{} x_1' \hat{} \ldots \hat{} d_m' \hat{} x_m'$ are analogous. By the third step, $\eta_m \in \mathcal{U}(x)$. Because $\mathcal{U}(x)$ is finite, and the set of orientations of each shock is finite, there are finitely many orientations of each shock in $\mathcal{U}(x)$. Thus, there exists an orientation o' of shock $\eta' \in \mathcal{U}(x)$ so that for infinitely many m, tuples $o \hat{} x \hat{} d_1 \hat{} x_1 \hat{} \ldots \hat{} d_m \hat{} x_m$, and $o_m \hat{} x \hat{} d_1' \hat{} x_1' \hat{} \ldots \hat{} d_m' \hat{} x_m'$ are analogous. By internal consistency, it must be that tuples $o \hat{} x \hat{} d_1 \hat{} x_1 \hat{} \ldots \hat{} d_m \hat{} x_m$, and $o_m \hat{} x \hat{} d_1' \hat{} x_1' \hat{} \ldots \hat{} d_m' \hat{} x_m'$ are analogous for all m. By the fourth step, it must be that $\eta' \in \mathcal{U}(d)$ for each $d \in D$, and $\eta' \notin \mathcal{U}(x')$ for each $x' \notin X \backslash D$. By the third step, $D(\eta') = D$.

If there are infinitely many sets D that are analogous to $D(\eta)$ relative to x, then there are infinitely many shocks η' such that $D(\eta')$ is analogous to D relative to x. Because $x \in D(\eta')$ for all such shocks, it must be that $\mathcal{U}(x)$ is infinite, which in turn leads to a contradiction with (3).

Appendix I. Proof of Lemma 11

For each $x, x' \in X$, let $x \triangle x'$ denote the symmetric difference of sets x and x', i.e., $x \triangle x' = x \backslash x' \cup x' \backslash x$. The symmetric difference is reflexive, symmetric, and transitive: for each $x, x', x'' \in X$,

$$x \Delta x = \varnothing,$$
$$x \Delta x' = x' \Delta x,$$
$$(x \Delta x') \Delta x'' = x \Delta (x' \Delta x'').$$

Because of the first and the last equality above, $x\Delta. : X \to X$ is a bijective mapping such that $x \Delta (x \Delta x') = x'$. One can check that two tuples $\bar{x}, \bar{x}' \in X^k$ are analogous if and only if there exists $x \in X$ such that

$$(x_1 \Delta x_1') \Delta x_l = x_l' \text{ for each } l \leq k. \tag{I.1}$$

Suppose that U is a local set. Take any $y \notin U$ and $z \in U$ and define

$$U' = U \cup (y\Delta z)\,\Delta U = \{x, y \triangle z \triangle x : x \in U\}.$$

Then, $|U'| \leq |2U|$.

We show that U' is local. It is enough to check that external consistency holds. Take any two analogous tuples $\bar{x}, \bar{x}'$ of elements of U'. Take any $x \in U$. Then, $\bar{x}'\hat{}\,((x_1\Delta x_1')\,\Delta x)$ is analogous to $\bar{x}\hat{}x$. We show that $x' = (x_1\Delta x_1')\,\Delta x \in U'$.

Notice that $x' = (x_1\Delta x_1')\,\Delta x$ is the unique element so that $x_1\hat{}x$ is analogous to $x_1'\hat{}x'$. Because U is local, if $x_1, x_1', x \in U$, it must be that $x' \in U$. For all $x_1, x_1', x \in U \cup y\Delta z\Delta U$, x' takes one of two forms:

$$x' = (w_1\Delta w_1')\,\Delta w, \text{ or}$$
$$x' = (y\Delta z)\,\Delta\left((w_1\Delta w_1')\,\Delta w\right),$$

for some $w_1, w_1', w \in U$. In particular, $x' \in U'$.

University of Texas at Austin, Department of Economics. *Email*: mpeski@gmail.com.
E-mail address: `mpeski@gmail.com`