BARGAINING WITH MECHANISMS: TWO-SIDED INCOMPLETE INFORMATION

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Abstract. I study a random proposer multi-round bargaining over a single good with transfers. In each round, a proposer may offer an arbitrary mechanism to determine the final allocation. If the offer is accepted, it is implemented and the game ends; otherwise, the next round commences. When there is a two-sided incomplete information with binary types for each player, the ex ante expected equilibrium payoffs are unique.

1. Introduction

Division of a bargaining surplus is one of the most important and the longest-studied questions in economic theory[1]. Since Nash Jr (1950), Nash (1953), and Rubinstein (1982), the literature has developed various cooperative and strategic approaches to the problem. Nevertheless, the crucial case of bargaining under incomplete information remains without a full and satisfactory solution. On one hand, many papers focused on the case of bargaining with one-sided incomplete information, where all the offers are made by the uninformed party. The assumption makes the model tractable by avoiding signaling issues, but it is restrictive. The solution typically depends on the suport of type distribution, with an uncomfortable difference between gap and no-gap cases. On the other hand, papers that study two-sided incomplete information typically establish a large folk-theorem-type set of equilibria[2]. The latter is due to a possibility of sustaining multiple equilibrium outcomes with punishing beliefs that is inherent to signaling models.

This paper shows that a natural and realistic modification of a standard bargaining model has a good solution: the payoffs are unique, simple to characterize, and continuous in the underlying parameters, including players' beliefs. I work with a random-proposer bargaining over a single good with transfers, and where each player has a privately known value of the good. In each round, a randomly selected player makes an offer. I refer to the probability of choosing a proposer as player's bargaining power. The model includes the possibility where all offers by one of the players. The

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[1]VERY PRELIMINARY AND INCOMPLETE. I am grateful for comments to seminar participants at U of Waterloo and UCL.

[2]If players are only able to offer simple allocations, bargaining games with both the informed and uninformed players making offers typically have multiple equilibria (Ausubel and Deneckere 1989, Gul and Sonnenschein (1988)). The uniqueness can sometimes can be restored by equilibrium refinements (Grossman and Perry (1986b)).

second player accepts or rejects the offer. If the offer is accepted, it is implemented and the game ends. Otherwise, the game moves to the next round. Players discount future payoffs.

I assume that players offer mechanisms: games, where actions and rules (i.e., outcome function) determine the allocation of the good and transfers. This assumption is a departure from a typical bargaining model, where players' offers take form of a particular outcome: a decision who gets the good, a price, etc. There are two reasons for the departure. First, offers in real-world negotiations often take more sophisticated forms that, instead of a final allocation, propose a process to find it. Such offers include menus of outcomes, menus of menus (for example, "you divide and I choose"), changes to bargaining protocol, deadlines, mediation or arbitration, etc. Second, on top of improved realism, I am motivated by a hypothesis that an expansion of the set of available offers may allow the players to successfully address signaling issues.

This model was introduced in Pęski (2022), where I showed that, under one-sided incomplete information case, all bargaining equilibria have the same payoffs. There are two difficulties in extending those results to two-sided incomplete information. First, the space of actions is much larger. With one-sided incomplete information, one can without loss of generality restrict attention to menus, or, more generally, dominant strategy mechanisms. Such mechanisms are easy to characterize and to work with. This restriction is not possible with two-sided incomplete information. Furthermore, because an offer may lead to change in players' beliefs, there is no revelation principle and a restriction to revelation mechanism would be with loss of generality. In fact, there is no a priori upper bound on the number of mechanism actions, even when the space of types for each player is finite. At the same time, some details, like action labels, should not play any role.

Second, two-sided incomplete information turns players' decisions into a fully fledged informed principal problem with a twist. In a standard formulation (see Myerson (1983), Maskin and Tirole (1990)), a principal proposes a feasible, incentive compatible, and individually rational allocation. The main concern is that the (on- or off-path) proposed mechanism may affect the beliefs of an agent and the principal and agent payoffs. For example, players may be stopped from a potentially profitable deviation because they are afraid of punishment with beliefs, i.e., a belief update which leads to an equilibrium of either the offer or continuation game with very unattractive payoffs. Because individual rationality is enforced at the design stage, the agent's decision to accept or reject the proposal is not explicitly modeled. On the contrary, in our model, the rejection leads to continuation bargaining game with payoffs that depend non-trivially on beliefs about both agents. When designing a mechanism, the proposer needs not only to think about the effect of the update of beliefs about her on her payoffs, but also about the update in beliefs about the other agent due to the equilibrium decision at the acceptance stage.

To alleviate both problems, I restrict the attention to incomplete information with two types $l_i < h_i$ for each player i. Further, in order to make the space of actions manageable, instead of explicitly modeling mechanisms as games, I represent them as their equilibrium payoff correspondences. More precisely, a (derived) mechanism is a correspondence that maps belief profiles to the sets of equilibrium payoffs in the original game. An important preliminary result is an implementation theorem which shows which payoff correspondences can be (virtually) implemented with proper games.

Two main results provide a characterization of equilibrium payoffs in the bargaining game:

(1) In each equilibrium, each type payoffs must be (weakly) larger than their random monopoly payoffs. The latter is defined as the probability of becoming a proposer multiplied by the payoff that a player would receive if she owned the good and was allowed to make a single take-it-or-leave it price offer to the other player.
(2) Assume w.l.o.g. $l_1 \leq l_2$. As $\delta \to 1$, each equilibrium maximizes player 1's ex ante expected payoffs, subject to feasibility, incentive compatibility, and the constraints from part (1).

The first result is valid for arbitrary discount factors. The result determines a lower bound on equilibrium payoffs. In some cases, including when one player has all the bargaining power, or when one player's type is known, this bound is sufficient, as there is only one feasible and incentive compatible payoff vector that satisfies the first condition. However, under a general two-sided incomplete information, there is a gap between random monopoly payoffs and interim efficient payoffs.

The second result says that, in equilibrium, the entirety of the gap goes towards less advantaged player 1, i.e., the player with a lower value of the lowest type. The payoffs of player 2 are determined uniquely. The payoffs of player 1 are determined uniquely only in the ex ante sense, i.e., in the expectation, before player 1 learns their type. The payoff uniqueness is true only for discount factors close to 1.

Two additional comments are in order. First, although the solution is always interim efficient, it is not always ex post efficient. For example, if the probability of types l_1 and h_2 is sufficiently high, the good sometimes goes to player 1 type l_1 even if it s socially efficient to allocate it to type l_2. Second, Myerson (1984) introduced a neutral solution in order to extend the Nash bargaining solution to environments with incomplete information. The neutral solution is the smallest solution that satisfies three axioms imposed on the class of all bargaining problems. Because the class of bargaining problems studied in this paper is narrower, it is difficult to verify the axioms directly. Nevertheless, the neutral solution is equal to the outcome of the bargaining if and only if the latter is ex post efficient. In the cases where ex post efficiency fails, the neutral solution (which is not ex post efficient as well) awards the entire gap between interim efficient payoffs and random monopoly payoffs to player 2.

The idea of the proof behind the first result is an extension of an argument from the complete information case: I show that if player i were to receive less than her random monopoly payoff, she would have a profitable deviation to reject all offers of player $-i$ and wait until she has an opportunity to make a carefully designed counteroffer. The counteroffer has two properties: First, regardless of the beliefs that player $-i$ may have about player i after seeing the counteroffer and possible rejections of $-i$'s offers, if accepted with probability 1, the expected payoff from the counteroffer is sufficiently high that each type of player i receives more than her existing payoff. Second, no matter how player $-i$ expects her beliefs to be updated after her acceptance or rejection decision, at least one of the positive probability types after rejection would rather accept. As a result, any equilibrium of the accept/reject decision of player $-i$ must lead to acceptance with probability 1. I refer to mechanisms that satisfy the second property as offers that cannot be refused and present a general characterization of such offers.

The two properties of the counteroffer are related to a Strong Undominated Pareto Optimal (SUPO) problem from Maskin and Tirole (1990). A SUPO mechanism is a solution to an informed principal problem where the agent's payoff offer is at least 0 due to individual rationality constraint. Unlike a SUPO offer, the counteroffer must ensure that the other player (i.e., the counterpart of the "agent") receives at least the continuation game payoffs. The latter are non-zero, and more importantly, they depend on continuation beliefs. In the language of Maskin and Tirole (1992), the belief-dependent individual rationality constraint turns the problem into a version of informed principal with common values. I do not know a general solution to this problem. For two types, I can construct explicit solution. The question whether mechanisms with the required properties exist in general type spaces beyond binary case, and whether the first result extends, remains open.

The proof of the second result is more complicated. I show that, by using a combination of various offers that cannot be refused, player 1 can extract the entirety of the gap. The details can be found in the paper.

This is not the first paper to use sophisticated offers in bargaining. Mechanisms as offers have been considered in axiomatic theories of bargaining in Harsanyi and Selten (1972), Myerson (1979), and Myerson (1984). Certain mechanisms, like menus, also appear in some work on strategic bargaining under one-sided incomplete information. With the exception of Jackson et al. 2020, all related papers that we are aware of work solely with two types. Sen (2000) (see also Inderst (2003)) studies a two-type alternating offer game, where players can offer menus but not general mechanisms, and demonstrates the existence of a unique outcome in a refinement of PBE (perfect sequential equilibrium due to Grossman and Perry (1986a)). The equilibrium behavior depends on whether the high type prefers her own complete information Nash payoff, or the Nash allocation of the low type. In a similar bargaining environment, Wang (1998) studies the Coasian bargaining model with one sided incomplete information, with the uninformed party making all the offers. He shows that, in the unique equilibrium, the uninformed player separates the other player's two types with an optimal screening contract. In particular, the Coase conjecture fails, as Bob retains all power subject to the incentive compatibility constraints. More recently, Strulovici (2017) assumes that, instead of ending the game, any accepted offer becomes the status quo for future bargaining. In that setting, in the unique equilibrium, the uninformed player is unable to offer an inefficient payoff to type u_1' in order to screen out the more extreme type u_1''.

Clippel, Fanning, and Rozen (2022) proposes a simple two-stage strategic bargaining model that implements neutral solution in a general class of environments with two-sided incomplete information and arbitrary type spaces They assume that types are ex-post verifiable. The latter assumption helps to eliminate incentive compatibility constraints, which makes a model more tractable. In contrast, the current paper does not require ex-post verifiability.

Jackson et al. 2020 considers a general bargaining environment. Although the authors allow for incomplete information on both sides, they make a strong assumption that the total value of bargaining surplus is commonly known. This assumption implies that there are no incentive problems that stop agents from truthfully revealing their information. In the unique equilibrium, the agents use menus to implement information revelation in a single round of bargaining. The result is robust to small perturbations of the common knowledge assumption.

2. Model

2.1. **Environment.** There are two players $i = 1, 2$ and a mediator. The mediator will have a very specific role and for most of the paper, one can ignore them. Each player has a finite types T_i; the main result of the paper applies to the binary type case $T_i = \{l_i, h_i\} \subseteq \mathbb{R}$. Players must decide on the allocation of a single good and transfers. A social outcome is a tuple $x = (q_1, q_2, \tau_1, \tau_2, \tau_M)$, where $q_i \in [0, 1]$ is probability that player i gets the good and $\tau_i \in [-L, L]$ is a transfer. I assume that the restriction $L < \infty$ is an arbitrarily large number and its purpose is to ensure that the space of social outcomes is compact. The outcome is feasible if $q_1 + q_2 \leq 1$ and $\tau_1 + \tau_2 + \tau_M \leq 0$ and I denote the set of feasible outcomes as X. Given outcome x, type t_i's payoff is equal to $q_i t_i + \tau_i$. Mediator's payoff is equal to τ_M.

2.2. **Payoffs.** Let ΔT_i be the space of beliefs about the types. Let $T = \times_i T_i$ be the space of type profiles, and let $\Delta T = \times_i \Delta T_i$ be the space of belief profiles. Let $U = \mathbb{R}^{\bigcup_i T_i}$ be the space of payoff vectors. Let $d_{\Delta T}, d_U$ be (Euclidean) metrics on the space of belief profiles and payoff vectors.

An *(ex post) allocation* is a mapping $\chi : T \to X$. The allocation is *incentive compatible given beliefs* $p \in \Delta T$ if, for each player i and type $t_i, s_i \in T_i$,

$$\tau(\chi, t_i) := \int \tau(\chi(t_i, t_{-i}), t_i)\, dp_{-i}(t_{-i}) \geq \int \tau(\chi(s_i, t_{-i}), t_i)\, dp_{-i}(t_{-i}).$$

Let

$$\mathcal{U}(p) = \left\{(\tau(\chi, t_i))_{i, t_i \in T_i} \in U : \chi \text{ is incentive compatible allocation}\right\}$$

be the set of payoff vectors in all incentive compatible allocations given prior p. Observe that correspondence $\mathcal{U} : \Delta T \rightrightarrows U$ is nonempty-valued (as $0 \in \mathcal{U}(p)$ for each p), convex-valued, and u.h.c.

For each payoff vector $u \in U$, each $p \in \Delta T$, denote $p_i \cdot u_i = \sum_{t_i} p_i(t_i) u(t_i)$. For each profile of welfare weights $\Lambda = (\Lambda_i(t_i))_{i,t_i}$ such that $\sum_{t_i \in T_i} \Lambda(t_i) = 1$ for each i, payoff vector u is Λ-*optimal under beliefs* p if

$$u \in \arg\max_{u \in \mathcal{U}(P)} \sum_i \Lambda_i \cdot u_i.$$

Payoff vector u is p-interim efficient if it is Λ-optimal for some weights $\Lambda \geq 0$ and beliefs p. Let $\mathcal{U}_{\text{eff}}(p) \subseteq \mathcal{U}(p)$ denote the set of all interim efficient payoffs.

2.3. **Payoff correspondences.** A *payoff correspondence* is any u.h.c. and nonempty-valued correspondence $e : \Delta T \rightrightarrows \mathbb{R}^{\bigcup_i T_i}$ such that $e(p) \subseteq \mathcal{U}(p)$ for each p. Let $\mathcal{E}_0$ be the space of payoff correspondences.

For any two correspondences $e, f \in \mathcal{E}_0$, say that $e \subseteq f$ if $e(p) \subseteq f(p)$ for each p.

Define an (asymmetric) distance function between payoff correspondences as:

$$d_0(e, f) = \max_{(p,u)\in e} \min_{(q,v)\in f} d_{\Delta T}(p, q) + d_U(u, v).$$

A sequence of correspondences e_n *approximates* f, $e_n \rightarrowtail f$, if $d_0(e_n, f) \to 0$.

