LOCAL STABILITY OF STATIONARY EQUILIBRIA
(VERY PRELIMINARY AND VERY INCOMPLETE)

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ABSTRACT. This paper characterizes stable stationary equilibria in large population dynamic games. Each player has a type which changes over time. A player's flow payoff as well as the evolution of her type depends on the distribution of population types and population strategies. A stationary equilibrium is called stable if, after perturbing the equilibrium strategies slightly, revision dynamics converge back to the equilibrium. We derive simple sufficient (and almost necesarry) conditions for stability. These conditions involve eigenvalues of a one-dimensional familiy of matrices. Moreover, in order to check whether an equilibrium is stable, it is enough to consider sine wave perturbations of the equilibrium.

1. INTRODUCTION

A large literature on learning is concerned with the convergence of boundedly rational best-response dynamics. Players receive opportunities to revise their actions and some information about the actions of others. Players ignore the fact that others will update their actions in the future, and best-responde to the current action profile as if it was never to change. This literature typically focuses on learning in normal form games. That is, the game played in each period is the same. In this paper, we seek to develop a theory of learning in dynamic games, where the game that agents face in each period is potentially different and its can depend on the population strategies.

In the specific model analysed in this paper, there is a continuum of agents. Each agent has a finite type space. The type of an agent changes over (continuous) time, and its evolution as well as the agent's payoffs are determined by the strategy of the agent, the strategies and types of the others. Because current actions affect not only current payoffs but also future distribution of types and payoffs, the agents have

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long-run incentives.[1] An equilibrium is called *stationary* if neither the population strategies, nor the distribution of types change over time. We seek to characterize the stable stationary equilibria of our model. An equilibrium is called *stable* if, after perturbing the equilibrium strategies slightly, revision dynamics imply convergence back to the equilibrium. In order to model revision dynamics, we assume that each player stochastically receives rare opportunities to update her strategy. Whenever a player has this opportunity, she forms a *prediction* regarding the population strategies and adjusts her strategy to best respond to the predicted environment.

Let us explain the main conceptual difficulty of extending the theory of learning to dynamic games. The standard assumption in the context of static games is that whenever an agent revises her behavior, she acts under a myopic assumption that the other players actions are never going to be updated in the future.[2] If the game is dynamic, an agent's strategy is a mapping from the set of future periods to actions. Even if the agent ignores the others' opportunities to revise their strategies, she still consider a possibility that the actions played will vary in time. Therefore, one must carefully model how the prediction about population strategies are formed.

We consider two scenarios regarding how players form their predictions. In the first scenario, whenever a player can revise her strategy she perfectly observes the strategies of the others. In other words, a player understands perfectly well how the action of others vary in time. The dynamic generated from these predictions is the straightforward conceptual generalization of the myopic best-response dynamics in normal form games and we refer to as *best-response dynamic*. Assuming that agents observe population log-run strategies is obviously problematic because at the moment, these strategies exist as intentions, yet to be executed. We treat the best-response dynamic as a benchmark. In the second scenario, players observe past actions of the others and forecast their strategies from detected patterns. Players can only detect patterns with certain fixed frequencies, including the zero frequency that corresponds to constant average actions. By varying the detectable frequencies, we can consider

[1]Our model is a continuous time version of dynamic games widely used across the economic theory (examples include **?**, **?**, **?**, and **?**).

[2]For the overview of the theory of learning in static games see see **?**, **?**, or **?**.

different levels of sophistication. For example, if only actions can be observed, the dynamic extends standard model of fictitious play. We refer to such a dynamic as *learning dynamic*.

The main result of this paper is a characterization of stable stationary equilibria of our model when the revision opportunities are rare. We first show that in order to check whether an equilibrium is stable, it is enough to consider sine wave perturbations of the equilibrium. A sine wave perturbation is essentially a repetitive oscillation around the equilibrium strategy. A particular perturbation can be parametrized by the frequency of the oscillation and the "amplitude" vector of actions. It turns out that the best response to such a sine wave perturbation of a certain frequency is well-approximated by another sine wave perturbation of the same frequency. Moreover, the relationship between the perturbation and the (approximate) best response is linear, and hence, the best-response operator can be identified with a matrix. Of course, this matrix depends on the frequency of the oscillation, and hence we need to consider a family of matrices parametrized by all possible frequencies. In the case of best-response dynamics, our main result is that a stationary equilibrium is stable if the real part of all the eigenvalues of each matrix in this family is negative. Conversely, if the real part of an eigenvalue of at least one matrix is strictly positive, the equilibrium is unstable. In the case of learning-dynamics, the real parts of the eigenvalues need to be negative only of those matrices which correspond to a frequency which is detectable by the players.

Our results are limited to the case when the revision opportunities are rare. More precisely, the sufficient conditions for stability may fail if the revision opportunities arrive fast (the necessary conditions remain necessary regardless of the speed of the dynamics). We view slow revision dynamics as more consistent with the spirit of learning and bounded rationality that motivates the literature.

It is instructive to compare our results with analogous results known in the static case. The best response dynamic in the dynamic games is closely related to the continuous time best response dynamic in static games; similarly, the learning dynamic is related to the models of fictitious play The asymptotic stability of a Nash equilibrium depends on the eigenvalues of the Jacobian of the best response function computed at the equilibrium. The main difference here is that the strategies in the dynamic models are much more complicated than in the static case. Instead of one, we need an entire one-dimensional family of matrices to characterize the stability. The role of the frequency parameter ω is inherently dynamic as it describes how the strategies change over time. It is worth to point out that the oscillations of strategies represent a different issue than the cycling of the dynamic. The latter appear if the eigenvalues have imaginary components both in the static (matching pennies, or Rock-Scissors-Paper game) and in the dynamic case (see Section 3) below.

The paper is divided as follows. Section 2 introduces the model and the definitions of the dynamics. Section 4 defines the constants that characterize the stability of an equilibrium. Section 3 discusses an example of a dynamic game with a unique stationary equilibrium that is not stable with respect to the revision dynamics. Sections 5, 6, and 7 characterize the stability of, respectively, the evolution type distribution with respect to the perturbations of the type distribution, and the stability of the stationary equilibria with respect to the two kinds of revision dynamics. Section 8 concludes. The Appendix contains all the proofs.

2. Model

Fix Banach space X. The norm on X is denoted with $\|.\|_X$ or simply $\|.\|$ whenever it does not lead to confusion. Let $\overline{X}$ to denote the Banach space of all continuous functions $\chi : R_+ \to X$ with the "sup" norm, $\|\chi\|_{\bar{X}} = \sup_{t\geq 0} \|\chi_t\|_X$. For each measure space $(C, \mathcal{C}, \mu)$, let $D((C, \mathcal{C}, \mu); X)$ denote the Banach space of (a.e. equivalence classes of) Bochner $\mathcal{C}$-measurable, square-integrable with respect to Lebesgue measure, mappings $w : \Omega \to X$ with the L^2-norm.[3] We also write $D(C; X)$ or $D(X)$ if the measurable structure or the entire measure space is known from the context. We assume that X is immersed in $\overline{X}$, and DX via constant mappings.

2.1. Dynamic game.

Continuous time is indexed with $t \in R$. The players discount future payoffs at constant rate $r > 0$.

[3] Bochner spaces $L^p(T; X)$ for $1 \leq p \leq \infty$ are generalization of standard L^p spaces of (equivalence classes of) measurable real-valued functions $f : T \to R$ to functions that take value in Banach space X. A function is Bochner measurable if it is an a.e. limit of countably-valued measurable functions.

There is a continuum of agents. In each period, each agent plays action $a \in A$, where the set of actions A is a finitely dimensional (Euclidean) space. Additionally, each agent has a private type θ that belongs to a finite set Θ. The types of the players may evolve throughout the game.

In each period, the agents payoffs and the dynamics of types depends on the current average distribution $e \in \Delta(A \times \Theta)$ over the actions and types in the population. The instantaneous payoffs of a player with action a and type θ are equal to

$$\sum_{b,\phi} e(b,\phi)\, g(a,\theta,b,\phi).$$

The type of a player evolves according to a Poisson process and its evolution is independent from the evolution of the types of the other players. The rate at which type θ of a player changes into type $\theta' \neq \theta$ is equal to

$$\sum_{b,\phi} e(b,\phi)\, \gamma(\theta'; a,\theta,b,\phi) \geq 0.$$

It is convenient to define the rate of "out" transitions out of state θ as

$$\gamma(\theta; a,\theta,b,\phi) := -\sum_{\theta' \neq \theta} \gamma(\theta'; a,\theta,b,\phi)$$

for all actions $a, b \in A$ and types θ, ϕ. Define a vector of transition rates

$$\gamma(a,\theta,b,\phi) = [\gamma(\theta'; a,\theta,b,\phi)]_{\theta' \in \Theta}. \tag{2.1}$$

Throughout the paper, we assume that functions g and γ are uniformly bounded, and twice continuously differentiable in (a,b) with uniformly bounded derivatives. [4]

Our model allows for the possibility that some types or their groups keep their population share fixed throughout the game, whereas the shares of other types (or

[4]The dependence of the payoffs and transition rates on the average distribution in the population can be interpreted as an outcome of random matching. The model is not restricted to uniform random matching as some other types of matching probabilities can be captured by appropriate adjustments in functions g and γ. For example, if types θ match with each other twice more often than with other types, this can be modeled by multiplying function $g(.,\theta,.,\theta)$by factor 2.