Let $d(e, f) = d_0(e, f) + d_0(f, e)$ be a metric on $\mathcal{E}_0$. Metric d is equivalent to the Hausdorff distance of correspondences e, f treated as subsets of $\Delta T \times U$. A sequence of correspondences e_n *converges to* f, $e_n \to f$, if $d(e_n, f) \to 0$. The space of payoff correspondences is compact under such metric.

A *payoff function* is a single-valued payoff correspondence. Because each payoff function is u.h.c., it is also necessarily continuous. A *Kakutani correspondence* is a payoff correspondence that is also convex-valued. A *Michael correspondence* is a payoff correspondence that can be approximated by a sequence of payoff functions. One shows using the Michael Selection Theorem that each Kakutani correspondence is a Michael correspondence (for example, see Pęski (2022)). Let $\mathcal{E} \subseteq \mathcal{E}_0$ be the set of Michael correspondences; it is compact under the Hausdorff distance.

2.4. **Mechanisms.** A *game* is a tuple $g = \left((A_i)_{i=1,2,M}, \xi\right)$, where A_i is a set of actions of players $i = 1, 2$ and player M (mediator) and $\xi : \times_{i=1,2,M} A_i \to X$ is an outcome function. A game is finite, if action sets are finite. In the game, players simultaneously choose actions $a_i \in A_i$ from (finite, compact) set A_i and outcome $\xi(a_i, a_{-i})$ is implemented. A game g and a belief profile p gives rise to a *game with incomplete information.* I assume that, in each such game, prior to taking their actions, players observe a public randomization device.

Following Pęski (2022), I assume that all relevant information about the game is contained in its payoff correspondence. In order to abstract from irrelevant details, I define a *mechanism* as a Michael correspondence. An example id a correspondence of all incentive compatible allocations $\mathcal{U}$. I refer to $\mathcal{U}$ as the *universal mechanism.* For another example, for each game g and belief profile p, let $E(p|g) \subseteq \mathcal{U}(p)$ denote the set of payoff vectors in Bayesian equilibria of the incomplete information game. If g is finite, the derived payoff correspondence $E(g)$ is Kakutani. As a result, $E(g)$ is a mechanism. [3]

The converse question, which mechanisms can be implemented by games, belongs to the implementation theory. Here, I show the following approximation result.

Theorem 1. *Each mechanism, i.e., each Michael correspondence, can be approximated by a sequence of payoff correspondences derived from finite games.*

Theorem 1 says that each mechanism can be approximated with a finite game. In other words, the space of Michael mechanisms is a closer of the space of payoff correspondences obtained from finite games. The proof of the Theorem focuses on mechanisms that are payoff functions, which is sufficient due to the definition of Michael correspondence. For arbitrary payoff function m, I construct a game between the two players and the mediator. The sole role of the mediator is to name player's beliefs p. Given beliefs p, I use virtual implementation result of Abreu and Matsushima (1992) to find a game that (approximately) implements $m(p)$.

I say that a mechanism m is *interim efficient* if $m \subseteq \mathcal{U}_{\text{eff}}$.

2.5. **Derived mechanisms.** Given one or more mechanisms, one can use them to create new ones. Suppose that m is a mechanism. Then,

- For each $\delta < 1$, let *discounted mechanism* δm be a mechanism obtained from mechanism m by multiplying all payoffs in m by δ.
- Let $w : \Delta T \to \mathbb{R}$ be an arbitrary continuous function. For each such w, mechanism $m +_i w$ is a mechanism, where, on top of m, if players beliefs are p, player $-i$ transfers $w(p)$ to player i. The transfer is from player to player, regardless of true types, hence it does not affect incentive compatibility of allocation in m.
- $\text{con}m$ is a mechanism obtained by preceding m by an observation of a public randomization: for each p, $(\text{con}m)(p) = \text{con}(m(p))$.
- Let $\mu \in \Delta\mathcal{E}$ be a probability distribution over mechanisms with a finite support. Define a *randomized mechanism* μ so that for each p
$$\mu(p) = \left\{ \int \xi(m)\, d\mu(m) \text{ where } \xi : \text{supp}\mu \to U \text{ is a selection } \xi(m) \in m(p) \right\}.$$
- A *menu of mechanisms* $MM_i(A)$ for a compact set $A \subseteq \mathcal{E}$ of mechanisms is a mechanism, where player i observes public randomization, chooses $m \in A$ and, additionally, sends a cheap-talk message from a rich Borel space S.[4] Player $-i$ observes the choices of player i and updates

[3]The restriction to upper-hemi continuous and convex-valued (due to public randomization) appears already in Maskin and Tirole (1990).

[4]Adding cheap talk allows to represent behavior using distributional strategies. This helps in establishing the compactness of strategy space, and plays an important role in the existence proof. Adding randomization convexifies the set fo payoffs, establishes that the payoff correspondence is Kakutani, hence Michael, which also plays a role in the existence.

their beliefs about i. Finally, a continuation equilibrium in mechanism m is implemented. The formal definition of $MM_i(A)$ and the proof that it is a proper mechanism are discussed in Appendix B.

- A *player i information revelation $I_i m$* is a mechanism, where player i observes public randomization, sends a cheap-talk message from a rich Borel space S, following which mechanism m is implemented. Formally, $I_i m = MM_i(\{m\})$, i.e., information revelation game is equivalent to a singleton menu of mechanisms.
- An *informed principal with continuation m, $IP_i(m)$*, is a mechanism, where, first, player i chooses an arbitrary mechanism $a \in \mathcal{E}$; second, player $-i$ chooses between mechanisms m and a:
$$IP_i(m) = MM_i(\{MM_{-i}(\{m, a\}) : a \in \mathcal{E}\}).$$
Mechanism n is interpreted as an offer made by the principal, and the choice of mechanism m is interpreted as a rejection of the offer.

2.6. **Random-proposer bargaining.** Finally, I formally define a *random-proposer bargaining* mechanism $\mathtt{B}$. The idea is to utilize the recursive nature of the bargaining game: In each period, a player i is randomly chosen with probability $\beta_i = \begin{cases} \beta & i = 1 \\ 1-\beta & i = 2 \end{cases}$ to be an informed principal with the continuation game equal to the discounted random-proposer bargaining. For a given discount factor δ, let $\mathtt{B}^\delta$ denote the mechanism corresponding to the bargaining game. Then, it must satisfy the following equation:

$$\mathtt{B}^\delta = \left(IP_1\left(\delta \mathtt{B}^\delta\right)\right)^{\beta_1} \left(IP_2\left(\delta \mathtt{B}^\delta\right)\right)^{\beta_2}.$$

I define correspondences $\mathtt{B}$ as the largest solution to the above equations. One shows that the largest solution is well-defined and $\mathtt{B}$ is a proper mechanisms.[5]

2.7. **Comments.** *Equilibrium.* It is instructive to compare the above definition of equilibrium payoffs with a more standard notion of equilibrium profile. Typically, in order to define a perfect equilibrium, one defines histories, strategies, beliefs, proposes a motion of measurability (which is an issue here due to the large space of actions), consistency, and proceeds to define an equilibrium as a profile of strategies that are best responses given the beliefs. The elements of such approach are contained in the definition of an equilibrium in the menu of mechanism game described in Appendix B. There is a key difference: Instead of characterizing strategic behavior in the continuation game, I replace it by a continuation payoff. The approach is modular: focus on the behavior in the game at hand and leave the continuation behavior for some other definition. One consequence is that such a definition assumes

[5]This follows from the following property of menus of mechanisms: Suppose that, for two compact sets of mechanisms A and B, there exists a measurable function $\phi : A \to B$ such that for each $m \in A$, $m \subseteq \phi(m)$. Then, $MM_i(A) \subseteq MM_i(B)$.

that the one-shot-deviation principle always holds. Another consequence is that the definition does not require that the continuation behavior in the mechanism is measurable with respect to the history in the game at hand, as long as the continuation payoffs and beliefs at the beginning of the mechanism are measurable.

The notion of consistent beliefs requires that, given player's strategy, with probability 1, the beliefs are updated according to the Bayes formula and, otherwise, the beliefs about player i's behavior change only following the action of player i. (So, for example, off-path behavior of player $-i$ does not affect the beliefs about player i.) The latter restriction is satisfied by notions like sequential equilibrium, at least in games where such a definition exist.

The definition of the equilibrium in the bargaining game is derived from the one just described, where, at each stage, I interpret (a) the continuation game as a mechanism, and (b) the action choice, as a choice of a mechanism in the menu of mechnisms.

Existence: The existence of an equilibrium in the menu of mechanisms is an issue due to the large action space. The proof follows the same ideas as the proof in Pęski (2022), henceforth it is omitted. I only point out is that, in one of the steps of the existence proof in Pęski (2022), I showed that Kakutani mechanisms can be approximated by payoff functions. Because this paper deals with Michael correspondences, the approximation is guaranteed by definition.

Mechanisms vs games: In the model, players propose mechanisms. The latter correspond to limits of finite games. There is a technical reason for doing so: the approach of this paper ensures that the space of players available is appropriately compact, with all the benefits it creates, the most important of all is the existence of equilibrium.

More importantly, for the main results of this paper, working with the limit games is sufficient. This is because the main results are negative - they show that an equilibrium with some payoffs cannot exist because some player would have a profitable deviation. The existence of a profitable deviation in the form of an abstract mechanism implies that at least one of the approaching games is a profitable deviation as well.

The limit approach is more problematic for the existence of equilibrium. This existence arguments that this paper relies on do not guarantee that the equilibrium choices involve mechanisms that can be implemented by games. There are few ways to address the issue: either look for stronger implementation results, or weaken the definition of equilibrium to an ε-equilibrium-like concept. The latter approach is standard in games with non-compact action spaces. Both solutions go beyond the scope of the current paper and are left for future research.

Environment: The single good environment described in Section 2.1 is very specific. Results of Section 5 apply much more widely. The quasi-linearity in transfers is only used in constructions of mechanisms like $a +_i \varepsilon$ for small $\varepsilon > 0$. But the assumption is not necessary, and the results apply to all environments where any mechanism can be approximated by a mechanism that is a strict improvement for all types of a given player. The results of Section 4 do not apply beyond the single-good-with-transfers environment.

Commitment: An important assumption of the model is that once the mechanism is offered and accepted, the players are committed to its implementation. Although this assumption is shared by the Coasian bargaining literature, and also the more recent literature on dynamic mechanism design with limited commitment (e.g., Skreta (2006), Doval and Skreta (2018), Liu et al. (2019)), I also allow for a wider range of mechanisms than this literature typically considers. For example, an agreement on negotiation protocol may force players to restrict their future options, set a deadline, or choose an ex-post inefficient outcome. In other words, I allow players to commit *jointly.* This approach to commitment is applicable in situations in which such a commitment is possible, either because the nature of dividing the surplus makes it impossible to divide it again, renegotiation is costly, or the agreement is enforced by an arbitrator or a court.

To make the point about commitment starker, consider a special case of the model in which only player 1 makes all the offers. As I explain below in section 3.2, such a model is equivalent to informal principal problem, where player 1 offers to selll the good to player 2 at the monopoly price that maximizes player 1's type payoff. If there is a complete information about type fo player 1, this result can be contrasted with the Coase conjecture, which predicts that the uninformed player sells at the price equal to the lowest possible value of the informed player, and the equilibrium is efficient. In the bargaining literature, the Coase conjecture has been typically associated with the "gap case" of the durable monopoly problem, with offers made only by the uninformed party (Gul, Sonnenschein, and Wilson (1986)), but the Coasian forces play a role also in alternating-offer models (Gul and Sonnenschein (1988)).

3. Benchmarks

In this section, I describe three relevant benchmarks.

3.1. **Complete information.** Assume that both player types are known, and, w.l.o.g., $t_1 < t_2$ are known. In such a case, the solution to the bargaining model is well-known. In equilibrium, the higher value player 2 gets the good and pays $\beta_1 t_2$ to player 1. The resulting payoffs of player i are equal to $\beta_i t_2$. In particular, the allocation is efficient, and the payoffs depend only on the type of the higher value player.

It is worthwhile to remind the argument behind this observation. Suppose that the lowest equilibrium payoff of player i is $x_i t_2$ for some $x_i < \beta_i$. Because the sum of payoffs available to players cannot be larger than t_2, player $-i$, if not a proposer, will always accept the offer that gives him at least $\delta(1-x_i)t_2$. Consider a player i strategy, where the player always rejects any offer of player $-i$, and whenever a proposer, always offers $\delta(1-x_i)t_2$ to player $-i$ and $(1-\delta(1-x_i))t_2$ for herself.

Because her offer will be accepted, such a strategy guarantees her the expected payoff of at least

$$\beta_i \sum_{t \geq 0} \left(\delta \left(1-\beta_i\right)\right)^t \left[1-\delta\left(1-x_i\right) t_2\right] = x_i t_2 + \left(\beta_i - t_i\right) \frac{1-\delta}{1-\delta+\delta\beta_i} t_2 > x_i t_2. \tag{3.1}$$

The last inequality leads to the contradiction with the definition of equilibrium.

3.2. **One-sided bargaining power (informed principal problem).**

When $\beta_i = 1$, player i makes all the offers. The model becomes equivalent to an informed principal problem with private values.Maskin and Tirole (1990) show that each player i type has unique equilibrium payoff equal to the monopoly payoff player i would receive if she owned the good and could make a single take-it-or-leave-it offer to player $-i$:

$$M\left(t_i; p_{-i}\right) = \max_{\tau} p_{-i}\left(t_{-i} \leq \tau\right) t_i + \left(1 - p_{-i}\left(t_{-i} \leq \tau\right)\right) \tau. \tag{3.2}$$

I refer to $M\left(t_i; p_{-i}\right)$ as a *monopoly payoff*. Player $-i$'s payoff for some types is equal to 0; if $-i$ type is strictly above the monopoly price of some types of player $-i$, his payoff is strictly above above 0. The allocation that induces such a payoff is typically not *ex post* efficient, but it is ex ante efficient.

Because player i's payoff is exactly the same as if her type was known, Maskin and Tirole (1990) conclude that incomplete information about informed principal type does not matter in private values, transferable utility case. It is worthwhile to emphasize that the definition of private value environment includes the fact that the payoffs after player $-i$ rejects player i's offer are equal to 00, or, more generally, do not depend on beliefs about i's types. This property of static informed principal models is not going to be satisfied in the informed principal with continuation mechanism IP, where, in general, continuation payoffs depend on beliefs about each player.

If $\delta = 1$, there is only one round of offers. If player i type t_i is chosen to be a proposer, she expects a payoff of $M\left(t_i; p_{-i}\right)$. Ex ante, she expects to get at least $\beta_i M_i\left(t_i; p_{-i}\right)$. I refer to this payoff as a random monopoly payoff. (Her equilibrium payoff may be higher if she expects non-zero payoff if she is not the proposer.)

3.3. **One-sided incomplete information.**

Suppose that type t_i of player i is known. Pęski (2022) showed that the equilibrium payoffs are unique. Player i is equal to her random monopoly payoff $\beta_i M_i\left(t_i; p_{-i}\right)$. Player $-i$ type t_{-i} receives at least $\beta_{-i} M_{-i}\left(t_{-i}; p_i\right) = \beta_{-i} \max\left(t_i, t_{-i}\right)$, and the payoffs are equal to the random monopoly payoff for all types below the monopoly price of player i. The proof extends the complete information argument by constructing a class of mechanisms $a^j(x)$ for each player j and each x such that, for each x,

- if accepted, the payoff of each type t_j in mechanism $a^j(x)$ is at least $xM_j\left(t_j; p_{-j}\right)$,

- if each type of player j expects to get at least $xM_j(t_j;p_{-j})$ in equilibrium, then, in any equilibrium, player $-j$ will always accept mechanism $a^j(1-\delta(1-x))$.

The second property is a counterpart of a simple observation in the complete information case. Extending it further to two-sided incomplete information is the key step of the proof of the first main result, Theorem 2.

4. Main results

In this section, I state the two main results of the paper. From now on, assume that each player has two types, $T_i = \{l_i, h_i\}$. W.l.o.g. assume that $l_1 \leq l_2$.

In the rest, the following notation is used: for each $x \in [0,1]$, I refer to $x_1 = x$ as a "player 1's share" and to $x_2 = 1 - x$ as "player 2's share". Abusing notation, I denote the probability of high type as $p_i = p_i(h_i)$.

The first result shows, in each equilibrium, for each discount factor, each type of each player gets at least their random monopoly payoff.

Theorem 2. *For each $\delta < 1$, each $u \in \mathtt{B}^\delta(p)$, each player i, each t_i, $u_i(t_i) \geq \beta_i M_i(t_i;p_{-i})$.*

Theorem 2 provides lower bounds on equilibrium payoffs. In some cases, the lower bounds are sufficient to determine the equilibrium payoffs: If either one of the players i has all the bargaining power ($\beta_i = 1$), or her type is known ($p_i \in \{0,1\}$), or when $h_1 > h_2$, there is a unique payoff vector that satisfies the payoff constraints. Such payoffs can be guaranteed in the following mechanism: player i gets the good with a probability equal to the probability of being chosen a proposer β_i and is allowed to make a single take-it-or-leave-it sell offer to player $-i$. (If such a mechanism is played, the low types gets exactly their random monopoly payoff; the high type may get a higher payoff if the opponent gets the good and sells it below the high type valuation.)

The proof of Theorem 2 follows the same logic as the argument described in Section 3.3. The main difficulty is in construction of mechanisms a^1 and a^2 with the required two properties. The construction is presented in Section 4.

I refer to mechanisms that satisfy the second property listed in Section 3.3 as mechanisms that cannot be refused given some continuation mechanism (see Section 5.2 for details). In particular, for each i, mechanism a^i is a mechanism that cannot be refused by player $-i$ given the continuation bargaining game.

When the lower bounds from Theorem 2 are not sufficient to determine payoffs, there is a gap between random monopoly payoffs and interim efficiency. To quantify such a gap, let

$$E(p,\beta) = \max_{u \in \mathcal{U}(p)} p_1 \cdot u_1 \text{ st. } \forall_{i,t_i} u_i(t) \geq \beta_i M_i(t_i;p) \tag{4.1}$$

denote the maximum expected payoff player 1 can get in an incentive compatible allocation that ensures that each type of player 2 gets his random monopoly payoff. Then, the gap is equal to

$$\text{Gap}\,(p,\beta) = E\,(p,\beta) - p_1 \cdot \beta_1 M_1\,(.;p_2)\,. \tag{4.2}$$

Section 5.4 provides a precise characterization of the gap. The gap is continuous in the parameters of the model, including beliefs and bargaining power. If $h_2 > h_1$ and both players have non-0trivial bargaining power, i.e., $\beta_i \in (0,1)$, the gap is maximized at interior beliefs. I show that the gap satisfies the following estimate:

$$\text{Gap}\,(p) \leq 6.25\% \cdot h_2 \text{ for all } p.$$

Mechanism a^2 is a mechanism that gives the entirety of the gap to player 1. There are other mechanisms with such property.

The second main result of this paper shows that, in equilibrium, the entirety of the gap goes to player 1:

Theorem 3. *For each p, $\lim_{\delta\leftarrow 1} \sup_{u\in \mathrm{B}^{\delta}(p)} |E\,(p,\beta) - p_1 \cdot u_1| = 0$.*

The characterization in Theorem 3 is tight enough to uniquely determine the allocation of the good (i.e., the probability with which players get the good, conditionally on their types), player 2 payoffs, and, for a generic subset of the parameter space, payoffs of player 1. In other cases, player 1 payoffs are determined up to their ex ante value.

In the proof, I show that, as $\delta \to 1$, for each $\varepsilon > 0$, player 1 has a strategy of form "reject all offers of player 1 and present a carefully designed counteroffer" that ensure that the player 1 gets all but, at most ε fraction of the gap. If given an opportunity to propose an offer, depending on the anticipated equilibrium continuation beliefs and her type, player 1's offer take form of one of two two mechanisms: a^1 and $a^2 - w$. The former is an offer that, as I explain above, cannot be refused given the continuation game. The latter mechanism is a version of mechanism a^2 but with additional payments w to player 1 to make a^2 an offer that cannot be refused by player 2. I show that, as $\delta \to 1$, the payments w needed can be made arbitrarily small. Asymptotically, a^2 is also a mechanism that cannot be refused. Thus, given the above comment, player 1 gets the entirety of the gap.

5. Proofs

In this section, I introduce main techniques and ideas behind the proofs of Theorems 2 and 3.

5.1. **Derived mechanisms.** I start with characterization of payoffs in two derived mechanisms: information revelation and menu of mechanisms (see Section 2.5). For each mechanism a, define its support as $Sa = \{(v,q) : v \in a\,(q)\}$.

Information revelation: For each probability distribution $\mu \in \Delta(\mathcal{U} \times \Delta T)$, let

$$p_i(\mu) = \int q d\mu(v, q)$$
$$u(t_i|\mu) = \frac{1}{p_i(t_i|\mu)} \int vq(t)\, d\mu(v, q) \text{ for each type } t_i \in T_i,$$

Say that distribution μ is i-splitting if

- μ only reveals information about player i types: there is p_{-i} such that $\mu \in \Delta(\mathcal{U} \times \Delta T_i \times \{p_{-i}\})$ and
- revelation of information is an equilibrium behavior: $\mu \in \Delta(\{v, q\} : v_i \leq u_i(q))$.

A standard lemma is presented without a proof:

Lemma 1. *For each mechanism a,*

$$I_i(a)(p) = \{u(\mu) : \mu \in \Delta Sa,\ \mu \textit{ is } i\textit{-splitting, and } p(\mu) = p\}.$$

Menu of mechanisms: In order to characterize payoffs in menus of mechanisms, the following two definitions will be useful. First, consider the following order on payoff vectors for player i. For each prior $p_i \in \Delta T_i$, let $u \leq_{p_i} u'$ (resp. $u <_{p_i} u'$) if and only if all strictly positive probability types of player i prefer payoffs u' to u: $u(t_i) \leq u'(t_i)$ for each t_i st. $p(t_i) > 0$ (resp. if, additionally, at least one of the inequalities for a strictly positive type is strict). In other words, $\leq_{p_i}$ is a standard vector comparison but applied only to payoffs associated with player i types that have strictly positive probability under p_i.

Second, for each mechanism m, each p and each player i, define the *upper contour of* m:

$$U_i m(p) = \{u \in \mathcal{U}(p) : u' \leq_{p_i} u \text{ for some } q_i \in \Delta T_i \text{ and } u' \in m(q_i, p_{-i})\}.$$

$U_i m(p)$ contains incentive compatible payoff vectors u in whcih player i has weakly higher payoffs than in some payoff vector u' taken from mechanism m, possibly, for different beliefs about player i type. An interpretation is that $U_i m$ is the set of payoffs resistant to such a veto threat by player i, who may reject current allocation and force mechanism m instead.

Lemma 2. *For each player i, each compact set of mechanisms A,*

$$MM_i(A) \subseteq con\left(I_i\left(\bigcup_{m \in A} m\right) \cap \bigcap_{m \in A} U_i m\right).$$

The Lemma shows that each payoff in the menu of mechanisms A (a) can be obtained in an information revelation game where the continuation payoff belongs to one of the mechanisms and (b) it must also belong to the upper contour of any mechanism in A.

Proof. Let S be the space of cheap-talk announcements. Suppose that $u \in MM_i(A)(p)$. Assume first that (u,p) is equilibrium tuple. Then, there exists $\alpha \in \Delta(A \times S)$ and measurable mappings $u, q : A \times S \to \bigcup_{m \in A} m$ such that $u(m,s) \in m(q(m,s))$ for each $m \in A$ and $s \in S$ and such that

$$u = \int u(m,s)\, d\alpha(m,s) \text{ and } u(t) \geq u(t|m,s) \text{ for all } m, s \text{ and } p_i\text{-almost all } t_i.$$

This implies that $u \in I_i\left(\bigcup_{m \in A} m\right)$ and that $u \in U_i m$ for each $m \in A$.

If (u,p) is an equilibrium tuple with randomization device, then the above claim applies to all tuples in the support of randomization. Hence, u is a convex combination of elements from sets $I_i\left(\bigcup_{m \in A} m\right)$ and $U_i m$ for each $m \in A$. □

5.2. **Offers one cannot refuse.** A proposer contemplating making an offer will have two concerns: what will the other side think about her after seeing her offer, and how the offer will be treated. I deal with the second concern first.

Because delay is inefficient, other things equal it would be beneficial to make offers that will not be refused. In order to design such offers, consider a mechanism approval game where player i chooses between mechanism m ("approval") and mechanism n ("rejection"). Ignoring public randomization and cheap-talk, any decision by player i can be represented by continuation $u^A \in m\left(p^A\right)$ and $u^R \in n\left(p^R\right)$. If rejection occurs in equilibrium with a positive probability, it must be that each p^R-positive probability type receives (weakly) higher payoff after rejection than after acceptance, $u^R \geq_{p^R} u^A$. The goal of the next definition is to exclude such a possibility:

Definition 1. Mechanism m is *an offer that player i (resp., strictly) cannot refuse given mechanism n*, if for each p_{-i}, each p_i, q_i, each $u \in m(p_i, p_{-i})$, each $v \in n(q_i, p_{-i})$, there is a type t_i such that $q_i(t_i) > 0$ and $u_i(t_i) \geq v_i(t_i)$ (resp., $u_i(t_i) > v_i(t_i)$).

Lemma 3. *Suppose mechanism m is an offer that player i strictly cannot refuse given mechanism n. Then, for each p,*

$$MM_i(\{m,n\})(p) \subseteq con(I_i(m))(p).$$

Proof. By Lemma 2, it is enough to show that $I_i(m \cup n)(p) \cap U_i m(p) \subseteq I_i m(p)$. Take $u \in I_i(m \cup n)(p) \cap U_i m(p)$. Let $\alpha \in \Delta(\{m,n\} \times S)$ be an equilibrium strategy and $u^{x,s}, q_i^{x,s}$ payoffs and beliefs $u^{x,s} \in x(q_i^{x,s}, p_{-i})$ for $x = m, n$ supporting u as an equilibrium of the revelation game $I_i(m \cup n)$,

Suppose that $u \notin I_i(m)(p)$. Then, $\alpha(n) > 0$ and $u(t) = u^{n,s}(t)$ for a α-positive probability signals s and $q_i^{n,s}$-all types t. It follows that $u \leq_{q_i^{n,s}} u^{n,s}$. There exists signal s such that $\mathrm{supp} q_i^{n,s} \subseteq \mathrm{supp} p_i$ (in fact, the support condition must be satisfied for probability 1 signals). Because $u \in U_i m(p)$, there exists $q_i' \in \Delta T_i$ and $u' \in m(q_i', p_{-i})$ such that $u' \leq_{p_i} u$. Together with the choice of s, I obtain that $u' \leq_{q_i^{n,s}} u$, and, by transitivity, $u' \leq_{q_i^{n,s}} u^{n,s}$. But the latter contradicts the assumption about m.Hence, $u \in I_i(m)(p)$. □

The information revelation game on the right-hand side appears due to inclusion of cheap-talk in the definition of menu of mechanisms game. Its effect disappears for mechanisms that *are closed with respect to revelation of information by player i:* $I_i(m) = m$. The convexification is due to public randomization device observed before player i make a choice. The effect of randomization disappears, and the bound becomes tighter, when $I_i(m)$ is a payoff function (i.e., single-valued correspondence).

Lemma 4. *Suppose that A is a finite set of offers that player $-i$ cannot refuse given n. Then,*

$$IP(i,n) \subseteq con\left(\bigcap_{a\in A} U_i\left(con\left(I_{-i}(a)\right)\right)\right).$$

Proof. For each $a \in A$, each $\varepsilon > 0$, find a sequence of payoff functions $m^{a,\varepsilon,n} \rightarrowtail a +_i \varepsilon\ m^{a,\varepsilon,n} a +_{-i} \varepsilon$. For sufficiently high $n(a,\varepsilon)$, it must be that $m^{a,\varepsilon,n(a,\varepsilon)}$ is an offer that player i strictly cannot refuse given mechanism n and it is at least ε-distant from a. (Otherwise, one obtains a contradiction with the definition of convergence and upper hemi-continuity of a.) Let $B^{\varepsilon} = \left\{m^{a,\varepsilon,n(a,\varepsilon)} : a \in A\right\}$. Because each mechanism $m \in B^{\varepsilon}$ is a payoff function, con$m = m$. By the definition of the informed principal mechanism and Lemma 2,

$$IP(i,n) \subseteq \mathrm{con}\left(\bigcap_{b\in B^{\varepsilon}} U_i b\right) \rightarrowtail \mathrm{con}\left(\bigcap_{b\in A} U_i b\right).$$

□

5.3. **Random monopoly payoffs.** This section presents the proof of Theorem 2. The first result constructs a class of helpful mechanisms. For each player i and each x, define mechanism

$$A_{i,x}(p) = \left\{u \in \mathcal{U}(p) : u_i \geq x M_i(.|p_{-i})\right\}.$$

Mechanism $A_{i,x}$ consists of all payoff vectors where player i receives at least her x-random monopoly payoffs.