Additionally, the model and the result can be generalized to non-linear dependence of the payoffs and transitions on the average distribution e. We avoid the generalization in this version of the paper to eliminate additional notation and definitions.

shares within the group) evolve depending on the strategies of the players. In the example from Section 3, there are two classes of players 1 and 2 with the population share fixed at $\frac{1}{2}$. Within each class, there are subtypes $\theta = -1, 1$ with evolving shares. This leads to four-element set of types $\Theta = \{(1,-1), (1,1), (2,-1), (2,1)\}$.

It will be useful to explicitly model the restrictions on the attainable type distributions. Let $\Lambda\Theta \subseteq \Delta\Theta$ be the subset of attainable probability distributions. Let $\Phi(\Theta) \subseteq R^{\Theta}$ be the linear subspace spanned by the vector transition rates $\gamma(a, \theta, b, \phi) \in R^{\Theta}$ for all actions a, b and types θ, ϕ. Then, for each $v, v' \in \Lambda\Theta$, $v - v' \in \Phi(\Theta)$, and $\Phi(\Theta)$ is the set of all possible directions of the evolution of the type distribution. In the first reading, the reader may assume that $\Lambda\Theta$ consists of all probability distributions and $\Phi(\Theta)$ is equal to the set of all vectors $v \in R^{\Theta}$ with coordinates that add up to 0.

2.2. **Strategies.** An agent plans her behavior by choosing a long-run strategy. Because the influence of each individual on the rest of the population is negligent, we assume that a strategy depend only on time and the agent's own type.[5] Let $\mathcal{A} = A^{\Theta}$ be the space of *generalized actions*, i.e., mappings $\alpha : \Theta \to A$ that assign proper actions to states. A *strategy* is a continuous mapping $\sigma : R_+ \to \mathcal{A}$ with the interpretation that $\sigma_t(\theta)$ is an action taken by the agent if her private type is θ. For each strategy σ, each $t > 0$, let $\sigma^{(t)}$ be the t-period continuation strategy defined $\sigma^{(t)}_s = \sigma_{s+t}$. The space of strategies is denoted with $\overline{\mathcal{A}}$.

A heterogeneous population is divided into cohorts $c \in C$. We assume that $(C, \mathcal{C}, \mu)$ is a non-atomic measure space of cohorts. A *strategy profile* is a measurable mapping $w \in D\left(\overline{\mathcal{A}}\right)$ with the interpretation that w_{ct} is the t-period (generalized) action played by the members of cohort c. For each $t > 0$, let $w^{(t)}$ be the continuation strategy

[5] Alternatively, the strategies may depend not only on time, but also on the actions and the types of the other agents. However, if we restrict the attention to the pure strategies, then all the best responses can be implemented with strategies that depend only on time (even if the strategies of the other agents are more complicated). That's not necessarily true when the agents use mixed strategies, but it will be true even with mixed strategies if we assume that the strategies do not depend on the actions or types of any countable set of agents.

profile after t. For each strategy profile w, let $w^E \in \overline{\mathcal{A}}$ be the average strategy defined as $w_t^E = \int w_{ct} d\mu(c)$.

Let $v_0 \in D(\Lambda\Theta)$ be the profile of type distribution among the members of the cohorts in period 0. A strategy profile w and the profile of type distributions v_0 determine the evolution of the type distributions $v(w, v_0) \in D\left(\overline{\Lambda\Theta}\right)$ as a solution to the following equation: $v_0(w, v_0) = v_0$, and for each t, each θ',

$$\begin{aligned} &\frac{d}{dt} v_{ct}(\theta'; w, v_0) \qquad (2.2) \\ &= \int_C \left(\sum_\phi \sum_{\theta \neq \theta'} \gamma(\theta'; w_{ct}(\theta), \theta, w_{st}(\phi), \phi) v_{ct}(\theta; w, v_0) v_{st}(\phi; w, v_0) \right) f(s) ds \\ &\quad - \int_C \left(\sum_\phi \sum_{\theta \neq \theta'} \gamma(\theta; w_{ct}(\theta'), \theta', w_{st}(\phi), \phi) v_{ct}(\theta'; w, v_0) v_{ts}(\phi; w, v_0) \right) f(s) ds. \end{aligned}$$

In other words, the increase of the mass of type θ' in cohort c is equal to the difference between the inflow and the outflow from the other types. Using vector notation (2.1), (2.2) can be written as

$$\frac{d}{dt} v_{ct}(w, v_0) = \int_C \sum_{\phi, \theta} \gamma(w_{ct}(\theta), \theta, w_{st}(\phi), \phi) v_{ct}(\theta; w, v_0) v_{st}(\phi; w, v_0) ds. \qquad (2.3)$$

One shows that the profile of type distribution paths $v(w, v_0)$ is uniquely defined given w and v_0. Moreover, the Markov property holds: for each $s \geq t > 0$,

$$v_{t+s}(w, v_0) = v\left(w^{(t)}, v_t(w, v_0)\right).$$

2.3. **Best responses and equilibrium.** Given a profile of strategies w and initial profile of type distributions v_0, the period t expected payoff of a player with type θ and strategy σ is equal to

$$\begin{aligned} &G_t(\theta, \sigma; w, v_0) \qquad (2.4) \\ &= \int_0^\infty \left(\int_C \sum_\phi v_{c,t+s}(\phi; w, v_0) \exp(-rt) E_{\theta_t = \theta, \sigma, w, v_0} g(\sigma_{t+s}(\theta_{t+s}), \theta_{t+s}, w_{c,t+s}(\phi), \phi) \right) ds, \end{aligned}$$

where the expectation is taken with respect to the distribution over future types induced by strategy σ, profiles of of strategies w and initial profile of type distributions

v_0, and given that initial state is equal to θ. A strategy σ is a *period t best response given w, v_0* if for each state θ,

$$G_t(\theta, \sigma, w, v) = \sup_{\sigma'} G_t(\theta, \sigma', w, v) \equiv V_t(\theta, w, v_0),$$

where $V_t(\theta, w, v)$ is the t-period value function of agent in state θ. A strategy is a *best response* if it is a best response for each period t. Let $V_t(w, v) \in R^{\Theta}$ be the vector of values for each type θ. By standard arguments (that rely on the Bellman's Principle of Optimality), there exists a continuous best response strategy that does not depend on the initial state θ. Moreover, the best responses must satisfy the Bellman equation (the standard proof is omitted):

Lemma 1. *If σ is a best response strategy given the profiles of strategies w and initial distributions v_0, then, for each t and type θ,*

$$\sigma_t(\theta) \in \arg\max_a \int \sum_{\phi} \Big(g(a, \theta, w_{ct}(\phi), \phi) + (V_t(w, v))^T \gamma(a, \theta, w_{ct}(\phi), \phi) \Big) v_{ct}(\phi)\, d\mu(c). \tag{2.5}$$

An *equilibrium* is a profile of strategies w^* and initial type distributions v_0^* such that the strategy of each cohort c is the best response given w^* and v_0^*. An equilibrium (w^*, v_0^*) is *stationary* if there exists a generalized action $\alpha^*$and a type distribution v^* such that $w_{ct}^* = \alpha^*$ and $v_{ct}(w, v_0^*) = v^*$for each cohort c and period t. Thus, in a stationary profile, all agents of the same type choose the same action and a type distribution in the population remains constant through the time. Let $\sigma_t^* = \alpha^*$ be the stationary strategy.

From now on, we assume the best responses are uniquely defined.[6] Let $b(w, v_0)$ be the unique best response given w, v_0. The Markov property implies that

$$b^{(t)}(w, v_0) = b\left(w^{(t)}, v_t(w, v_0)\right).$$

2.4. **Evolution of type distribution.** In a stationary equilibrium, the type distribution in each cohort is equal to v^*. We are interested in the stability of the type distribution with respect to small initial perturbations given that the players follow the equilibrium strategies. We say that *type distribution is stable at the stationary equilibrium* if there exists $\epsilon > 0$ such that for each initial perturbation of the type distribution $v_0 \in D\left(\overline{\Lambda\Theta}\right)$, if $\|v_0 - v^*\| \leq \epsilon$, then

$$\lim_{t\to\infty} \|v_t(\sigma^*, v_0) - v^*\| = 0.$$

The type distribution is *unstable* if there exists $\epsilon > 0$, such that for each $\delta > 0$, there exists $v_0 \in D\left(\overline{\Lambda\Theta}\right)$ and t such that $\|v_t(\sigma^*, v_0) - v^*\| \geq \epsilon$.

We view the stability of the type distribution as a necessary condition for the robustness of the stationary equilibrium.

2.5. **Revision dynamics.** The main result of the paper characterizes the stability of a stationary equilibrium with respect to two types of revision dynamics. The dynamics start with a period 0 perturbation of strategies and type distributions. The initial perturbation modifies the long-run strategies of some, possibly all agents as well as the initial type distribution. We assume the perturbation is close to the stationary equilibrium strategy σ^* and the type distribution v^*. Each agent is aware of her new perturbed strategy as it represents her plan to act in all future periods. In the same time, the perturbed strategy does not have to be a best response against the strategies of the other players. In particular, the initial perturbation does not have to constitute an equilibrium.