Lemma 5. *For each player i, each x, there exists a family of mechanisms $\left\{a^i(x) : x \in [0,1]\right\}$ with the following properties: for each x*

1. *$a^i(x) \subseteq A_{i,x}$,*
2. *$a^i(x)$ is an offer player $-i$ cannot refuse given $A_{i,x}$,*
3. *$a^i(x)$ is closed with respect to revelation of information for player i: $I_i\left(a^i(x)\right) = a^i(x)$.*

Additionally, for each type t_1 of player 1, payoffs $a_1^2(t_1; x; p_1, p_2)$ of this type under mechanism $a^2(x)$ are convex in p_1.

The next result is the key step of the proof and it applies to any class of mechanisms $a(.)$ with the above properties.

Lemma 6. *Suppose that for some $x < \beta_i$, $\mathtt{B}^\delta \subseteq A_{i,x}$ and that $a(.)$ is a class of mechanisms with the properties from Lemma 5. Then, for each $y < 1 - \delta(1-x) =: x^{(\delta)}$,*

$$MM_{-i}\left(a(y), \delta\mathtt{B}^\delta\right) \subseteq a(y).$$

Proof. Notice first that $\delta\mathtt{B}^\delta$ is weakly $(-i)$-dominated by $A_{i,x^{(\delta)}}$. Indeed, let m be a mechanism, where agent i is given the monopoly power: she or he owns the good and can sell it at a price chosen by them. For each $u \in \mathtt{B}^\delta(p)$, construct a vector $u' = (1-\delta)m(p) + \delta u$. Then, $u' \in A_{i,x^{(\delta)}}$ and $u'_{-i} \geq \delta u_{-i}$.

Next, notice that for each $y < x^{(\delta)}$, $a^i(y) > a^i(x)$[6], which implies that $a^i(y)$ is an offer player $-i$ cannot refuse given $A_{i,x^{(\delta)}}$, and, as an implication, $a^i(y)$ is an offer player $-i$ cannot refuse given $\delta\mathtt{B}^\delta$. The result follow from Lemma 3. □

Let

$$x_0 = \sup\left\{x \geq 0 : \mathtt{B}^\delta \subseteq A_{i,x}\right\}$$

be the largest bound on x that ensures that player i gets at least her x-random monopoly payoffs in any equilibrium of the bargaining game. Suppose that $x_0 < \beta_i$. Consider a strategy, where player i rejects any offer of player $-i$ and, whenever given an opportunity to propose, she offers $a^i(y)$ for some $y < x_0^{(\delta)}$ and some $\varepsilon > 0$. Because of Lemma 6, if she becomes the proposer for the first time in period t, her payoff is at least equal to

$$yM_i\left(.|q^t_{-i}\right),$$

where $q^t_{-i}$are updated beliefs about player $-i$'s types. The expected payoff is equal to

$$\begin{aligned}
&\beta_i \sum_{t\geq 0} \left(\delta\left(1-\beta_i\right)\right)^t y\, \mathbb{E}\, M_i\left(.|q^t_{-i}\right) \\
\geq &\beta_i \sum_{t\geq 0} \left(\delta\left(1-\beta_i\right)\right)^t \left(1-\delta\left(1-x_0\right)\right) \mathbb{E}\, M_i\left(.|q^t_{-i}\right) - \left(x_0^{(\delta)} - y\right)\max\left(h_1, h_2\right)
\end{aligned}$$

where the inequality follows from the fact that monopoly payoffs are not larger than the largest value in the game and the expectation is over beliefs about player $-i$ types. Because monopoly payoffs are convex in the beliefs, and the Bayes formula implies that $\mathbb{E}\, q^t_{-i} = p_{-i}$, the above is not smaller than

$$\geq x_0 M_i\left(.|p_{-i}\right) + \left(\beta_i - x_0\right)\frac{1-\delta}{1-\delta+\delta\beta_i} M_i\left(.|p_{-i}\right) - \left(x_0^{(\delta)} - y\right)\max\left(h_1, h_2\right).$$

By taking y sufficiently close to $x^{(\delta)}$, player i's strategy ensures that her payoffs strictly larger than $x_0 M_i\left(.|p_{-i}\right)$. The contradiction with the choice of $x_0 < \beta_i$ demonstrates that $x_0 \geq \beta_i$ and concludes the proof of Theorem 2.

[6]More precisely, it is true unless player $-i$ type is $t_{-i} = 0$ and beliefs are $p_i(t_i = 0) = 1$. This case would have to be dealt with separately. For now, assume that $l_i > 0$.

5.4. **The Gap.** In this section, I present the characterization of the gap (4.2). I divide the analysis into two relevant cases:

Case $l_1 < l_2 < h_1 < h_2$. In order to describe the maximum payoffs of player 1 subject to random monopoly payoff constraints, it is necessary to divide the space of belief profiles into few regions. The first division corresponds to changing solutions to the monopoly problem (3.2). Let

$$p_2^* = \frac{l_2 - l_1}{h_2 - l_1}$$

be the threshold belief over type h_2 at which type l_1 is indifferent between two monopoly prices $\tau = l_2$ and $\tau = h_2$. All other types have a single belief-independent monopoly price.

Let

$$p_1^*(x) = \frac{x}{x + (1-x)\frac{h_2 - h_1}{h_2 - l_2}}$$

be a belief threshold such that for, $p_1 \geq p_1^*(x)$, the following allocation is incentive compatible:

$$q_1, \tau_1(t) = \begin{cases} 0, xl_2 & t = l_1 l_2 \\ 0, xh_2 & t = l_1 h_2 \\ 1, -(1-x)h_1 & t = h_1 l_2 \\ 0, xh_2 & t = h_1 h_2 \end{cases}.$$

Let

$$f_i(p_i; x) = \min\left(\frac{p_i}{p_i^*(x)}, \frac{1-p_i}{1-p_i^*(x)}\right) \text{ for each } i, \text{ and } f(p; x) = \prod_i f_i(p_i; x),$$

where we take $p_2^*(x) = p_2^*$. Finally, let

$$\Delta(x) = x(1-x)\frac{h_2 - l_2}{h_2 - l_1}\frac{h_2 - h_1}{h_2 - xl_2 - (1-x)h_1}(l_2 - l_1).$$

Proposition 1. *Suppose $l_1 < l_2 < h_1 < h_2$. The expected (ex ante) payoffs of player 1 in such allocations are equal to*

$$E(p, x) = p_1 \cdot (xM_1(p_2)) + f(p; x)\Delta(x).$$

The above payoffs are attained by mechanism $a^2(x)$ with payoffs described in Table 1. The figure below the Table shows the division of the space of belief profiles into three zones. The right-most column describes the optimality coefficients which make the allocations in mechanism $a^2(x)$ Λ-optimal for a given belief profile.

Case $l_1 < h_1 < l_2 < h_2$. Additionally to the belief threshold p_2^* defined in the previous case, let

$$p_2^{**} = \frac{l_2 - h_1}{h_2 - h_1} < p_2^*$$

	type h_2 payoffs	$q_1(l_1,h_2)$	$q_1(h_1,h_2)$	$\Lambda_1(h_1)$
	type l_2 payoffs	$q_1(l_1,l_2)$	$q_1(h_1,l_2)$	$\Lambda_2(h_2)$
		type l_1 payoffs	type h_1 payoffs	
A	$(1-x)M_2(h_2)$	$0, xh_2$	$0, xh_2$	
	$(1-x)M_2(l_2)$	$x-(1-x)\frac{p_1}{1-p_1}\frac{h_2-h_1}{h_2-l_2}$, $(1-x)\frac{p_1}{1-p_1}\frac{h_2-h_1}{h_2-l_2}l_1$	1, $-(1-x)h_1+(1-x)\frac{h_2-h_1}{h_2-l_2}(l_2-l_1)$	p_1 $p_2-(1-p_2)\frac{p_2^*}{1-p_2^*}$
		$xM_1(l_1)$	$xM_1(h_1)$ $+(1-x)(1-p_2)\frac{h_2-h_1}{h_2-l_2}(l_2-l_1)$	
B	$(1-x)M_2(h_2)$ $-(1-x)p_1(h_2-h_1)$ $+x(1-p_1)(h_2-l_2)$	$0, xl_2$	0, $xh_2+(1-x)(h_2-h_1)$	p_1
	$(1-x)M_2(l_2)$	$0, xl_2$	$1, -(1-x)h_1$	0
		$xM_1(l_1)$	$xM_1(h_1)$ $+(1-x)p_2(h_2-h_1)$	
C	$(1-x)M_2(h_2)$	$0, xh_2$	$0, xh_2$	p_1
	$(1-x)M_2(l_2)$	$0, xl_2$	$1, -(1-x)h_1$	p_2
		$x((1-p_2)l_2+p_2h_2)$	$xM_1(h_1)$	

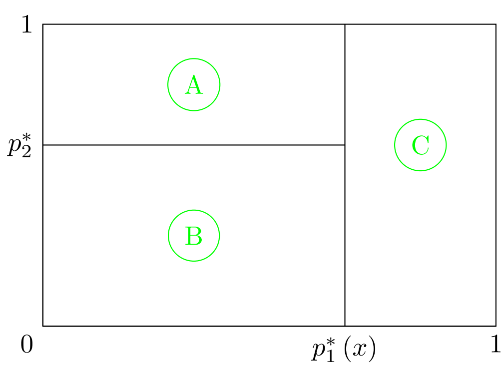

TABLE 1. Payoffs, good allocation, and transfers in mechanism a^2 in case $l_2 < h_1$.

be the probability at which type h_1 is indifferent between between monopoly prices $\tau = l_2$ and $\tau = h_2$. All other types have a single (belief-independent) monopoly price.

Let

$$p_1^*(x) = x.$$

Let $f_1(p_1;x)$ be defined as in the previous case and let

$$f_2(p;x) = \max\left(0, \min\left(\frac{p_i - p_2^{**}}{p_2^* - p_2^{**}}, \frac{1-p_2}{1-p_2^*}\right)\right).$$

Let $f(p;x) = f_1(p_1;x) f_2(p;x)$ be as above. Finally, let

$$\Delta(x) = x(1-x)\frac{(h_2 - l_2)(h_1 - l_1)}{h_2 - l_1}.$$

Proposition 2. *Suppose $l_1 < h_1 < l_2 < h_2$. The expected (ex ante) payoffs of player 1 are equal to*

$$E(p,x) = p_1 \cdot (xM_1(p_2)) + f(p;x)\Delta(x).$$

The gap payoffs are attained by mechanism $a^2(x)$ with payoffs described in Table 2.

5.5. **Proof of Theorem 3.** This section contatins the proof of Theorem 3. To simplify the notation, I write $a^i(p)$, instead of $a^i\left(p|\beta^\delta\right)$ for $i = 1,2$, $f_1(p_1)$ instead of $f_1\left(p|\beta^{(\delta)}\right)$, p_1^* instead of $p_1^*\left(\beta^{(\delta)}\right)$, and Δ instead of $\Delta\left(\beta^{(\delta)}\right)$, etc. The following properties of $a^2(x)$ mechanism will play a role in the proof:

- Mechanisms a^1 and a^2 are closed with respect to revelation of information by player 2.
- For each p_1, payoffs $a_1^2(t_i|p_1,p_2)$ of each type t_1 player 1 are convex in p_2.
- Player 1 payoffs in mechanism a^2 (as described in Tables 1 and 2) are equal to

$$u_1(l_1) = \beta^{(\delta)} M_1(l_1|p_2) + \mathbf{1}_{p_1 < p_1^*}\frac{f_1(p_1)}{p_1} f_2(p_2)\Delta, \tag{5.1}$$

$$u_1(h_1) = \beta^{(\delta)} M_1(h_1|p_2) + \mathbf{1}_{p_1 > p_1^*}\frac{f_1(p_1)}{1-p_1} f_2(p_2)\Delta,$$

  which implies that

$$p_1 \cdot a_1^2(p) = E_1\left(p_1, \beta^{(\delta)}\right) = p_1 \cdot \beta^{(\delta)} M_1(.|p_2) + f_1(p_1) f_2(p_2|x)\Delta(x). \tag{5.2}$$

To quantify the distance between $E_1\left(p_1, \beta^{(\delta)}\right)$ and player 1's expected equilibrium payoffs, let

$$w_0 = \max_{p,u\in \mathtt{B}^\delta(p)} \frac{1}{f_1(p_1)}\left(\frac{\beta}{\beta^{(\delta)}} E_1\left(p_1,\beta^{(\delta)}\right) - p_1 \cdot u_1\right).$$

Then, for each belief profile p and each equilibrium payoff $u \in \mathtt{B}^\delta(p)$, it must be that

$$p_1 \cdot u_1 \geq \frac{\beta}{\beta^{(\delta)}} E_1\left(p_1,\beta^{(\delta)}\right) - w_0 f_1(p_1),$$

and there is a sequence of equilibrium payoffs $u^n \in \mathtt{B}^\delta(p^n)$ such that $p_1^n \cdot u_1^n$ converges to the the right-hand side.

A

$$\begin{array}{r|cc}
(1-x)\,M_2\,(h_2) & 0, xh_2 & 0, xh_2 \\
(1-x)\,M_2\,(l_2) & \frac{x-p_1}{1-p_1}, \left(x - \frac{x-p_1}{1-p_1}\right) l_1 & 1, -(1-x)\,l_1 \\
\hline
 & xM_1\,(l_1) & \begin{array}{l} xM_1\,(h_1) \\ +(1-x)\,(1-p_2)\,(h_1-l_1) \end{array}
\end{array}
\qquad p_1,\ p_2-(1-p_2)\,\frac{p_2^*}{1-p_2^*}$$

B

$$\begin{array}{l|cc}
(1-x)\,M_2\,(h_2)+(x-p_1)\,(h_2-l_2) & 0, xl_2 & 0, xh_2+(1-x)\,(h_2-l_2) \\
(1-x)\,M_2\,(l_2) & 0, xl_2 & 1, -(1-x)\,l_2 \\
\hline
 & xM_1\,(l_1) & \begin{array}{l} xM_1\,(h_1) \\ +(1-x)\,(p_1h_2+(1-p_2)\,h_1-l_2) \end{array}
\end{array}
\qquad p_1,\ p_2-(1-p_2)\,\frac{l_2-h_1}{h_2-l_2}$$

C

$$\begin{array}{r|cc}
(1-x)\,M_2\,(h_2) & 0, xh_2 & 0, xh_2 \\
(1-x)\,M_2\,(l_2) & 0, xh_1 & 1, -x\frac{1-p_1}{p_1}h_1 \\
\hline
 & x\,((1-p_2)\,h_1+p_2h_2) & xM_1\,(h_1)
\end{array}
\qquad p_1,\ 0$$

D

$$\begin{array}{l|cc}
(1-x)\,M_2\,(h_2)+x\,(h_2-l_2) & 0, xl_2 & 0, xl_2 \\
(1-x)\,M_2\,(l_2) & 0, xl_2 & 0, xl_2 \\
\hline
 & xM_1\,(l_1) & xM_1\,(h_1)
\end{array}
\qquad p_1,\ 0$$

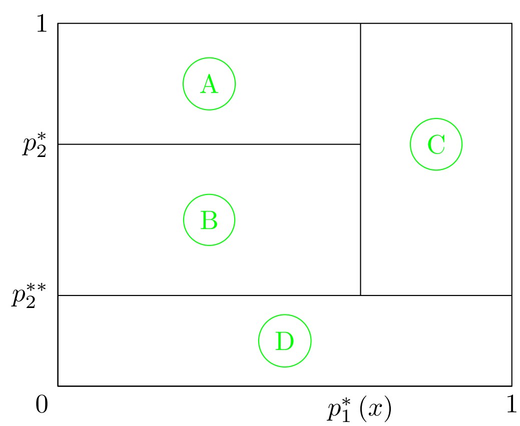

TABLE 2. Payoffs, good allocation, and transfers in mechanism a^2 in case $h_1 < l_2$.

Next, we choose the smallest payment scheme $w\,(p_1)$ such that mechanism $a^w = a^2 +_2 w$ cannot be refused by player 2 given the continuation bargaining game $\delta\mathtt{B}^\delta$. For this purpose, for each p_1, let

$$w\,(p_1) = \min\left\{w : \forall_{p_2, u\in\delta\mathtt{B}^\delta(p_1,p_2)}\exists_{t_2:p_2(t_2)>0}\left(1-\beta^{(\delta)}\right)M_2\,(t_2|p_1)+w > u\,(t_2)\right\}.$$

The above formula and the definition of mechanisms that cannot be refused implies that a^w is such a mechanism. The key step of the proof is the following result proven in the next subsection:

Lemma 7. *There exists $w_0^*(\delta) > 0$ and $\varepsilon(\delta) > 0$ such that $\lim_{\delta\to 1} w_0(\delta) = 0$ and if $w_0 \geq w_0(\delta)$, then, for each p such that $p_1 \in (0,1)$, each $u \in \left(U_1 a^w \cap U_1 a^1\right)(p)$,*

$$p_1 \cdot u_1 \geq p_1 \cdot a_1^2(p) - \frac{\beta^{(\delta)}}{\beta} w_0 f_1(p_1) + \varepsilon(\delta).$$

Corollary 1. *The above inequality holds for each $u \in IP\left(i, \delta \mathtt{B}^\delta\right)(q_1, p_2)$ for each $q_1 \in (0,1)$.*

Proof. The claim follows from the definition of operator U_1 and from Lemma 4. □

Take any equilibrium $v \in \mathtt{B}^\delta(p)$ such that

$$p_1 \cdot v_1 < \frac{\beta}{\beta^{(\delta)}} E_1\left(p_1, \beta^{(\delta)}\right) - w_0 f_1(p_1) + \frac{\beta}{\beta^{(\delta)}} \varepsilon(\delta).$$

Consider a strategy, where player 1 never accepts any offer from player 2, and, instead, awaits the possibility of playing the informed principal with continuation game $\delta\mathtt{B}^\delta$. By the time player 1 becomes a proposer, beleifs may change. First, due to possibly out-of-equilibrium updating, player 2 beliefs may change to q_1. Second, because of player 2 equilibrium revealation of information, player 1 beliefs may get updated to p_2'. For the latter, the martingale property ensures that $\mathbb{E}_{p_2} p_2' = p_2$. The expected payoff from such a strategy is not smaller than

$$\begin{aligned} p_1 \cdot u_1 &\geq \sum_{t\geq 0} \beta\left(\delta(1-\beta)\right)^t \left[\mathbb{E}_{p_2} p_1 \cdot a_1^2(p_1, p_2) - \frac{\beta^{(\delta)}}{\beta} w_0 f_1(p_1) + \varepsilon(\delta)\right] \\ &\geq \frac{\beta}{\beta^{(\delta)}} p_1 \cdot a_1^2(p_1, p_2) - w_0 f_1(p_1) + \frac{\beta}{\beta^{(\delta)}} \varepsilon(\delta) > p_1 \cdot v_1. \end{aligned}$$

But the last inequality contradicts the choice of v as an equilibrium outcome.

5.6. **Proof of Lemma 7.** I start with a preliminary estimate:

Lemma 8. *There exists a constant C_0 that depends only on the payoff parameters of the model and a function $\psi(p)$ st. $\psi(p_1^*) = 0$ and $|\psi(p_1)| \leq C_0(1-\delta) f_1(p_1)$ such that for each p,*

$$w(p_1) \leq \delta w_0 f_1(p_1) + \psi(p_1).$$

Proof. Take any $u \in \delta\mathtt{B}^\delta(p)$. Then, $u' = \frac{1}{\delta} u \in \mathtt{B}^\delta(p)$. Function

$$\psi(p) = \delta p_1 \cdot a_1^2(p|\beta) - \delta \frac{\beta}{\beta^{(\delta)}} p_1 \cdot a_1^2(p)$$

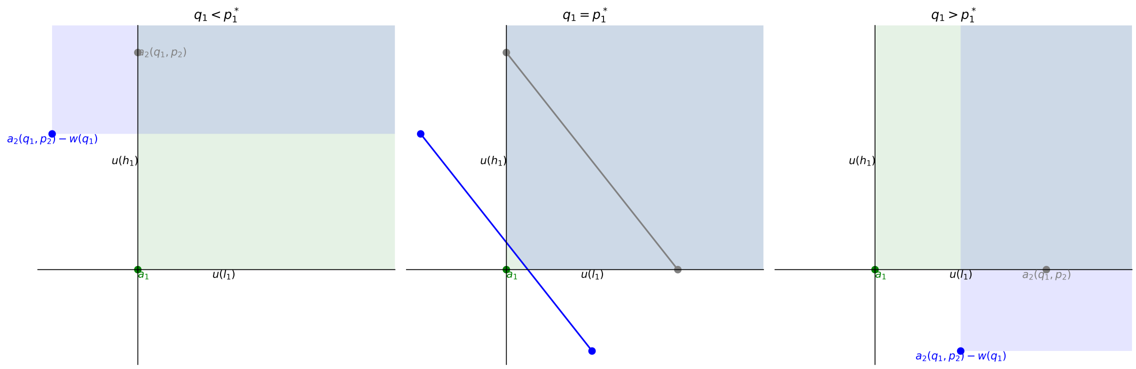

FIGURE 5.1. Mechanisms a^1(green), a^2 (grey), and $a^w - w$ (blue). For each q_1, the green shaded area describes payoffs u_1 that 1-dominate $u' \in a^1(q_1, p_2)$ and the blue shaded area contains payoffs that dominate $u' \in a^w(q_1, p_2)$.

has the required properties (it can be verified through direct calculations based on formulas from Tables 1 and 2.) The definition of w_0 implies that

$$\begin{aligned} p_1 \cdot u_1' &\geq p_1 \cdot \frac{\beta}{\beta^{(\delta)}} a_1^2(p) - w_0(p_1) f_1(p_1) \\ &\geq p_1 \cdot a_1^2(p|\beta) - \left[w_0(p_1) f_1(p_1) + \frac{1}{\delta}\psi(p) \right], \end{aligned}$$

Because $a^2(p|\beta)$ is a solution to the problem of maximizing $p_1 \cdot u_1$ subject to the random monopoly payoff constraint for player 2, there exists t_2 such that

$$u_2'(t_2) \leq (1-\beta) M_2(t_2|p_1) + w_0(p_1) f_1(p_1) + \frac{1}{\delta}\psi(p).$$

Due to $1 - \beta^{(\delta)} = \delta(1-\beta)$, the above implies that

$$\begin{aligned} u_2(t_2) &= \delta u_2'(t) \\ &\leq \left(1 - \beta^{(\delta)}\right) M_2(t_2|p_1) + \delta w_0(p_1) f_1(p_1) + \psi(p). \end{aligned}$$

Because u and p were arbitrary, the last inequality and the definition of payment scheme $w_1(.)$ implies that, for each p_1, $w_1(p_1) < \delta w_0 f_1\left(p_1|\beta^{(\delta)}\right) + \psi(p)$.

I move to the proof of Lemma 7. Take $u \in U_1 a^w(p) \cap U_1 a^1(p)$. Because $u \in U_1 a^1(p)$,

$$u_1(t_1) \geq \beta^{(\delta)} M_1(t_1|p_2). \tag{5.3}$$

Because $u \in U_1 a^w(p)$, there is q_1 and $u' \in a^w(q_1, p_2)$ such that $u_1(t_1) \geq u_1'(t_1)$ for each t_1. Let

$$g_0(p_1, q_1) = \begin{cases} p_1 & q_1 < p_1^* \\ f_1(p_1) & q_1 = p_1^* \\ 1 - p_1 & q_1 > p_1^* \end{cases} \text{ and } g_1(p_1, q_1) = \begin{cases} \frac{p_1}{p_1^*} & q_1 \leq p_1^* \\ \frac{1-p_1}{1-p_1^*} & q_1 \geq p_1^* \end{cases}.$$

I will show that □

$$p_1 \cdot u_1 \geq p_1 \cdot \beta^{(\delta)} M_1(.|p_2) - g_0(p_1, q_1) w(q_1) + g_1(p_1, q_1) f_2(p_2) \Delta. \tag{5.4}$$

Indeed, if $q_1 \neq p_1^*$, the definition of mechanism a^w as well as the characterization of mechanism a^2 in equations (5.1) imply that

$$u_1(h_1) \geq \mathbf{1}_{q_1 < p_1^*} \frac{f_1(q_1)}{q_1} f_2(p_2) \Delta + \beta^{(\delta)} M_1(h_1|p_2) - w(q_1),$$
$$u_1(l_1) \geq \mathbf{1}_{q_1 > p_1^*} \frac{f_1(q_1)}{1 - q_1} f_2(p_2) \Delta + \beta^{(\delta)} M_1(l_1|p_2) - w(q_1).$$

Together with (5.3), the above inequalities imply (5.4) for $q_1 \neq p_1^*$.

If $q_1 = p_1^*$, $u_1' \in \text{con}\left\{\lim_{p_1 \nearrow p_1^*} a_1^w(p), \lim_{p_1 \searrow p_1^*} a_1^w(p)\right\} \cap A_{1,\beta^\delta}(p_1^*, p_2)$, which implies that

$$\begin{aligned} p_1 \cdot u_1 \geq & \min_{u_1' \in \text{con}\left\{\lim_{p_1 \nearrow p_1^*} a_1^w(p), \lim_{p_1 \searrow p_1^*} a_1^w(p)\right\} \cap A_{1,\beta^\delta}(p_1^*, p_2)} p_1 \cdot u_1' \\ = & \begin{cases} \frac{p_1}{p_1^*} f_1(p_1^*) f_2(p_2) \Delta + p_1 \cdot \beta^{(\delta)} M_1(h_1|p_2) - \frac{p_1}{p_1^*} w(p_1^*) & p_1 < p_1^* \\ \frac{1-p_1}{1-p_1^*} f_1(p_1^*) f_2(p_2) \Delta + p_1 \cdot \beta^{(\delta)} M_1(l_1|p_2) - \frac{1-p_1}{1-p_1^*} w(p_1^*) & p_1 > p_1^*. \end{cases} \end{aligned}$$

Inequality (5.4) follows.

For future reference, let

$$D = \min\left(\frac{1}{p_1^*}, \frac{1}{1 - p_1^*}\right) \leq \frac{g_1(p_1, q_1)}{g_0(p_1, q_1)},$$

and find $\kappa > 0$ such that $D \frac{1}{1+\kappa} > 1$. Let C_0 be the constant from Lemma 8 and let $w_0^*(\delta) = \frac{C_0}{\kappa}(1 - \delta)$ for some small $\kappa > 0$.

Due to Lemma 8 and because$u_1 \in U_1 a^1(p)$,

$$\begin{aligned} p_1 \cdot u_1 \geq & p_1 \cdot \beta^{(\delta)} M_1(.|p_2) \\ & + \max(0, g_1(p_1, q_1) f_2(p_2) \Delta - g_0(p_1, q_1) f_1(q_1) \delta w_0 - g_0(p_1, q_1) \psi(q_1)). \end{aligned}$$

I estimate

$$\begin{aligned}
&p_1 \cdot u_1 - \left(p_1 \cdot a_1^2 (p) - \frac{\beta^{(\delta)}}{\beta} w_0 f_1 (p_1) \right) \\
\geq &\max \left(0, g_1 (p_1, q_1) f_2 (p_2) \Delta - g_0 (p_1, q_1) f_1 (q_1) \delta w_0 - g_0 (p_1, q_1) \psi (q_1)\right) \\
&+ \frac{\beta^{(\delta)}}{\beta} w_0 f_1 (p_1) .
\end{aligned}$$

Consider three cases:

- The above is larger than 0 for $q_1 = p_1^*$ due to the definitions of the functions, because $\beta^{(\delta)} > \beta > \delta\beta$, and due to $\psi (p_1^*) = 0$.
- Suppose that $q_1 \neq p_1^*$, and

  $$g_1 (p_1, q_1) f_2 (p_2) \Delta \leq g_0 (p_1, q_1) f_1 (q_1) \delta w_0 + g_0 (p_1, q_1) \psi (q_1) ,$$

  Using bounds on function ψ, I get

  $$w_0 \geq \frac{g_1 (p_1, q_1) f_2 (p_2) \Delta}{g_0 (p_1, q_1) f_1 (q_1) \left(\delta + \frac{1}{w_0^*} C (1 - \delta) \right)} \geq \frac{1}{1 + \kappa} D f_2 (p_2) \Delta.$$

  Hence,

  $$\begin{aligned}
  &p_1 \cdot u_1 - \left(p_1 \cdot a_1^2 (p) - \frac{\beta^{(\delta)}}{\beta} w_0 f_1 (p_1) \right) \\
  \geq &f_1 (p_1) \left[\frac{\beta^{(\delta)}}{\beta} w_0 - f_2 (p_2) \Delta \right] \\
  \geq &\left[\frac{1}{1 + \kappa} D - 1 \right] f_1 (p_1) f_2 (p_2) \Delta > 0.
  \end{aligned}$$

- If $q_1 \neq p_1^*$, and

  $$g_1 (p_1, q_1) f_2 (p_2) \Delta \geq g_0 (p_1, q_1) f_1 (q_1) \delta w_0 + g_0 (p_1, q_1) \psi (q_1) ,$$

  then

  $$\begin{aligned}
  &p_1 \cdot u_1 - \left(p_1 \cdot a_1^2 (p) - \frac{\beta^{(\delta)}}{\beta} w_0 f_1 (p_1) \right) \\
  \geq &\left(\frac{\beta^{(\delta)}}{\beta} f_1 (p_1) - \delta g_0 (p_1, q_1) f_1 (q_1) \right) w_0 - g_0 (p_1, q_1) f_1 (q_1) \psi (q_1) \\
  &+ (g_1 (p_1, q_1) - f_1 (p_1)) \frac{g_0 (p_1, q_1)}{g_1 (p_1, q_1)} (f_1 (q_1) \delta w_0 + \psi (q_1)) \\
  \geq &f_1 (p_1) \left(\frac{\beta^{(\delta)}}{\beta} - \frac{1}{D} (1 + \kappa) \right) w_0 > 0.
  \end{aligned}$$

6. Conclusions

TBA

Appendix A. Proof of Theorem 1

Due to the definition of a Michael mechanism, it is enough to show that each payoff function can be approximated by a sequence $E(g_n)$, where g_n are games. Suppose that m is a payoff function.