[6]More precisely, the analysis of this paper holds if the best responses are unique in some neighborhood of the stationary equilibrium (w^*, v_0^*). Lemma 12 in the Appendix describes the conditions on the fundamenals that guarantee the local uniqueness of the best responses. Essentially, there are two conditions: (a) the hypotheses of Theorem 1 holds (which is sufficient and almost necessary for the type distribution to be stable), and (b) semi-negative definite matrix M^*_{AA} (defined in Section 4) is negative definite.

In each period $t > 0$, agents may receive an opportunity to revise her strategy. The opportunities arrive independently at constant Poisson rate $\lambda > 0$ (that, in particular, does not depend on actions and types of the players). Given a revision opportunity in period t, an agent chooses a best response strategy given the current type distribution and one of two assumptions about the future behavior of the agents. The best response becomes the new strategy of the revising agent.

We present a formal definition of the revision dynamics. We begin with the space of cohorts. To allow for initial heterogeneity, we assume that in period 0 the agents are divided into cohorts $c_0 \in C_0$, where $(C_0, \mathcal{C}_0, \mu_0)$ is a measure space. The initial cohorts are further divided into the groups of agents with the same history of revision opportunities. Let $C = \{(c_0, t_1, t_2, ...) : c_0 \in C_0, \text{ and } 0 < t_1 < t_2 < ...\}$ be the space of cohorts, where c_0 determines the initial strategy and t_i is the period of ith revision opportunity. The measure μ^λ on C (with the Borel σ-algebra) is defined in the following way: c_0 are distributed according to μ_0, and for each $i \geq 1$, the conditional probability density of the waiting time $t_i - t_{i-1}$ for the ith revision opportunity given $c_0, t_1 ..., t_{i-1}$ is equal to $e^{-\lambda(t_i - t_{i-1})}$, where we take $t_0 = 0$.

For each cohort $c = (c_0, t_1, ...)$, each $t \geq 0$, let $d^t(c) = \max(t_i : t_i \leq t)$ be the most recent revision period for cohort c. Let $\mathcal{C}^t$ be the smallest σ-algebra on C such that all functions d^s for $s \leq t$ are measurable. Let $\mu^{\lambda,t}$ be the restriction of μ^λ to σ-algebra $\mathcal{C}^t$.

In each period τ, the state of the dynamics is given by a profile of continuation strategies $w^\tau \in D\left((C, \mathcal{C}^\tau, \mu^\tau), \overline{\mathcal{A}}\right)$ and a profile of τ-period intra-cohorts type distributions $v^\tau \in D\left((C, \mathcal{C}^\tau, \mu^\tau), \overline{\mathcal{A}}\right)$. We interpret $w^\tau_{cs}(\theta)$ is an action that the members of cohort c plan in period τ to be played in period $\tau + s$. The initial perturbation is given by w^0 and v^0.

The type-distributions evolve according to the following equation:

$$\frac{d}{d\tau} v^\tau_c = \int_C \sum_{\phi,\theta} \gamma\left(w^\tau_{c0}(\theta), \theta, w^\tau_{\chi 0}(\phi), \phi\right) v^\tau_c(\theta) v^\tau_\chi(\phi) d\mu^\lambda(\chi). \tag{2.6}$$

Finally, for each τ, the strategy w_c^τ of cohort c is equal to the continuation of the best response strategy chosen in period $d^\tau(c)$:

$$w_c^\tau = b^{(\tau - d^\tau(c))}\left(w^{P,d^\tau(c)}, v^{d^\tau(c)}\right).$$

Here, $w^{P,t}$ is a profile of strategies that represents the period t prediction about the future behavior of the population.

Best response dynamic. We consider two versions of the revision dynamic that differ with respect to the assumptions that players make about future behavior. In the best response dynamic, each agent observes the current state of the population, including the (long-run) strategies and the type distributions of the players, i.e

$$w^{P,t} = w^t.$$

To fix attention, we simply assume that the players strategies are chosen in a publicly visible way. (Because there is continuum of agents, there are no strategic reasons not to disclose their strategies truthfully.) This assumption might not be particularly realistic in many situations. We treat it as a useful benchmark to compare the best response dynamic with the learning dynamic.

The players in the dynamic are myopic in the sense that they do not anticipate the future evolution of the dynamics when they revise their strategies. However, they rationally anticipate the future actions and the type-evolution of the rest of the players under the (incorrect) assumption that their strategies are not going to be revised.

Learning dynamic. In the learning dynamic, an agent observes correctly the current distribution of types. She does not observe the strategies that the other players intend to play in the future. Instead, she observes the past actions of the other agents, she tries to detect patterns. The forecaster can only detect patterns that reoccur with a certain frequency. Specifically, let $\Omega \subseteq R_+$ be the set of detectable frequencies. We assume that $0 \in \Omega$. For each τ and cohort c, choose coefficients $a_{c,\sin}^\tau(\omega)$ and $a_{c,\cos}^\tau(\omega)$ for each $\omega \in \Omega$ so to minimize

$$\int_0^\tau \left(w_{c0}^s - \sum_{\omega\in\Omega} a_{c,\sin}^\tau(\omega)\sin(2\pi\omega s) - \sum_{\omega\in\Omega} a_{c,\cos}^\tau(\omega)\cos(2\pi\omega s)\right)^2 ds. \tag{2.7}$$

In other words, the agents regress past actions on the space of functions spanned by repeatable patterns with frequencies in set Ω. The forecast is defined as the sum of extrapolated patterns observed in the past actions

$$w_{cs}^{P,\tau} = \sum_{\omega\in\Omega} a_{c,\sin}^{\tau}(\omega)\sin(2\pi\omega(s+t)) + \sum_{\omega\in\Omega} a_{c,\cos}^{\tau}(\omega)\cos(2\pi\omega(s+t)).$$

When $\Omega = \{0\}$, then $w_{cs}^{P,\tau}$ is equal to the average action played by the members of cohort c before period. In such a case, the learning dynamics are equivalent to the fictitious play. In general, different sets of detectable frequencies Ω correspond various levels of forecasting sophistication.

Stability. We say that the stationary equilibrium (σ^*, v^*) is *stable with respect to the λ-best response dynamics* (or, *λ-stable*) , if there exists $\epsilon > 0$, such that for any best response path (w^τ, v^τ) such that $\|w_0 - \sigma^*\| \leq \epsilon$ and $\|v_0 - v^*\| \leq \epsilon$,

$$\lim_{\tau\to\infty} \|w^\tau - \sigma^*\| = \lim_{\tau\to\infty} \|v^\tau - v^*\| = 0.$$

We say that the the stationary equilibrium is *unstable with respect to the λ-best response dynamics* (or, *λ-unstable*) if there exists $\eta > 0$ such that for each $\epsilon > 0$, there exists an initial perturbation (w_0, v_0) such that $\|w_0 - \sigma^*\| \leq \epsilon$ and $\|v_0 - v^*\| \leq \epsilon$ and τ so that for the induced best response path (w^τ, v^τ),

$$\text{either } \|w^\tau - \sigma^*\| \geq \eta \text{ or } \|v^\tau - v^*\| \geq \eta.$$

Similarly, we define the stability and instability with respect to the (Ω, λ)-learning dynamics.

3. Example

In this section, we use an example to illustrate the methodology and the main ideas of this paper.

3.1. **Example.**

We describe a dynamic version of matching pennies games. There are two classes of players, 1 and 2, both with equal and constant shares in the population. Each player has one of two types, $k \in \{-1, 1\}$. The time is continuous. The players discount future payoffs at instantaneous discount rate $r > 0$.

In each period t, each player chooses an action from set $A = [-1, 1]$. After the actions are chosen, players 1 and 2 are randomly and uniformly matched in pairs. If player j type $k_1 \in \{1, -1\}$ with action a_1 meets player $-j$ type k_2 and with action a_2, their payoffs are equal to

$$-a_1^2 + k_1 a_2 \text{ for player 1, and}$$
$$-a_2^2 - k_2 a_1 \text{ for player 2.}$$

The class of the player does not change. At each period, the type may change with the Poisson arrival rate equal to $1 - \gamma k_j a_j$, where j is a class of the player and $\gamma > 0$ is a parameter.

In other words, player 1 is rewarded if his type matches the action of player 2 and player 2 is rewarded if her type mismatches the action of player 1. Moreover, each player can increase the chance of being type 1 by choosing higher a. The manipulation is costly, and absent any dynamic considerations, the player would prefer to choose the natural transition rate $a = 0$.

The model has a unique stationary equilibrium $\alpha^* (\theta) = 0$ for each type with stationary distribution is $v (\theta) = \frac{1}{2}$ stationary payoff 0 for all players and all types. This is also the unique efficient outcome among all stationary strategies.

3.2. **Stability.** It is easy to show that the stationary equilibrium is stable with respect to initial perturbations in which the actions are constant over time. The logic is similar to the standard argument in the matching pennies game. First, observe that all players class $i = 1, 2$ have the same strict incentives regardless of their type k_i. Moreover, the players i incentives depend only on the future average actions used by players $-i$. Suppose that after the initial perturbation, the average action of players 2 is positive. Players 1 best respond with positive actions. The new best response behavior of players 1 leads to more positive types of player 1. In turn, this creates incentives for player 2 to choose negative actions, which leads to winding down of the initial perturbation.