Take $\varepsilon > 0$. Because m is a continuous function on a compact domain, it is uniformly continuous, and there exists a decreasing function $\phi_0 > 0$ such that if $d_{\Delta T}(p, q) \leq \phi_0$, then $d_U(m(p), m(q)) \leq \frac{1}{6}\varepsilon$.

For each $\varepsilon > 0$, and each p, use Abreu and Matsushima (1992) to find a two-player (no need for a mediator) game g_p such that $d_U(m(p), E(g_p)(p)) < \frac{1}{6}\varepsilon$. (Note that the measurability condition is trivially satisfied in independent private value environment with different preferences of each type over X. In fact, my model belongs to the Special case from Abreu and Matsushima (1992).) Because g_p implements $E(g)(p)$ as a strict equilibrium, for each $\varepsilon > 0$, there is $0 < \eta(p) \leq \phi_0$ such that $d_U(E(g_{p,\varepsilon})(p), E(g_{p,\varepsilon})(q)) \leq \frac{1}{6}\varepsilon$ for each q such that $d_{\Delta T}(q, p) \leq \eta(p, \varepsilon)$.

Let $V_p = \left\{q : d_{\Delta T}(p, q) < \frac{1}{2}\eta(p)\right\}$ be an open ball on ΔT and consider a covering $\mathcal{V} = \{\{q : d_{\Delta T}(p, q) \leq \eta(p)\} : p \in \Delta T\}$ As space ΔT is compact, there is a finite set $P_0 \subseteq \Delta T$ and a subcover $\mathcal{V}^* = \{V_p : p \in P\}$ of ΔT. Let $\eta_0 = \min_{p \in P} \eta(p) > 0$.

For each $q_i \in \Delta T_i$, let $Q_i(q_i, t_i) = q_i(t_i) - \sum_s q_i^2(s)$ be the Brier scoring rule. Let $Q(q, p) = \sum_i \sum_{t_i} p_i(t_i) Q_i(q_i, t_i)$. It is well known that for each p, p is the unique solution to the maximization problem $\max_q Q(q, p)$. I show that there exists a finite set $P_1 \subseteq \Delta T$ such that for each p, each $q' \in \arg\max_{q \in P_1} Q(q, p)$, $d_{\Delta T}(q', p) \leq \frac{1}{2}\eta_0$.

For each $p \in \Delta T$,let $P_0(p) = \{q : p \in V_q\}$. For each $p \in \Delta T$, let $P_1(p) = \arg\max_{q \in P_\varepsilon} Q(q, p)$. Let $P(p) = \bigcup_{q \in P_1(p)} P_0(p)$. Then, for each $q \in P(p)$, $d_{\Delta T}(p, q) \leq \eta(q, \varepsilon)$. By the triangle's inequality

$$\begin{aligned}
& d_U(E(g_{q,\varepsilon})(p), m(p)) \\
\leq & d_U(E(g_{q,\varepsilon})(p), E(g_{q,\varepsilon})(q)) + d_U(E(g_{q,\varepsilon})(q), m(q)) + d_U(m(q), m(p)) \\
\leq & \frac{1}{2}\varepsilon.
\end{aligned}$$

I will construct a finite game g such that for each $p \in \Delta T$,

$$\max_{u \in E(g)(p)} \min_{v \in \bigcup_{q \in P(p)} E(g_{q,\varepsilon})(p)} d_U E(u, v) \leq \frac{\varepsilon}{2}.$$

The above bound implies that $d_U(E(g)(p), m(p)) \leq \varepsilon$ for each p.

Let g_0 be a finite game between the two players, in which type t_i of each player i has an ex post strict best response $a_i(t_i)$, such that $a_i(t_i) \neq a_i(t_i')$ for $t_i \neq t_i'$.[7] Assume that $C \geq 1$ is an upper bound on all payoffs $u \in E(g_r)(p)$, where $r \in P_0 \cup \{0\}$, for any p, $\max_{i,t_i} |u_i(t_i)| \leq C$. Construct game g:

(1) Action sets:
   (a) player i chooses action $a_{0,i} \in T_i$ in game g_0 and, for each q, action $a_{q,i} \in A_q$ in game g_q,
   (b) the mediator chooses action $(p_M, q_M) \in A_M = \{(p,q) : p \in P_1, q \in P_0(p)\}$,
(2) Payoffs:
   (a) with probability $\frac{\varepsilon}{4C}$, player i receives payoffs $g_i(t_i, t_{-i})$, and, with the remaining probability $1 - \frac{\varepsilon}{4C}$, payoff $g_{q_M}(a_{q_M,i}, a_{q_M,-i}) - \frac{\varepsilon}{4}$,
   (b) the mediator receives payoff $\frac{\varepsilon}{8}\sum_i Q_i(p_{M,i}, a_{0,i})$.

The claim follows from straightforward calculations.

Appendix B. Equilibrium and existence in Menu of Mechanisms

I present a formal definition of equilibrium in a menu of mechanisms M. The definition follows the definition from Pęski (2022), but it is adjusted for the two-sided incomplete information case. For clarity, I present the definition in two steps. I start with a notion of equilibrium without randomization, with a focus on the behavior. Next, I add randomization.

Definition 2. A tuple $(u,p) \in U \times \Delta T$ is an *equilibrium tuple* in menu of mechanisms A, if there exists a measurable strategy $\sigma_i : U \to \Delta(\mathcal{E} \times A)$, measurable continuation payoffs $v : \mathcal{E} \times A \to U$, and, if $i = A$, measurable belief function $q_i : M \times A \to \Delta T_i$, such that the following conditions hold:

- payoff consistency:
$$u_i(t_i) = \int v_i(t_i|m,a)\,\sigma_i(d(m,a)|t_i) \text{ for each } t_i \in T_i$$
$$u_{-i}(t_{-i}) = \sum_{t_i} p_i(t_i) \int v_{-i}(t_{-i}|m,a)\,\sigma_i(d(m,a)|t_j) \text{ for each } t_i \in T_i$$
- best response: for each m, a,
$$v_i(t_i|m,a) \leq u_i(t_i) \text{ for each } t_i \in T_i$$
- belief consistency: for each continuous function $f : U \times M \times A \to \mathbb{R}$, we have
$$\sum_{t_i} p_i(t_i) \int f(s,m,a)\,q_i(ds|m,a)\,\sigma_i(d(m,a)|t_i) = \sum_{t_i} p_i(t_i) \int f(t_i,m,a)\,\sigma_i(d(m,a)|t_i),$$

[7]For example, suppose that suppose that, with a probability $\frac{1}{2}$, player i is able to choose between (a) buying the good at price $\frac{l_i+h_i}{2}$ (the payment may, but it does not have to go to player 2), or (b) refusing to buy the good, in which player 2 gets the good for free.

- continuation payoffs: for each m, a, we have

$$v(m, a) \in m(q_i(m, a)).$$

We refer to the tuple (σ, q, v) as a (perfect Bayesian) equilibrium of menu of mechanisms $MM_i(A)$.

A tuple (u, p) is an *equilibrium tuple with randomization device (e.t.r.d.)* in menu A if there is a probability distribution $\gamma \in \Delta U$ such that (u', p) is an equilibrium tuple in menu A for γ-all u' and $u = \int u' d\gamma(u')$.

For each $p \in \Delta T$, let

$$\begin{aligned} MM_i(A)(p) &= \{u : (u, p) \text{ is e.t.r.d. in menu } A\} \\ &= \operatorname{con}\{u : (u, p) \text{ is equilibrium tuple in menu } A\}. \end{aligned} \tag{B.1}$$

The equality in the second line is due to the Choquet Theorem.

The best response condition, together with the payoff consistency condition ensure that all p_i-positive probability types of player i best respond and receive payoffs as in u. The remaining 0-probability types may either receive a lower payoff, or have no well-defined best response. This feature is without loss of generality, as we can always modify the equilibrium object to ensure the maximization for 0-probability types.

The next result establishes regularity properties (including the existence of equilibria) of menus of mechanisms.

Proposition 3. *For each closed set of mechanisms $A \subseteq \mathcal{E}$ is closed, the set of menus of mechanisms $\{MM_i(B) : B \subseteq A$ and B is compact$\}$ is closed as well. In particular, $MM_i(A)$ is a mechanism (hence, non-empty-valued payoff correspondence), and, for any mechanism m, $\{MM_i(\{m, n\}) : n \in \mathcal{E}\}$ is a mechanism.*

The proof is almost identical to an analogous proof in Pęski (2022). The key difference is that Pęski (2022) works with Kakutani mechanisms, for which it is necessary to prove that they can be approximated by a payoff function. Here, this step is not necessary, as the approximation by payoff functions is ensured by the definition of a Michael mechanism.

Appendix C. Interim efficiency

The goal of this section is to characterize optimal payoff vectors under the assumption that each player has two types. Denote

$$\Delta_i = h_i - l_i \text{ for each } i \text{ and } R = l_2 - l_1.$$

For each belief $p = (p_1, p_2)$, denote $p_i(h) = p_i = 1 - p_i(l)$.

An allocation is defined as probabilities $\left(q_i^{xy}\right)_{i,x,y}$ (where q_i^{xy} is interpreted as a conditional probability that player i gets the good conditionally on player i type being x_i and player $-i$ type being y_{-i}) and transfers $\tau_i^l = \tau_i$ and $\tau_i^h = \tau_i + \Delta\tau_i$ such that two condition hold

- feasibility condition

$$q_i^{xy} + q_{-i}^{yx} = 1 \text{ for each } x, y, \text{ and} \tag{C.1}$$

- ex ante budget balance:$\sum_i \tau_i + p_i \Delta\tau_i = 0.$

Let $q_i^x = \sum_y q_i^{xy} p_{-i}\left(y\right).$

The allocation is incentive compatible if $q_i^l \leq q_i^h$ and $\Delta\tau_i \in \left[q_i^l \Delta_i, q_i^h \Delta_i\right]$. Say that $IC_i\left(h\right)$ constraint is binding if $\Delta\tau_i = q_i^l \Delta_i$ and $IC_i\left(l\right)$ constraint is binding if $\Delta\tau_i = q_i^h \Delta_i$.

In the rest of this section, I write Λ_i instead of $\Lambda_i\left(h_i\right)$ and take $\Lambda_i\left(l_i\right) = 1 - \Lambda_i$.

Proposition 4. *Fix p and suppose that $\Lambda_1 = p_1$. For any $\Lambda_2 \in [0,1]$, payoff vector u is Λ-optimal under beliefs p if and only if u is a payoff vector obtained from some incentive compatible allocation $\left(q_i, \tau_i\right)$ such that the following conditions are satisfied:*

(1) Optimal allocation:

$$\begin{aligned}
q_2^{ll} =& 1 - q_1^{ll} = \begin{cases} 1 & \Lambda_2 > p_2 - \left(1 - p_2\right)\frac{l_2 - l_1}{h_2 - l_2} \\ 0 & \Lambda_2 < p_2 - \left(1 - p_2\right)\frac{l_2 - l_1}{h_2 - l_2} \end{cases}, \\
q_2^{lh} =& 1 - q_1^{hl} = \begin{cases} 1 & \Lambda_2 > p_2 - \left(1 - p_2\right)\frac{l_2 - h_1}{h_2 - l_2} \\ 0 & \Lambda_2 < p_2 - \left(1 - p_2\right)\frac{l_2 - l_1}{h_2 - l_2} \end{cases}, \\
q_2^{hl} =& 1 - q_1^{lh} = 1, \\
q_2^{hh} =& 1 - q_1^{hh} = \begin{cases} 1 & \max\left(\frac{1}{p_2}\Lambda_2, 1\right)\Delta_2 > h_1 - l_2 \\ 0 & \max\left(\frac{1}{p_2}\Lambda_2, 1\right)\Delta_2 < h_1 - l_2 \end{cases}.
\end{aligned} \tag{C.2}$$

,

(2) Incentive constraints: For each player i,
   (a) if $\Lambda_i > p_i$, then $IC_i\left(l\right)$ constraint is binding,
   (b) if $\Lambda_i < p_i$, then $IC_i\left(h\right)$ constraint is binding,

Corollary 2. *Fix p weights Λ st. $\Lambda_1 = p_1$. Suppose that $\left(q_i\right)$ is an allocation that satisfies (C.2) and payoff vector u is such that:*

(1) Expected payoffs: $\sum_i \left(1 - p_i\right) u_i\left(l_i\right) + p_i u_i\left(h_i\right) = W\left(q\right) = \sum_{i,t_i,t_{-i}} t_i q_i^{t_i,t_{-i}}$,

(2) Incentives: for each i

$$u_i(h_i) - u_i(l_i) \in \left[q_i^l(h_i - l_i), q_i^h(h_i - l_i)\right],$$
$$u_i(h_i) - u_i(l_i) = q_i^l(h_i - l_i) \text{ if } \Lambda_i < p_i,$$
$$u_i(h_i) - u_i(l_i) = q_i^h(h_i - l_i) \text{ if } \Lambda_i > p_i.$$

Then, payoff vector u is Λ-optimal.

Proof. Given Theorem 4, we need to verify that u can be obtained from incentive compatible allocation (q_i, τ_i) for some transfers τ_i. Define $\tau_i^x = x_i q_i - X_i$. Feasibility conditions are satisfied due to the expected payoff equality. □

C.1. **Proof.** Denote the expected payoffs of each type as

$$\begin{aligned} L_i =& q_i^l l_i - \tau_i^l, \\ H_i =& q_i^h h_i - \tau_i^l - l_i\left(q_i^h - q_i^l\right) - \alpha_i \Delta_i \Delta q_i \\ =& L_i + \Delta_i q_i^h - \alpha_i \Delta_i \Delta q_i, \end{aligned}$$

where we denote $\Delta q_i = q_i^h - q_i^l$.

The budget balance implies that the expected welfare must be equal to the expected utility from allocations:

$$\sum_i L_i + \sum p_i(H_i - L_i) = \sum(1 - p_i) q_i^l l_i + \sum p_i q_i^h h_i.$$

After substitutions and some algebra, we get

$$\begin{aligned} \sum_i L_i =& \sum_i \left[(1 - p_i) q_i^l l_i + p_i q_i^h h_i - p_i \Delta_i q_i^h + p_i \alpha_i \Delta_i \Delta q_i\right] \\ =& \sum_i \left[q_i^l l_i + p_i \Delta q_i (l_i + \alpha_i \Delta_i)\right]. \end{aligned} \tag{C.3}$$

Consider the welfare maximization problems with weights $\Lambda_i \in [0, 1]$:

$$\max \sum_i (1 - \Lambda_i) L_i + \Lambda_i H_i \text{ st. feasibility and IC constraints.}$$

Using the formula (C.3), the objective function can be rewritten as

$$\begin{aligned} & \sum_i (1 - \Lambda_i) L_i + \Lambda_i H_i \\ =& \sum_i \left[q_i^l l_i + p_i \Delta q_i (l_i + \alpha_i \Delta_i)\right] + \sum_i \Lambda_i \left(\Delta_i q_i^l + (1 - \alpha_i) \Delta_i \Delta q_i\right) \\ =& \sum_i [l_i + \Lambda_i \Delta_i] q_i^l + \sum_i (p_i h_i + (1 - \alpha_i)(\Lambda_i - p_i) \Delta_i) \Delta q_i. \end{aligned}$$

Thus, if $\Delta q_i > 0$, then $\alpha_i = 0$ if $\Lambda_i > p_i$ and $\alpha_i = 1$ if $\Lambda_i < p$. Conversely, if $\Delta q_i = 0$, the value α_i does not matter. Further, the above is equal to

$$
\begin{aligned}
=&\sum_i \left[l_i + \Lambda_i \Delta_i\right] q_i^l + \sum_i \left(p_i l_i + \Lambda_i \Delta_i - \alpha_i \left(\Lambda_i - p_i\right) \Delta_i\right) \Delta q_i \\
=&\sum_i \left[\left(1-p_i\right) l_i + \alpha_i \left(\Lambda_i - p_i\right) \Delta_i\right] q_i^l + \sum_i \left[p_i \left(l_i + \alpha_i \Delta_i\right) + \left(1-\alpha_i\right) \Lambda_i \Delta_i\right] q_i^h \\
=&\sum_i \left[\left(1-p_i\right) l_i + \alpha_i \left(\Lambda_i - p_i\right) \Delta_i\right] \left(\left(1-p_{-i}\right) q_i^{ll} + p_{-i} q_i^{lh}\right) + \sum_i \left[p_i \left(l_i + \alpha_i \Delta_i\right) + \left(1-\alpha_i\right) \Lambda_i \Delta_i\right] \left(\left(1-p_{-i}\right) q_i^{hl} + p_{-i} q_i^{hh}\right).
\end{aligned}
$$

Recalling the feasibility conditions, the above is equal to

$$
\begin{aligned}
=&\left[\left(1-p_1\right) l_1 + \alpha_1 \left(\Lambda_1 - p_1\right) \Delta_1\right] \left(1 - \left(1-p_2\right) q_2^{ll} - p_2 q_2^{hl}\right) + \left[p_1 h_1 + \left(1-\alpha_1\right)\left(\Lambda_1 - p_1\right) \Delta_1\right] \left(1 - \left(1-p_2\right) q_2^{lh} - p_2 q_2^{hh}\right) \\
&+ \left[\left(1-p_2\right) l_2 + \alpha_2 \left(\Lambda_2 - p_2\right) \Delta_2\right] \left(\left(1-p_1\right) q_2^{ll} + p_1 q_2^{lh}\right) + \left[p_2 h_2 + \left(1-\alpha_2\right)\left(\Lambda_2 - p_2\right) \Delta_2\right] \left(\left(1-p_1\right) q_2^{hl} + p_1 q_2^{hh}\right) \\
=&l_1 + \Lambda_1 \Delta_1 \\
&+ q_2^{ll} \left(\left(1-p_1\right)\left(1-p_2\right) l_2 + \left(1-p_1\right) \alpha_2 \left(\Lambda_2 - p_2\right) \Delta_2 - \left(1-p_1\right)\left(1-p_2\right) l_1 - \left(1-p_2\right) \alpha_1 \left(\Lambda_1 - p_1\right) \Delta_1\right) \\
&+ q_2^{lh} \left(p_1 \left(1-p_2\right) l_2 + p_1 \alpha_2 \left(\Lambda_2 - p_2\right) \Delta_2 - p_1 \left(1-p_2\right) h_1 + \left(1-p_2\right)\left(1-\alpha_1\right)\left(\Lambda_1 - p_1\right) \Delta_1\right) \\
&+ q_2^{hl} \left(\left(1-p_1\right) p_2 h_2 + \left(1-p_1\right)\left(1-\alpha_2\right)\left(\Lambda_2 - p_2\right) \Delta_2 - \left(1-p_1\right) p_2 l_1 - p_2 \alpha_1 \left(\Lambda_1 - p_1\right) \Delta_1\right) \\
&+ q_2^{hh} \left(p_1 p_2 h_2 + p_1 \left(1-\alpha_2\right)\left(\Lambda_2 - p_2\right) \Delta_2 - p_1 p_2 h_1 - p_2 \left(1-\alpha_1\right)\left(\Lambda_1 - p_1\right) \Delta_1\right)
\end{aligned}
$$

Because $\Lambda_1 = p_1$, the above simplifies to

$$
\begin{aligned}
=&\left(1-p_1\right) l_1 + p_1 h_1 \\
&+ q_2^{ll} \left(l_2 - l_1 + \alpha_2 \frac{\Lambda_2 - p_2}{1-p_2} \Delta_2\right) \left(1-p_1\right)\left(1-p_2\right) \\
&+ q_2^{lh} \left(l_2 - h_1 + \alpha_2 \frac{\Lambda_2 - p_2}{1-p_2} \Delta_2\right) p_1 \left(1-p_2\right) \\
&+ q_2^{hl} \left(h_2 - l_1 + \left(1-\alpha_2\right) \frac{\Lambda_2 - p_2}{p_2} \Delta_2\right) \left(1-p_1\right) p_2 \\
&+ q_2^{hh} \left(h_2 - h_1 + \left(1-\alpha_2\right) \frac{\Lambda_2 - p_2}{p_2} \Delta_2\right) p_1 p_2.
\end{aligned}
\tag{C.4}
$$

The result follows from direct calculations. For example, notice that the coefficient multiplying q_2^{hl} is always positive, which implies that $q_2^{hl} = 1$ in Λ-optimal allocation.

C.2. **Proof of Proposition 1.** Proposition 1 follows from the following result:

Proposition 5. *Suppose $l_1 < l_2 < h_1 < h_2$. Table 1 shows the good allocation, optimality weights, and player 2 payoffs for all incentive compatible allocations that induce a solution of the maximization* *problem (4.1). The expected (ex ante) payoffs of player 1 in such allocations are equal to*

$$E(p,x) = p_1 \cdot (xM_1(p_2)) + f(p;x)\,\Delta(x)\,.$$

First, notice that, in each of the case listed in Table 1, the optimality of allocations for specified weights follows from Proposition 4. Second, notice that, in each case, if player 2's payoffs $u_2(t_2)$ satisfy (**??**), then

$$W(q) - \sum_{t_2} p_2(t_2)\,u_2(t_2) = \sum_{t_1} p_1(t_1)\,x_1 M_1(t_1;p_2) + f(p;x)\,\Delta(x)\,.$$

(I omit tedious but straightforward calculations.)

Third, in each case, I am going to find a payoff vector u such that (a) it satisfies the conditions of Corollary 2, (b) the payoffs of each type of player 2 are equal to (**??**), and (c) u satisfies the random monopoly payoff constraints of problem (4.1) for player 1 .

- Case A: Let u be such that (b) holds and

$$u_1(l_1) = xM_1(l_1;p_2) \text{ and } u_1(h_1) = xM_1(h_1;p_2) + \frac{1}{p_1} f(p;x)\,\Delta(x)\,.$$

The incentive conditions for player 1 are satisfied because $x(1-p_2) \leq q_1^l \leq q_1^h \leq 1-p_2$, and

$$\begin{aligned} u_1(h_1) - u_1(l_1) &= x\left(M_1(h_1;p_2) - M_1(l_1;p_2)\right) + \frac{1}{p_1} f(p;x)\,\Delta(x) \\ &= x(1-p_2)(h_1-l_1) + (1-x)(1-p_2)\frac{h_2-h_1}{h_2-l_2}(l_2-l_1) \\ &\in \left[x(1-p_2)(h_1-l_1),\,(1-p_2)(h_1-l_1)\right], \end{aligned}$$

where the upper bound is due to $\frac{h_2-h_1}{h_2-l_2}(l_2-l_1) < h_1 - l_1$. Similarly, for player 2,

$$u_2(h_2) - u_2(l_2) = x\left(M_2(h_2;p_2) - M_2(l_2;p_2)\right) +$$

$$p_2^* = \frac{l_2 - l_1}{h_2 - l_1}$$

Finally, notice that property (b) implies that either the random monopoly payoff constraints for this type are binding or $\Lambda_2(t_2) = 0$. Together with (c), it further implies that u is feasible for problem (4.1). Finally, property (a) means that u satisfies the first -order conditions with Lagrangian multiplier on the player 2 payoff constraints equal to Λ_2. This concludes the proof of the Proposition.

C.3. **Proof of Proposition 2.** Proposition 2 follows from the following result:

Proposition 6. *Suppose $l_1 < h_1 < l_2 < h_2$. Table 4 shows the good allocation and optimality weights for all incentive compatible allocations that induce a solution of the maximization problem (4.1). Additionally, the expected (ex ante) payoffs of player 1 are equal to*

$$E(p,x) = p_1 \cdot (xM_1(p_2)) + f(p;x)\,\Delta(x)\,, \text{ and}$$

for each type t_2 of player, their payoff is equal to $u_2(t_2) =$

$$\begin{cases} x_2 M_2(h_2; p) + x(h_2 - l_2) & t_2 = h_2, p_2 < p_2^{**} \\ x_2 M_2(h_2; p) + (x - p_1)(h_2 - l_2) & t_2 = h_2, p_1 < x, p_2 \geq p_2^{**} \\ x_2 M_2(t_2; p) & \textit{otherwise.} \end{cases} \tag{C.5}$$

The proof follows the same lines as the proof of Proposition 1.

Appendix D. Mechanisms

In This Appendix, I describe mechanism(s) a^1 in two different parameter cases. Mechanism a^2 is described in the main text.

D.1. **Description.**

Case $l_2 < h_1$. Define

$$p_1^*(x) = \frac{x}{x + (1-x)\frac{h_2 - h_1}{h_2 - l_2}} \text{ and } p_1^{**}(x) = \frac{x}{x + (1-x)\frac{h_2 - h_1}{h_2 - l_1}}$$

Case $h_1 < l_2$. Let

$$p_1^*(x) = x \text{ and } p_1^{**}(x) = \frac{x}{x + (1-x)\frac{h_2 - h_1}{h_2 - l_1}}.$$

D.2. **Proof of Lemma 5.** Consider mechanisms described in Appendix D. Condition (1) is trivially satisfied. In order to verify condition (3), notice that for any i, any p_i, any $p_{-i} < q_{-i}$, either $a^i(p_i, p_{-i}|x) = a^i(p_i, q_{-i}|x)$, or the payoffs of types of player $-i$ are such that

$$a_{-i}^i(l_{-i}|p_i, p_{-i}, x) < a_{-i}^i(l_{-i}|p_i, q_{-i}, x),$$
$$a_{-i}^i(h_i|p_i, p_{-i}, x) > a_{-i}^i(h_i|p_i, q_{-i}, x).$$

In the former case, information revelation does not add new equilibria. In the latter case, the inequalities insure that there is no equilibrium of the information revelation game in which posteriors $p_{-i} < q_{-i}$ are attained with a strictly positive probability.

Define

$$A_{i,x}^*(p) = \left\{u_{-i} : \exists u' \in A_{i,x}(p) \text{ st. } u_{-i} < u'_{-i}\right\}, \text{ and}$$
$$E_{-i}(\Lambda_{-i}, p) = \max_{u \in A_{i,x}^*(p)} \Lambda_{-i} \cdot (u - x_{-i} M_{-i}(.; p_i)). \tag{D.1}$$

Set $A_{i,x}^*$ consists of all player $-i$'s payoffs that are strictly worse (type by type) than player $-i$ payoffs that allow player i to receive her random monopoly payoffs.