A key observation in the above argument is that best responses have opposite form to the original strategies. It turns out that, for certain values of parameters, the best responses to some non-stationary strategies are similar to these strategies. In such a

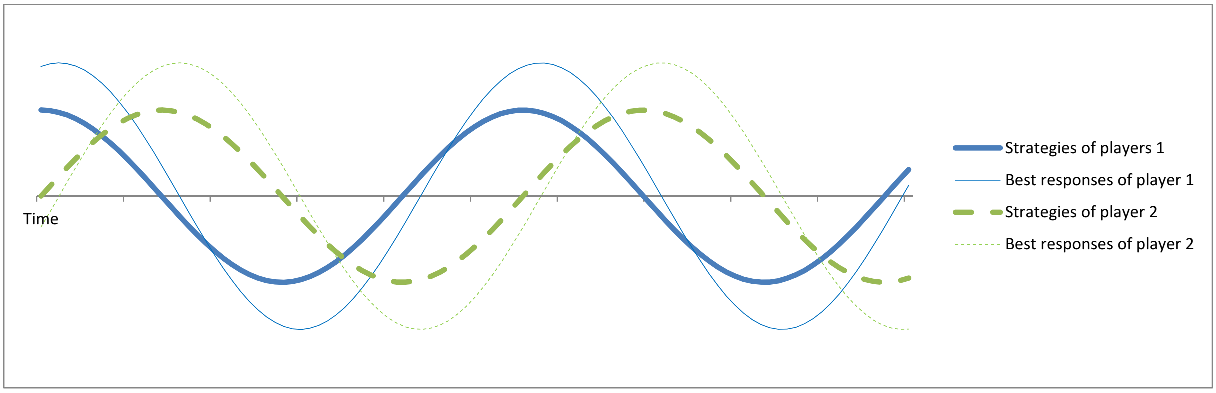

FIGURE 3.1. Initial strategies and the best responses. (The values of parameters are $r = 1, \gamma = 20$, and $\omega = 2$.)

case, the best response reinforces the initial perturbation. In result, the best response dynamic may diverge.

We describe an example of such strategies. Let the strategies of players 1 and 2 be equal to, respectively, $a\cos(2\pi\omega t)$ and $a\sin(2\pi\omega t)$ for some small amplitude $a > 0$ and frequency parameter ω. Figure 3.1 plots the strategies as functions of time.

Consider the best response of players 1 in period $t = 0$. These players expect the strategy of players 2 to oscillate between the positive and negative actions with the period of oscillation equal to $\frac{1}{|\omega|}$. Due to the discounting, they put more weight on the earlier fluctuations. Because the earliest fluctuation makes player 2 choose positive actions, it increases the value of positive type for players 1, which, in turn, creates incentives to choose a strictly positive action. If the cost of the manipulation of the transition rate γ is small enough, the best response positive action may be higher than the action $a\cos(0)$ prescribed by the initial strategy.

When t increases, players 1 anticipated payoff from positive type eventually starts decreasing. In fact, if the discount factor is not too high, players 1 become roughly indifferent between their types when $t \approx \frac{1}{4}\frac{1}{|\omega|}$, or when the actions prescribed by the current strategies are small. As a result, their best response action is also close to 0. For higher t, players 1 anticipate a larger share of negative actions of players 2 in the near future. This increases the value of negative type, which reduces the best response action of player 2. The rest of the best response strategy closely follows the initial strategy of player 1. (As one can notice on 3.1, the phase of the best response is shifted relative to the initial strategy. The shift is due to the discounting and it will disappear when $r \to 0$.)

A similar argument shows that the best responses of player 2 closely resemble the strategies of player 2. Because the strategies of all players have the same period $\frac{1}{\omega}$, the best responses are periodic with the same period length.

the cost of non-zero action is relatively low, and its impact on the transition probabilities high. Any small differences between the expected payoffs from negative and positive types lead to high differences in the best response action. In particular, the amplitude of the best response oscillations can be much higher than the amplitude of the initial strategies. Because the best response strategies resemble the original strategies, one may expect that the best response dynamics lead to ever increasing amplitude of the oscillations, which leads to the divergence of the dynamics.

3.3. **Approximate best response dynamics.** In order to introduce the main ideas of the paper, we are going to discuss the stability of the best response dynamics more carefully. The idea is to derive a linearized approximation to the best response dynamics, describe the stability of the approximate dynamics, and show that the same conditions hold for the original best response dynamics.

As a first step, we derive an approximation to the best response function. Let w be a strategy profile and let v^0 be the initial type distribution. Bellman equations imply that the best response strategy $b(\sigma, v_0)$ must satisfy first-order conditions:

$$2b_t\left(j,k;\sigma,v^0\right) = \gamma\left(V_t\left(j,1;w,v^0\right) - V_t\left(j,-1;w,v^0\right)\right), \tag{3.1}$$

where $V_t(j,k;.)$ is a period t continuation value of players class j and type 1. The above equation has a natural interpretation (compare also with a more general formula (6.3) below). The marginal cost of increasing action b_t is equal to $2b_t$. The marginal benefit is proportional to the difference between the continuation values of types 1 and -1 multiplied by the rate at which an increase in the action affects the rate of transitions from type -1 to 1.

Next, we compute an approximation to the continuation value function. Because an envelope theorem applies in our setting, we can approximate the continuation value of an agent assuming that she uses an equilibrium strategy $\alpha^*(\theta) = 0$ instead of her best response. Thus, the continuation values for players class 1 are approximately equal to

$$V_t\left(1, k; w, v^0\right) \approx \int_t^\infty e^{-r(u-t)} \left(kp_u^{1,k} + (-k)\left(1 - p_u^{1,k}\right)\right)\left(\sum_l \alpha_u^{2,l} w_u^E(2,l)\right) du, \quad (3.2)$$

with an analogous equation for players class 2. Here, $p_u^{j,k}$ is the probability that player j who plays the stationary strategy is going to have type k in period u if in period tu her type is k, $\alpha_u^{j,l}$ is a period u fraction of players with class j and type k, and $w_u^E(j,l) = \int w_{iu}(j,l)\, d\mu(i)$ is the average action played by agents class j and type l in period u. In the neighborhood of stationary equilibrium, $p_u^{j,k} \approx \frac{1}{2} + \frac{1}{2}e^{-2(u-t)}$ and $\alpha_u^{j,k} \approx \frac{1}{2}$. Substituting to (3.2) and then to (3.1), we obtain

$$b_t\left(1, k; w, v^0\right) \approx \gamma \int_t^\infty e^{-(r+2)(u-t)} \left(\frac{1}{2}\sum_l w_u^E(2,l)\right) du. \quad (3.3)$$

with an analogous equation for players 2:

$$b_t\left(2, k; w, v^0\right) \approx \gamma \int_t^\infty e^{-(r+2)(u-t)} \left(\frac{1}{2}\sum_l w_u^E(2,l)\right) du$$

(see a general formula (6.4) below). In other words, the best response action of players class j is equal to the discounted and scaled average of the future actions of players class 2. In particular, note that the approximate best response depends only on the average actions of the other players and not on the details of the profile. Moreover, the approximate best response in period t does not depend on the initial type distribution v^0 and the actions of the other players played before t. (As we explain below, the first feature holds generally. The latter issue is specific to our example.) To shorten the subsequent notation, we write $b(w)$ for the (approximate) best response to strategy profile w, and for future reference, notice that we can rewrite the approximation (3.3) together with an analogous equation for players 2 as

$$b_t(w) \approx \gamma \int_t^\infty e^{-(r+2)(u-t)} A w_u du,$$

where we treat generalized actions $b_t(w)$ and w_u as vectors, and A is a matrix equal to

$$A=\begin{bmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 0 & 0 \end{bmatrix}.$$

We can describe an approximation to the best response dynamics. Indeed, suppose that the average strategy in period τ is equal to $w^{\tau,E}$. In particular, the agents plan to play action $w^{\tau,E}_{t-\tau}$ in period $t\geq\tau$. Between periods τ and $\tau+d\tau$, approximately $\lambda d\tau$ of randomly drawn agents receive a revision opportunity. These agents replace their current strategies by $b\left(w^{\tau,E}\right)$ (we drop the dependence on the initial distribution as it is not important in our example). Because the agents are chosen at random, the heuristic evolution of the average actions that players intend to play in period $t+\tau$ is given by

$$\begin{aligned}\frac{dw^{\tau,E}_{t-\tau}}{d\tau} &\approx \lambda\left(b_{t-\tau}\left(w^{\tau,E}\right)-w^{\tau,E}_{t-\tau}\right) \\ &\approx \lambda\left(\gamma\int_{t-\tau}^{\infty}e^{-(r+2)(u-t+\tau)}Aw^{\tau,E}_{u}du-w^{\tau,E}_{t-\tau}\right)\text{ for each }t\geq\tau.\end{aligned}\tag{3.4}$$

Because the best responses depend only one the average strategies , the approximate dynamics are well-defined by the above equation. Moreover, due to the approximation (3.3), the right-hand side of the evolution equation is linear in the average strategies.where we treat $w^{\tau,E}_{t}$ as a vertical vector, and matrix A is defined as follows

3.4. **Necessary conditions for stability.** Next, we are going to present a necessary conditions for the stability of the dynamics (3.4). The idea is to consider sine-wave strategies. It is convenient to describe the sine-way strategies using complex numbers. Specifically, let $a=(a(j,k))$ be a vector of complex numbers, and let

$$\begin{aligned}\sigma^a_t(j,k) =&\text{Re}\left(a_{j,k}\right)\cos\left(2\pi\omega t\right)+\text{Im}\left(a_{j,k}\right)\sin\left(2\pi\omega t\right).,\\ =&\text{Re}\left(a\left(j,k\right)e^{-i2\pi\omega t}\right)\end{aligned}\tag{3.5}$$

where $i=\sqrt{-1}$ is the imaginary unit, $a_{j,k}\sim 0$ is a (complex or real) amplitude coefficient, ω is a real frequency parameter that is common to all players, and the equality follows from the Euler's formula[7] .