Zone	Payoffs, good allocation, and transfers	Beliefs
A	$\begin{array}{c\|cc} (1-x)M_2(h_2) & 0, xh_2 & 0, xh_2 \\ (1-x)M_2(l_2) + (1-x)p_1\frac{h_2-h_1}{h_2-l_1}(l_2-l_1) & x-(1-x)\frac{p_1}{1-p_1}\frac{h_2-h_1}{h_2-l_1},\ (1-x)\frac{p_1}{1-p_1}\frac{h_2-h_1}{h_2-l_1}l_1 & 1, -(1-x)h_1 \\ \hline & xM_1(l_1) & xM_1(h_1) \end{array}$	p_1 $p_2-(1-p_2)\frac{p_2^*}{1-p_2^*}$
B	$\begin{array}{c\|cc} (1-x)M_2(h_2) + x(1-p_1)(h_2-l_2) & 0, xl_2 & 0, xh_2 \\ (1-x)M_2(l_2) & 0, xl_2 & 1, -(1-x)h_1 \\ \hline & xM_1(l_1) & xM_1(h_1) \end{array}$	p_1 p_2
C	$\begin{array}{c\|cc} (1-x)M_2(h_2) & 0, xh_2 & 0, xh_2 \\ (1-x)M_2(l_2) + x(1-p_1)(l_2-l_1) & 0, xl_1 & 1, -(1-x)h_1 \\ \hline & xM_1(l_1) & xM_1(h_1) \end{array}$	p_1 p_2

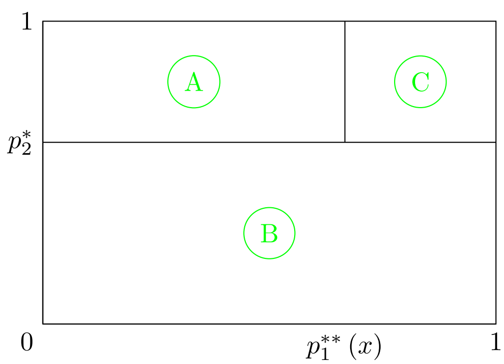

TABLE 3. Payoffs, good allocation, and transfers in mechanism a^1 in case $h_1 > l_2$.

Condition (2) is verified on a case by case basis. Below, let X denote the set of beliefs corresponding to zone X. For any pair of belief profiles $p = (p_i, p_{-i}) \in X$, and $p' = \left(p_i, p'_{-i}\right) \in X'$, I find an allocation $q : T \to [0,1]$, transfers $\tau : T \to \mathbb{R}$, and optimality weights Λ such that

$$u'_i(t_i) = \sum p'_{-i}(t_{-i})\left(q(t_i, t_{-i})\, t_i + \tau(t_i, t_{-i})\right) \le x_i M_i\left(t_i \middle| p'_{-i}\right) \text{ for each } t_i \text{ st. } \Lambda_i(t_i) > 0, \text{ and}$$

$$u'_{-i}(t_{-i}) = \sum p_i(t_i)\left(\left(1 - q(t_i, t_{-i})\right) t_{-i} - \tau(t_i, t_{-i})\right) \le a_{-i}(t_{-i}|x, p) \text{ for each } t_{-i} \text{ st. } \Lambda_{-i}(t_{-i}) > 0,$$

and such that the resulting payoffs $u' = (u'_i)$ are Λ-optimal under beliefs p'. The allocation q and transfers τ are often taken as the allocation and transfers implementing payoff vector $a_{-i}(x,p)$,\, ie. zone X allocation and transfers, which are described in tables in Section D. In such a case, the second

A

$$\begin{array}{l|cc}
(1-x)\,M_2\,(h_2) & 0, xh_2 & 0, xh_2 \\
(1-x)\,M_2\,(l_2) + (1-x)\,p_1 \frac{(h_2-l_2)(h_1-l_1)}{h_2-l_1} & \begin{array}{c} x-(1-x)\,\frac{p_1}{1-p_1}\frac{h_2-h_1}{h_2-l_1}, \\ (1-x)\,\frac{p_1}{1-p_1}\frac{h_2-h_1}{h_2-l_1} l_1 \end{array} & 1, -(1-x)\,h_1 \\
\hline
 & xM_1\,(l_1) & xM_1\,(h_1)
\end{array}
\qquad
\begin{array}{l} p_1, \\ p_2-(1-p_2)\,\frac{p_2^*}{1-p_2^*} \end{array}$$

B

$$\begin{array}{l|cc}
(1-x)\,M_2\,(h_2) + x\,(1-p_1)\,(h_2-l_2) & 0, xl_2 & 0, xh_2 \\
(1-x)\,M_2\,(l_2) & 0, xl_2 & x, 0 \\
\hline
 & xM_1\,(l_1) & xM_1\,(h_1)
\end{array}
\qquad
\begin{array}{l} p_1, \\ p_2-(1-p_2)\,\frac{p_2^{**}}{1-p_2^{**}} \end{array}$$

C

$$\begin{array}{l|cc}
(1-x)\,M_2\,(h_2) & 0, xh_2 & 0, xh_2 \\
(1-x)\,M_2\,(l_2) + x\,(1-p_1)\,(h_1-l_1) & 0, xl_1 & 1, -(1-x)\,h_1 \\
\hline
 & xM_1\,(l_1) & xM_1\,(h_1)
\end{array}
\qquad
\begin{array}{l} p_1, \\ p_2-(1-p_2)\,\frac{p_2^*}{1-p_2^*} \end{array}$$

D

$$\begin{array}{l|cc}
(1-x)\,M_2\,(h_2) + x\,(h_2-l_2) & 0, xl_2 & 0, xl_2 \\
(1-x)\,M_2\,(l_2) & 0, xl_2 & 0, xl_2 \\
\hline
 & xM_1\,(l_1) & xM_1\,(h_1)
\end{array}
\qquad
\begin{array}{l} p_1, \\ 0 \end{array}$$

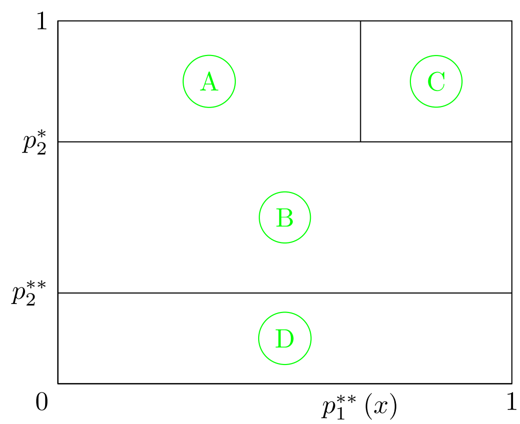

TABLE 4. Payoffs, good allocation, and transfers in mechanism a^1 in case $h_1 < l_2$.

set of inequalities above are satisfied trivially. Notice that cases where $X = X'$, i.e., the two belief profiles belong to the same zone, can be ignored due to the fact that payoffs in mechanism $a\,(x)$ are interim efficient and payoffs a_{-i} do not depend on p_{-i} within a zone.

Let $u = a_{-i}\,(t_{-i}|x, p)$. Consider the following cases:

- case $l_2 < h_1$, $i = 1$:

- $p \in A, p' \in B$: Let $q = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$[8], $\tau = \begin{bmatrix} \gamma h_2 + (x-\gamma) l_2 & x h_2 \\ \gamma l_1 + (x-\gamma) l_2 & -(1-x) h_1 \end{bmatrix}$, where $\gamma = (1-x) \frac{p_1}{1-p_1} \frac{h_2 - h_1}{h_2 - l_1}$, and $\Lambda_2 = 0$. γ is the probability that player 1 gets the good conditionally on types (l_1, l_2) in mechanism a^1 and it is chosen so that the incentive constraint of type h_2 is binding. (q, τ) are Λ_2-optimal for beliefs p'. Moreover, one verifies that that $u_2'(l_2) = u_2(l_2)$, $u_1'(l_1) \le x l_2$ for each $p_2' \le p_2^{**}$, and $u_1'(h_1) = x M_1(h_1 | p_2')$,
- $p \in C, p' \in B$: Let q and τ be the zone C allocation and transfers, and $\Lambda_2 = p_2'$. (q, τ) are Λ_2-optimal for beliefs $p_2' \le p_2^*$. Moreover, $u_1'(l_1) = x((1-p_2') l_1 + p_2' h_2) \le x l_2$ and $u_1'(h_1) = x M_1(h_1 | p_2')$ for all $p_2' \le p_2^*$,
- $p \in B, p' \in A \cup C$: Let q and τ be the zone B allocation and transfers, and $\Lambda_2 = p_2'$. (q, τ) are Λ_2-optimal for beliefs $p_2' \ge p_2^*$. Moreover, $u_1'(l_1) = x l_2 \le x M_1(l_1 | p_2')$ and $u_1'(h_1) = u_1(h_1)$ for all $p_2' \ge p_2^*$,

- case $l_2 < h_1$, $i = 2$:
  - $p \in A, p' \in C$: Let q and τ be the zone A allocation and transfers for $p_1 = p_1^*(x)$, i.e., $q = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $\tau = \begin{bmatrix} x h_2 & x h_2 \\ x l_1 & -(1-x) h_1 + \gamma \end{bmatrix}$, where $\gamma = x \frac{1 - p_1^*(x)}{p_1^*(x)} (l_2 - l_1)$, and $\Lambda_2 = p_2$. γ is chosen so that the payoff of type t_2 is smaller than his random monopoly payoff $(1-x) M_2(l_2 | p_1')$ for all beliefs $p_1' \ge p_1^*(x)$. (q, τ) are Λ_2-optimal for beliefs $p_1' \ge p_1^*(x)$. Moreover, $u_1 = u_1'$, $u_2'(h_2) = (1-x) h_2 = u_2(h_2)$, and $u_2'(l_2) \le u_2(l_2)$ for $p_1' \ge p_1^*(x)$,
  - $p \in B, p' \in C$: Let q and τ be the zone B allocation and transfers, and $\Lambda_2 = 0$. (q, τ) are Λ_2-optimal for beliefs p'. Moreover, $u_2'(l_2) = (1-x) M_2(l_2 | p_1')$ and $u_2'(h_2) = (1-x) M_2(h_2 | p_1')$ for $p_1' \ge p_1^*(x)$,
  - $p \in C, p' \in A \cup B$: Let q and τ be the zone C allocation and transfers, and $\Lambda_2 = p_2'$. (q, τ) are Λ_2-optimal for beliefs p' for a relaxed problem, where the incentive condition of player 2 type h_2 is ignored. Moroever, $u_2'(t_1) = (1-x) M_2(t_2 | p_1')$ for each t_2,
- case $l_2 > h_1$, $i = 1$:
  - $p \in A, p' \in B \cup D$: Let $q = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$[9], $\tau = \begin{bmatrix} \gamma h_2 + (x-\gamma) l_2 & x h_2 \\ \gamma l_1 + (x-\gamma) l_2 & -(1-x) h_1 \end{bmatrix}$, where $\gamma = (1-x) \frac{p_1}{1-p_1} \frac{h_2 - h_1}{h_2 - l_1}$, and $\Lambda_2 = 0$. γ is the probability that player 1 gets the good conditionally on types (l_1, l_2) in mechanism a^1 and it is chosen so that the incentive

[8] Allocation and transfers are presented in a matrix with cells corresponding to type profiles $\begin{bmatrix} l_1 h_2 & h_1 h_2 \\ l_1 l_2 & h_1 l_2 \end{bmatrix}$ and contents of the cells correspond to, respectively, allocation or transfer to player 1.

[9] Allocation and transfers are presented in a matrix with cells corresponding to type profiles $\begin{bmatrix} l_1 h_2 & h_1 h_2 \\ l_1 l_2 & h_1 l_2 \end{bmatrix}$ and contents of the cells correspond to, respectively, allocation or transfer to player 1.

constraint of type h_2 is binding. (q,τ) are Λ_2-optimal for beliefs p'. Moreover, one verifies that that $u_2'(l_2)=u_2(l_2)$, $u_1'(l_1)\le xl_2$ for each $p_2'\le p_2^{**}$, and $u_1'(h_1)=xM_1(h_1|p_2')$,

- $p\in C, p'\in B\cup D$: Let q and τ be the zone C allocation and transfers, and $\Lambda_2=p_2'$. (q,τ) are Λ_2-optimal for beliefs $p_2'\le p_2^*$. Then, $u_1'(l_1)=x((1-p_2')l_1+p_2'h_2)\le xl_2$ and $u_1'(h_1)=u_1(h_1)\le xM_1(h_1|p_2')$ for all $p_2'\le p_2^*$,
- $p\in B$, $p'\in A\cup C$: Let q and τ be the zone B allocation and transfers, and $\Lambda_2=p_2'-(1-p_2)\frac{p_2^{**}}{1-p_2^{**}}$. (q,τ) are Λ_2-optimal for beliefs $p_2'\ge p_2^*$. Moreover, $u_1'(l_1)=xl_2\le xM_1(l_1|p_2')$ and $u_1'(h_1)=u_1(h_1)$ for all $p_2'\ge p_2^*$,
- $p\in B$, $p'\in D$: Let q and τ be the zone D allocation and transfers, and $\Lambda_2=0$. (q,τ) are Λ_2-optimal for beliefs $p_2'\le p_2^{**}$. Moreover, $u_1'(l_1)=xM_1(l_1|p_2')$ and $u_1'(h_1)=xM_1(l_1|p_2')$ for all $p_2'\ge p_2^*$. Finally, $u_2'(l_2)=u_2(l_2)$,
- $p\in D$, $p'\in A\cup B\cup C$: Let q and τ be the zone D allocation and transfers, and $\Lambda_2=p_1$. (q,τ) are Λ_2-optimal for any beliefs p_2'. Moreover, $u_1'(l_1)=xl_2\le xM_1(l_1|p_2')$ and $u_1'(h_1)=xl_2\le xM_1(h_1|p_2')$ for all $p_2'\ge p_2^*$,

- case $l_2>h_1$, $i=2$:
  - $p\in A, p'\in C$: Let q and τ be the zone A allocation and transfers for $p_1=p_1^*(x)$, i.e., $q=\begin{bmatrix}0&0\\0&1\end{bmatrix}$, $\tau=\begin{bmatrix}xh_2 & xh_2\\ xl_1 & -(1-x)l_1\end{bmatrix}$, and $\Lambda_2=p_2'-(1-p_2')\frac{p_2^*}{1-p_2^*}$. (q,τ) are Λ_2-optimal for beliefs $p_1'\ge p_1^*(x)$. Moreover, $u_1=u_1'$, $u_2'(h_2)=(1-x)h_2=u_2(h_2)$, and $u_2'(l_2)\le u_2(l_2)$ for $p_1'\ge p_1^*(x)$,
  - $p\in B, p'\in C$: Let q and τ be the zone B allocation and transfers, and $\Lambda_2=0$. (q,τ) are Λ_2-optimal for beliefs p'. Moreover, $u_2'(l_2)=(1-x)M_2(l_2|p_1')$ for all beliefsp_1',
  - $p\in C, p'\in A\cup B$: Let q and τ be the zone C allocation and transfers, and $\Lambda_2=p_2'-(1-p_2')\frac{p_2^{**}}{1-p_2^{**}}$. (q,τ) are Λ_2-optimal for beliefs p' for a relaxed problem, where the incentive condition of player 2 type h_2 is ignored. Moreover, $u_2'(t_1)=(1-x)M_2(t_2|p_1')$ for each t_2,

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