The approximate best response of class 1 players functions given σ are equal to

$$\begin{aligned} b_t\left(1,k;\sigma^a,v^0\right) &\approx \gamma\int_t^\infty e^{-(r+2)(u-t)}\frac{1}{2}\sum_l \operatorname{Re}\left(a_{2,l}e^{-i2\pi\omega u}\right)du \\ &= \operatorname{Re}\left(\gamma\left(\frac{1}{2}\sum_l a_{2,l}\right)\left(\int_0^\infty e^{-(r+2+2\pi i\omega)u}du\right)e^{-2\pi i\omega t}\right) \\ &= \operatorname{Re}\left(\gamma\frac{1}{r+2+2\pi i\omega}\left(\frac{1}{2}\sum_l a_{2,l}\right)e^{-2\pi i\omega t}\right), \end{aligned}$$

with an analogous equation for class 2 players. In particular, the best response function is a sine-wave strategy

$$b\left(\sigma^a,v^0\right)=\sigma^{(K(\omega)+I)[a]},$$

where linear operator $K(\omega)$ is equal to

$$K(\omega)=-I+\gamma\frac{1}{r+2+2\pi i\omega}A. \tag{3.6}$$

Because of the linearity of the dynamics (3.4), if the initial strategy has sine-wave form (3.5), then the entire subsequent dynamics will have the sine-wave form. Moreover, the dynamics have an explicit solution

$$w_t^{\tau,E}=\sigma^{e^{\lambda\tau K(\omega)}\left[a^0\right]}=\left(e^{\lambda\tau K(\omega)}\left[a^0\right]\right)e^{-2\pi i\omega t}, \tag{3.7}$$

where $e^{\lambda\tau K(\omega)}$ is an operator exponential of linear operator $\lambda\tau K(\omega)$ (see Appendix A.3 for details). The long-run convergence properties of the solution (3.7) are well-understood. Specifically, we compute the eigenvalues of operator $K(\omega)$:

$$\psi=\pm\gamma\left(\frac{2\pi\omega+(2+r)\,i}{(2+r)^2+\omega^2}\right)-1.$$

If there exists a frequency ω such that $K(\omega)$ has an eigenvalue with a strictly positive real part, then there exists a corresponding eigenvector a^0 such that (a) (3.7) diverges

[7]Recall that the Euler's formula states that for each real x, $e^{ix}=\cos x+i\sin x$.

away from 0, and (b) the initial perturbation σ^{a^0} leads to a divergent dynamic. This gives us a necessary condition for stability.

It turns out that if the real parts of all eigenvalues are strictly negative, then the dynamic is stable regardless of the form of the initial perturbation. The argument is somehow more complicated and we postpone it to Section 6.1.

4. Constants

In this section, we define all the constants that we use in the characterization of the stability of stationary equilibria. From now on, we fix the stationary equilibrium (α^*, v^*) of the dynamic game. The subsequent definitions and notations are divided into four parts. The first part is devoted to a general terminology on linear operators. The next three parts deal with the constants that are associated with the transition rates, the payoffs, and the local characterization of the best response function.

4.1. **Linear operators.** For any two finitely dimensional vector spaces E and E', let $L(E, E')$ denote the space of linear operators $A : E \to E'$ with the operator norm $\|A\| = \max_{e:\|e\|\leq 1} \|Ae\|$. For example, $L(E, R)$ is the dual space of E. We write $L(E)$ instead of $L(E, E)$. For all operators $A, \in L(E, E')$ and $B \in L(E', E'')$, we write $B \circ A \in L(E, E'')$ to denote the composition of A and B.

It will be convenient to extend the definitions of linear operators to vectors spaces over complex numbers. For each vector space E, let $E^C = E \oplus iE$ denote the *complexification* of E.[8] Let $L^C\left(E^C, E'^C\right)$ be the space of (complex) linear operators between complex vector spaces E^C and E'^C. The standard linear operators between vector spaces E and E' uniquely extend to (complex) linear operators between E^C and E'^C, i.e., $L(E, E') \subseteq L\left(E^C, E'^C\right)$.[9]

For any finitely dimensional space E, we say that operator $A \in L(E)$ is *stable* if each eigenvalue λ of operator A has strictly negative real part, $\Re(\lambda) < 0$. Operator A is *unstable*, if it has an eigenvalue λ with a strictly positive real part, $\Re(\lambda) > 0$.

[8]In other words, $E \oplus iE = \{(e_1, e_2) : e_i \in E\}$ is a vector space with the standard vector addition and multiplication by complex scalar given by $(a + ib) \cdot (e_1, e_2) = (ae_1 - be_2, ae_2 + be_1)$.

[9]For any $A \in L(E, E')$, we define $((a + ib) A)[(e_1, e_2)] = (aAe_1 - bAe_2, aAe_2 + bAe_1)$.

Family of operators $B \subseteq L(E)$ is *uniformly stable* if there exists $\gamma > 0$ such that for each $A \in B$, each eigenvalue λ of B, $\Re(\lambda) \leq -\gamma$.

We assume that $|\Theta|$-dimensional space R^{Θ} and its subspaces $\Lambda\Theta$ and $\Phi(\Theta)$ are equipped with the "sup" norm. It is convenient to interpret vectors $V \in R^{\Theta}$ as the elements of the dual spaces $L(\Phi(\Theta), R)$ and $L(\Lambda\Theta, R)$ in the natural way: for each $\upsilon \in \Phi(\Theta)$, let $v[\upsilon] = v \cdot \upsilon$.

4.2. **Transitions.** Let $\gamma_{max} < \infty$ be an upper bound on the absolute values of function γ as well its first, and second-order derivatives.

Let $\gamma_{a;\theta,\phi}, \gamma_{b;\theta,\phi} \in L(A, \Phi(\Theta))$ be the derivatives of function $\gamma : A \times \Theta \times A \times \Theta \to \Phi(\Theta)$ with respect to, respectively, a and b (once) evaluated at $(\alpha^*(\theta), \theta, \alpha^*(\phi), \phi)$.

Define linear operators:

- $\Gamma^* \in L(\Phi(\Theta), \Phi(\Theta))$: for each $\upsilon \in \Phi(\Theta)$, let
$$\Gamma^*[\upsilon] = \sum_{\theta,\phi} \gamma(\alpha^*(\theta), \theta, \alpha^*(\phi), \phi)\, \upsilon(\theta)\, \upsilon^*(\phi).$$
Operator Γ^* measures the effect of the perturbation in player's own distribution of types on transition rates. The domain of Γ^* can be extended to $\Lambda\Theta$. The stationarity of distribution υ^* implies that $\Gamma^*[\upsilon^*] = 0$, and $\Gamma^*[\Lambda\Theta] \subseteq \Phi(\Theta)$,
- $\Gamma^{**}_{\Theta} \in L(\Phi(\Theta), \Phi(\Theta))$: for each $\upsilon \in \Phi(\Theta)$, let
$$\Gamma^{**}_{\Theta}[\upsilon] = \sum_{\theta,\phi} \gamma(\alpha^*(\theta), \theta, \alpha^*(\phi), \phi)\, \upsilon^*(\theta)\, \upsilon(\phi).$$
Operator Γ^{**}_{Θ} measures the effect of the perturbation in population on the transition rates at the stationary distribution,
- $\Gamma^*_{+\Theta} = \Gamma^* + \Gamma^{**}_{\Theta} \in L(\Phi(\Theta), \Phi(\Theta))$. Operator $\Gamma^*_{+\Theta}$ measures the combined effect of the same perturbation in player's own distribution and in the general population,
- $\Gamma^*_A \in L(\mathcal{A}, L(\Phi(\Theta), \Phi(\Theta)))$: for each generalized action α, each $\upsilon \in \Phi(\Theta)$,
$$(\Gamma^*_A[\alpha])[\upsilon] = \sum_{\theta,\phi} (\gamma_{a;\theta,\phi}[\alpha(\phi)])\, \upsilon(\theta)\, \upsilon^*(\phi).$$
Operator Γ^*_A measures the first-order effect of the perturbation in one's own actions on the evolution of type distribution,

- $\Gamma_B^* : L(\mathcal{A}, L(\Phi(\Theta), \Phi(\Theta)))$: for each generalized action α, each $\upsilon \in \Phi(\Theta)$,

$$(\Gamma_B^*[\alpha])[\upsilon] = \sum_{\theta,\phi} (\gamma_{b;\theta,\phi}[\alpha(\theta)])\, \upsilon(\theta)\, \upsilon^*(\phi).$$

  Operator Γ_B^* measures the effect of the perturbation in the population's actions on the dynamics of the type distribution.
- $\Gamma_{A+B}^* = \Gamma_A^* + \Gamma_B^* \in L(\mathcal{A}, L(\Phi(\Theta), \Phi(\Theta)))$. Operator$\Gamma_{A+B}^*$ combines two first-order effects of the perturbation in actions on the evolution of type distributions,
- $\Gamma_A^{**} \in L(\mathcal{A}, \Phi(\Theta))$: for each generalized action α,

$$\Gamma_A^*[\alpha] = \sum_{\theta,\phi} (\gamma_{a;\theta,\phi}[\alpha(\phi)])\, \upsilon^*(\theta)\, \upsilon^*(\phi).$$

  Operator Γ_A^* measures the first-order effect of the perturbation in one's own actions on the evolution of type distribution,
- $\Gamma_B^{**} \in L(\mathcal{A}, \Phi(\Theta))$: for each generalized action α,

$$\Gamma_A^*[\alpha] = \sum_{\theta,\phi} (\gamma_{a;\theta,\phi}[\alpha(\phi)])\, \upsilon^*(\theta)\, \upsilon^*(\phi).$$

  Operator Γ_A^* measures the first-order effect of the perturbation in one's own actions on the evolution of type distribution,
- $\hat{\Gamma}_A^* \in L\left(L(\Phi(\Theta), R), (L(A, R))^\Theta\right)$: for each vector $V \in R^\Theta$, each type θ,

$$\left(\left(\hat{\Gamma}_A^*[V]\right)(\theta)\right)[a] = \sum_{\phi} (V \circ \gamma_{a;\theta,\phi}[a])\, \upsilon^*(\phi).$$

  Operator $\hat{\Gamma}_A^*$ measures the impact of the change in the actions on the continuation payoffs V. It plays a role in the characterization of the first order conditions.

4.3. **Payoffs.** Let

$$g^* = \left[\sum_{\phi} g(\alpha^*(\theta), \theta, \alpha^*(\phi), \phi)\, \upsilon^*(\phi)\right]$$

be the vector of instantaneous payoffs in the stationary equilibrium.

Let $V^* \in R^\Theta$ be the value function in the stationary equilibrium. Using the fact that $r > 0$, we can compute

$$V^* = V(\sigma^*, \upsilon^*) = \int_0^\infty \exp(-rt)\left[g^* \circ \exp(\Gamma^* t)\right] dt = g^* \circ (rI - \Gamma^*)^{-1}.$$

Given our interpretation, we treat g^* and V^* as the elements of the dual spaces $L(\Phi(\Theta), R)$ and $L(\Lambda\Theta, R)$.

4.4. **Best responses.** Define function $M : A \times \Theta \times A \times \Theta \to R^\Theta$: so that for all actions a, b and types θ, ϕ,

$$M(a, \theta, b, \phi) = g(a, \theta, b, \phi) + V^*\left[\gamma(a, \theta, b, \phi)\right]. \tag{4.1}$$

Function M combines two payoff effects of actions: the first term is the direct effect on instantaneous payoffs g, and the second term captures the effect on the type evolution, which in turn affects the future continuation payoffs. Function M is closely related to the terms of the Bellman equation (2.5) and it plays an important role in the local characterization of the best response function.

Let $M^*_{a;\theta,\phi}, M^*_{b;\theta,\phi} \in L\left(A, R^\Theta\right)$ and $M^*_{aa;\theta,\phi}, M^*_{ab;\theta,\phi} \in L(A, L(A, R))$ be the derivatives of function M with respect to, respectively, a (once), b (once), a (twice), and a and b (once each) evaluated at $(\alpha^*(\theta), \theta, \alpha^*(\phi), \phi)$. Define linear operators:

- $M^*_B \in L(\mathcal{A}, L(\Phi(\Theta), R))$ so that for each $\beta \in \mathcal{A}$, each $\upsilon \in \Phi(\Theta)$, we have

  $$(M^*_B[\beta])(\upsilon) = \sum_{\phi,\theta} M^*_{b;\theta,\phi}\left[\beta(\phi)\right] \upsilon(\theta)\, \upsilon^*(\phi),$$

  Operator M^*_B describes the effect of the average change in the population actions on the equilibrium value of function M,
- $M^*_\Theta \in L\left(\Phi(\Theta), L\left(\Phi(\Theta), R^\Theta\right)\right)$: for each $\nu, \upsilon \in \Phi(\Theta)$, each type θ, we have

  $$(M^*_\Theta[\nu])[\upsilon](\theta) = \sum_\phi M(\alpha^*(\theta), \theta, \alpha^*(\phi), \phi)\, \upsilon(\theta)\, \nu(\phi).$$

  Operator M^*_Θ describes the effect of the average change in the population type distribution on M,

- $M_{AB}^* \in L\left(\mathcal{A}, (L(A,R))^{\Theta}\right)$: for each $\beta \in \mathcal{A}$, for each type θ, let

$$M_{AB}^*[\beta](\theta) = \sum_{\phi} \left(M_{ab;\theta,\phi}^*[\beta(\theta)]\right) v^*(\phi).$$

  (Notice that that the derivative M_a with respect to a can be treated as an element of the vector space A, and that $M_{ab;\theta,\phi}^*$ can be understood as a linear operator on A.) Operator M_{AB}^* captures the effect of the change in the average action in the environment β on the equilibrium first-order conditions,
- $M_{A\Theta}^* \in L\left(\Phi(\Theta), (L(A,R))^{\Theta}\right)$: for each $\upsilon \in \Phi(\Theta)$, for each type θ, let

$$M_{A\Theta}^*[\upsilon](\theta) = \sum_{\phi} M_{a;\theta,\phi}^* \upsilon(\phi).$$

  Operator $M_{A\Theta}^*$ captures the effect of the change in the average type on the equilibrium first-order conditions,
- $M_{AA}^* \in L\left(\mathcal{A}, (L(A,R))^{\Theta}\right)$: for each $\alpha \in \mathcal{A}$, for each type θ, let

$$(M_{AA}^*[\alpha])(\theta) = \sum_{\phi} \left(M_{aa;\theta,\phi}^*[\alpha(\theta)]\right) v^*(\phi).$$

  We show below that the stationary equilibrium conditions imply that operator M_{AA}^* is negatively semi-definite. If M_{AA}^* is negative definite, it has an inverse $M_{AA}^{-1} \in L\left((L(A,R))^{\Theta}, \mathcal{A}\right)$.

5. Stability of type distribution

Our first result characterizes the stability of the type distribution dynamics with respect to initial perturbation of the type distribution away from the stationary distribution v^*.

Theorem 1. *Suppose that linear operator Γ^* is stable. If operator $\Gamma_{+\Theta}^*$ is stable, then the type distribution is stable at the stationary equilibrium. If operator $\Gamma_{+\Theta}^*$ is unstable, then the type distribution is unstable.*

The Theorem assumes that the eigenvalues of Γ^* have strictly negative real parts (notice that the real parts of the eigenvalues of Γ^* are always non-positive because Γ^* is a stochastic matrix). If operator $\Gamma_{+\Theta}^*$ is stable, then all (sufficiently small) initial perturbations in the type distributions disappear in the long-run. If the operator $\Gamma_{+\Theta}^*$

is unstable, then we can find an arbitrarily small initial perturbation υ_0 that leads to the diverging dynamics.

We sketch an argument behind Theorem 1. Take any $\upsilon_0 \in \Phi(\Theta)$ and let $\upsilon_{0c} = v^* + \upsilon_0$ for each cohort c. Because all cohorts behave in the same way, we may assume that there is only one representative cohort. The type distribution evolves according to the following differential equation

$$\frac{d}{dt}\upsilon_t = \sum_{\theta,\phi} \gamma\left(\alpha^*(\theta), \theta, \alpha^*(\phi), \phi\right) \upsilon_t(\phi) \upsilon_t(\theta) \text{ for each } t \geq 0, \tag{5.1}$$

with initial conditions $\upsilon_0 = v^* + \upsilon_0$.

Equation (5.1) typically does not have a closed-form solution and we consider a linearized approximation. Notice that for each $t \geq 0$,

$$\begin{aligned}
\frac{d}{dt}\upsilon_t =& \sum_{\theta,\phi} \gamma\left(\alpha^*(\theta), \theta, \alpha^*(\phi), \phi\right) \upsilon_t(\phi) \upsilon_t(\theta) \\
=& \sum_{\theta,\phi} \gamma\left(\alpha^*(\theta), \theta, \alpha^*(\phi), \phi\right) v^*(\phi) v^*(\theta) \\
&+ \sum_{\theta,\phi} \gamma\left(\alpha^*(\theta), \theta, \alpha^*(\phi), \phi\right) \left(v^*(\phi)\left(\upsilon_t(\theta) - v^*(\theta)\right) + \left(\upsilon_t^*(\phi) - v^*(\phi)\right) v^*(\theta)\right) \\
&+ \sum_{\theta,\phi} \gamma\left(\alpha^*(\theta), \theta, \alpha^*(\phi), \phi\right) \left(\left(\upsilon_t(\theta) - v^*(\theta)\right)\left(v^*(\phi) - v^*(\phi)\right)\right) \\
\approx& \sum_{\theta,\phi} \gamma\left(\alpha^*(\theta), \theta, \alpha^*(\phi), \phi\right) \left(v^*(\phi) \upsilon_t(\theta) + \upsilon_t(\phi) v^*(\theta)\right) = \Gamma^*_{+\Theta}\left[\upsilon_t\right],
\end{aligned}$$

where the approximate equality comes from the fact that the first term disappears because v^* is a stationary distribution, and the last term is of second-order magnitude. The linearized equation has a closed-form solution

$$\upsilon_t \approx \upsilon_0 + \exp\left(t\Gamma^*_{+\Theta}\right)\left(\upsilon_0 - v^*\right),$$

where $\exp(.)$ is the matrix exponential function (see Appendix A.3for details).

By the properties of the matrix exponential, if operator $\Gamma^*_{+\Theta}$ is stable, then

$$\lim_t \left\| \exp\left(t\Gamma^*_{+\Theta}\right) \upsilon_0 \right\| = 0$$

for any initial $\upsilon_0 = \upsilon_0 - v^*$. If it is unstable, then there exists an eigenvector υ_0 of $\Gamma^*_{+\Theta}$ such that $\lim_t \left\| \exp\left(t\Gamma^*_{+\Theta}\right) \upsilon_0 \right\| = \infty$. In particular, the solution to the linearized equation diverges. Appendix E shows that the instability of the solution extends to the original equation (5.1).

6. Stability of stationary equilibria with respect to best response dynamic

In this section, we describe the sufficient and (almost) necessary conditions for the stability of stationary equilibria with respect to the best response dynamics.

Suppose that operator matrix M^*_{AA} is negative semi-definite and operators Γ^* and $\Gamma^*_{+\Theta}$ are stable. Then, for each $\omega \in R$, (complex) linear operators

$$M^*_{AA}, 2\pi i\omega I_{\Phi(\Theta)} - \Gamma^*_{+\Theta}, (r + 2\pi i\omega) I_{\Phi(\Theta)} - \Gamma^*$$

have well-defined inverses. Define a family of (complex) operators $K(\omega) \in L^C\left(\mathcal{A}^{\mathcal{C}}, \mathcal{A}^{\mathcal{C}}\right)$: for each $\alpha \in \mathcal{A}$,

$$\begin{aligned}
&(K(\omega))[\alpha] \qquad (6.1)\\
=& - I_{\mathcal{A}} - M^{-1}_{AA} \circ M^*_{AB}[\alpha]\\
&- M^{-1}_{AA} \circ M^*_{A\Theta} \circ \left(2\pi i\omega I_{\Phi(\Theta)} - \Gamma^*_{+\Theta}\right)^{-1} \circ \left(\Gamma^*_{A+B}[\alpha]\right)\\
&- M^{-1}_{AA} \circ \hat{\Gamma}^*_A \circ (M^*_B[\alpha]) \circ \left((r + 2\pi i\omega) I_{\Phi(\Theta)} - \Gamma^*\right)^{-1}\\
&- M^{-1}_{AA} \circ \hat{\Gamma}^*_A \circ \left(M^*_\Theta\left[\left(2\pi i\omega I_{\Phi(\Theta)} - \Gamma^*_{+\Theta}\right)^{-1} \circ \left(\Gamma^*_{A+B}[\alpha]\right)\right]\right) \circ \left((r + 2\pi i\omega) I_{\Phi(\Theta)} - \Gamma^*\right)^{-1}.
\end{aligned}$$

Theorem 2. *Suppose that operator M^*_{AA} is negative definite and operators Γ^* and $\Gamma^*_{+\Theta}$ are stable. If family $K(\omega)$ is uniformly stable, then there exists $\lambda^* > 0$ such that for each $\lambda < \lambda^*$, the stationary equilibrium is λ-stable. If there exists ω such that $K(\omega)$ is unstable, then the stationary equilibrium is λ-unstable for any λ.*
If family $\{K(\omega) : \omega \in \Omega\}$ is uniformly stable, then there exists $\lambda^* > 0$ such that for each $\lambda < \lambda^*$, the stationary equilibrium is (Ω, λ)-stable. If there exists $\omega \in \Omega$ such that $K(\omega)$ is unstable, then the stationary equilibrium is (Ω, λ)-unstable for any λ.

The Theorem provides a sufficient and almost necessary condition of the stability of stationary equilibrium. In order to verify the conditions, one needs to find eigenvalues of a one-dimensional family of linear operators $K(\omega)$.

The Theorem requires three assumptions. The two first assumptions are not restrictive. First, a local characterization of stationary equilibrium (see Lemma 1 and step 2 of the proof of Lemma 12 in the Appendix) implies that operator M^*_{AA} is always negative semi-definite. The Theorem makes a slightly stronger assumption that operator M^*_{AA} is negative definite. Second, noticeΓ^* that always has eigenvalues with non-positive real part because it is a stochastic operator. The Theorem assumes that the eigenvalues of Γ^* have strictly negative real parts. Finally, the only restrictive assumption is that that $\Gamma^*_{+\Theta}$ is stable. However, as we have already demonstrated, together with the stability of Γ^*, this assumption is sufficient and almost necessary for the stability of the evolution of the type distribution (Theorem 1). Together, the assumptions of the Theorem ensure that the best response strategy is uniquely defined for profiles of strategies and initial distributions that are in some neighborhood of the stationary equilibrium values.

It is instructive to compare Theorem 2 with an analogous result in the case of static games. Recall that a static game equilibrium is stable if all eigenvalues of the Jacobian of the best response function computed at the equilibrium have real parts that are strictly smaller than 1. In the case of dynamic games, the space of strategies is infinitely dimensional and it would not be possible (or helpful) to state the result in terms of a "derivative" of the best response function. However, each linear operator $K(\omega) + I_{\mathcal{A}}$ can be interpreted as the Jacobian of the best response function *restricted* to a finitely dimensional class of wave-like strategies with frequency ω. Specifically, consider a class of strategies $\sigma(\omega, \alpha)$ indexed by a (real) frequency parameter $\omega \in R$ and a (complex) amplitude vector $\alpha \in \mathcal{A}^C$:

$$\begin{aligned}\sigma_t(\theta; \omega, \alpha) =& \sigma^*_t + \operatorname{Re}\left(\alpha(\theta) e^{-i2\pi\omega t}\right). \\ =& \sigma^*_t + \operatorname{Re}(\alpha(\theta)) \cos(-2\pi\omega t) - \operatorname{Im}(\alpha(\theta)) \sin(-2\pi\omega t).\end{aligned} \tag{6.2}$$

The second equality comes from the Euler's formula (see footnote 7). If $\omega = 0$, then $\sigma(\omega, \alpha)$ is a constant strategy; otherwise, it is periodic with period length $\frac{1}{\omega}$. The amplitude vector contains information about the generalized actions played throughout the wave-like strategy and also about the phase of the strategy. Its norm, $\|\alpha\|$, measures the deviation of strategy $\sigma(\omega, \alpha)$ away from the stationary strategy σ^*. Below (see equation (6.3)), we argue that if a population of players uses strategy $\sigma(\omega,\alpha)$ with sufficiently small vector α, then the best response strategy belongs to wave-like class with frequency ω: for high t,

$$b_t(\sigma, v_0) \approx \sigma(\omega, (K(\omega) + I_{\mathcal{A}})\alpha)$$

(We explain below that for high t, the best response does not depend on the initial distribution v_0.) Thus, linear operator $K(\omega) + I_{\mathcal{A}}$ contains an information about the link between an amplitude vector of a wave-like strategy, and the amplitude of the best response.

A consequence of the proof of the Theorem is that to test the stability it is not necessary to consider all possible initial perturbations, and it is enough to restrict the analysis to a class wave-like perturbations of form (6.2).

The proof of the Theorem is based on ideas similar to those that are developed in Sections 3.3-**??**. We sketch key steps below and leave the details for the Appendix.

6.1. **Sketch of the proof of Theorem 2: sufficient conditions.** We explain the argument behind the first part of the Theorem. The idea is to derive a stability test for a simplified version of the best response dynamics and then show that the same test applies to the original dynamics. . We proceed through a series of approximations. The first step is to find a linear approximation to the best response function.

Approximation of the best response function. Take a profile of strategies w and a profile of initial type distributions v^0. We are going to show that we can approximate $b_t(w,v)$ in the neighborhood of the stationary equilibrium with a linear function of w and v^0. As an intermediary step, we use Lemma 1 and notation from Section 4 to show that

$$\begin{aligned} b_t^{A0}(w, v_0) - \sigma_t^* \approx & - M_{AA}^{-1} \circ M_{AB}^* \left[w_t^E - \alpha^* \right] \\ & - M_{AA}^{-1} \circ M_{A\Theta}^* \left[v_t^E(w, v_0) - v^* \right] \\ & - M_{AA}^{-1} \circ \hat{\Gamma}_A^* \left[V_t(w, v_0) - V \right]. \end{aligned} \tag{6.3}$$

(For details, see step 2 of the proof of Lemma 12 in Appendix D.) In particular, the t-period best responses depend on the average actions played in the population, the average type distributions, and the value functions in period t.

It is instructive to notice that the terms of the approximation (6.3) with the terms of linear operator $K(\omega)$.

As a next step, we derive linear approximations to the evolution of the type distribution $v_t^E(w, v_0)$ and the value function $V_t(w, v_0)$. The former depends on the actions played before period t and the initial distribution v^0 for details, see Lemma 9 in Appendix C). The latter depends on the actions played after period t, and the type distributions in periods $s > t$, which in turn depend on the actions played before period s and the initial distribution v_0 (see step 4 of Lemma 12 in Appendix D). As we do not have closed-form formulas for the two approximations, we delay their formal statement to the Appendix. Using the approximations together with (6.3), we show that that there exists a (real) operator valued function $\kappa : R \to L(\mathcal{A}, \mathcal{A})$ such that for all profiles w of strategies and v^0 of initial type distributions, for high t,

$$b_t(w, v_0) - \sigma_t^* \approx -\left(M_{AA}^{-1} \circ M_{AB}^*\right)\left[w_t^E - \alpha^*\right] + \int_{-\infty}^{\infty} \kappa(u)\left[w_{t-u}^E - \sigma^*\right] du. \tag{6.4}$$

(To simplify the notation, we assume that $w_u^E = \sigma_u^*$ for each $u \leq 0$.) In particular, the best response is approximately equal to the weighted average of the present, past and future average actions in the population. Notice that the right-hand side of (6.4) does not depend on the initial distribution v_0. This is because of the ergodicity of the evolution of types (more precisely, due to the stability of linear operators Γ^* and $\Gamma_{+\Theta}^*$), the effect of the initial distribution disappears for high t.

The operator-valued function $\kappa(u)$ aggregates the past (for positive u) and future (for negative u) actions. For further use, it is important to describe two properties of $\kappa(.)$. First, $\kappa(.)$ has exponentially decaying tails, i.e., $\|\kappa(u)\| \leq Pe^{\rho u}$ for some constants P and ρ. The interpretation is that the impact of actions far away in the future or in the past on the present best response action decays exponentially. Second, although we do not have a closed-form formula for κ, we can show that its Fourier transform[10] is equal to

$$\mathcal{F}\kappa\left(\omega\right)=\int_{-\infty}^{\infty}e^{2\pi i\omega u}\kappa\left(u\right)du=K\left(\omega\right)+I_{\mathcal{A}}+M_{AA}^{-1}\circ M_{AB}^{*}, \tag{6.5}$$

where linear operator $K\left(\omega\right)$ is defined in (6.1).

Approximate linear dynamics. Let $\left(w^{\tau},v^{\tau}\right)$ be the best response dynamics initiated by perturbation $\left(w^{0},v^{0}\right)$. For each $t>\tau$, define a generalized action

$$\varpi_{t}^{\tau}=w_{t-\tau}^{\tau,E}-\alpha^{*}=\int\left(w_{c,t-\tau}^{\tau}-\alpha^{*}\right)d\mu^{\lambda}\left(c\right).$$

Thus, ϖ_{t}^{τ} is equal to the difference between the average action that τ-period agents plan to play in period $t>\tau$ and the stationary equilibrium action.

We are going to describe the dynamics of ϖ_{t}^{τ}. Recall that the average action ϖ_{t}^{τ} changes with τ because in each period τ, a fraction of players replace their strategies by the best response $b\left(w^{\tau},v^{\tau}\right)$. Because the arrival of a revision opportunity is independent from the agents' behavior, and because the mass of the players that receive the revision opportunity is equal to $\lambda d\tau$, the change in the average action is equal to

$$\frac{d\varpi_{t}^{\tau}}{d\tau}=\lambda\left(b_{t-\tau}\left(w^{\tau},v^{\tau}\right)-\varpi_{t}^{\tau}\right)\text{ for }t\geq\tau. \tag{6.6}$$

We use approximation (6.4) to derive a linearized version of the above dynamics: for $t\geq\tau$,

$$\frac{d\varpi_{t}^{\tau}}{d\tau}\approx\lambda\left(\left(-I_{\mathcal{A}}-M_{AA}^{-1}\circ M_{AB}^{*}\right)\left[\varpi_{t}^{\tau}\right]+\int_{-\infty}^{\infty}\kappa\left(u\right)\left[\varpi_{t-u}^{\tau}\right]du\right). \tag{6.7}$$

We need to be careful here because approximation (6.4) is only valid for high t. However, it turns out that, if λ is sufficiently low, the approximation (6.4) is good enough. The intuition is that if the revision opportunities are rare, then from the point of view of period t, the bulk of actions played in period t were chosen as the best responses in periods $s\ll t$ and they are well approximated by (6.4).

The dynamics (6.7) are still too complicated for a head-on analysis. We make two additional simplifications. First, suppose that actions ϖ_{t}^{τ} and equation (6.7) holds for all all t, including $t\leq\tau$. Second, suppose that ϖ_{t}^{0} is square-integrable in t. In

[10]See Appendix A.2 for the formal definition and the properties of the Fourier transform.

what follows, we first analyze the dynamics (6.7) with the two simplifications, and next, we explain how to dispense with them.

Dynamics of Fourier coefficients. The two assumptions imply that ϖ_t^τ can be treated as square-integrable functions of $t \in R$. In particular, we can apply Fourier transform to the both sides of (6.7): for each $\omega \in R$,

$$\begin{aligned}\frac{d\mathcal{F}\varpi^\tau(\omega)}{d\tau} =& \mathcal{F}\left(\frac{d\varpi_\cdot^\tau}{d\tau}\right)(\omega)\\ =& \mathcal{F}\left(\lambda\left(\left(-I_{\mathcal{A}} - M_{AA}^{-1}\circ M_{AB}^*\right)[\varpi_\cdot^\tau] + \int_{-\infty}^{\infty}\kappa(u)\left[\varpi_{\cdot-u}^\tau\right]du\right)\right)(\omega)\\ =& \lambda\left(\left(-I_{\mathcal{A}} - M_{AA}^{-1}\circ M_{AB}^*\right)[\mathcal{F}\varpi^\tau(\omega)] + \mathcal{F}\kappa\left[\mathcal{F}\varpi^\tau(\omega)\right]\right)\\ =& \lambda K(\omega)\left[\mathcal{F}\varpi^\tau(\omega)\right]. \end{aligned} \tag{6.8}$$

The first and the third equality follows from the linearity of the Fourier transform. The last equality comes from (6.5).

Equation (6.8) is a linear time-homogeneous first-order (vector-valued) differential equation that can be analyzed separately for each ω and it has an explicit solution

$$\mathcal{F}\varpi^\tau(\omega) = e^{t\lambda K(\omega)}\left[\mathcal{F}\varpi^0(\omega)\right], \tag{6.9}$$

where $e^{t\lambda K(\omega)}$ is a matrix exponential function.[11]

The convergence properties of (6.9) are well-understood. In particular, if the real parts of all eigenvalues of operators $K(\omega)$ are uniformly bounded away from 0, then $\mathcal{F}\varpi^\tau(\omega) \to 0$ when $\tau \to \infty$ at exponential rate.

Back to the original dynamics. Because the Fourier transform has a continuous inverse, the convergence of the Fourier coefficients $\mathcal{F}\varpi^\tau(\omega)$ implies that the average actions converge to 0. (We gloss over some technical issues: The convergence of Fourier coefficients implies the convergence of $\varpi^\tau(\omega) \to 0$ in the square-integrable norm L^2. An additional argument is required to show that the average actions converge to 0 uniformly, or that $\varpi^\tau(\omega) \to 0$ in the "sup", L^∞-norm. That argument relies on the fact that function κ is uniformly bounded and has exponential tails.)

[11]See Appendix A.3 for details.

We briefly discuss the issues related to extending the convergence of the linearized dynamics (6.7) to the original dynamics. First, we want to show that the convergence holds for all functions ϖ_t^τ uniformly bounded (in t) and not only square-integrable. For this purpose, for each t, we take some sufficiently large $\Delta > 0$ and divide the initial perturbation ϖ^0 into two parts $\varpi^0 = \varpi^{0,\text{near } t} + \varpi^{0,\text{distant } t}$, where $\varpi^{0,\text{near } t} = \varpi_s^0 \mathbf{1}_{|\mathbf{s}-\mathbf{t}|\leq\mathbf{\Delta}}$ corresponds to near past and future, and $\varpi^{0,\text{distant } t} = \varpi_s^0 \mathbf{1}_{|\mathbf{s}-\mathbf{t}|>\mathbf{\Delta}}$ corresponds to distant past and future. Because (6.7) is linear, we can separately consider the dynamics initiated by each of the two parts. The "near" part is square integrable, hence the dynamics converges by the above argument. The speed of the convergence depends on the square-norm of $\varpi^{0,\text{near } t}$, hence it decreases with Δ. Because of the exponentially decaying tails of function κ, the effect of the distant part on the dynamics of ϖ_t^τ is limited as long as $\tau < \tau^*$ for some threshold τ^* that increases with Δ. We can choose Δ so to guarantee that $\left\|\varpi_t^{\tau^*}\right\|$ is smaller than $\|\varpi_t^0\|$ uniformly across all t. We repeat the argument to show that $\varpi_t^\tau \to 0$ uniformly across all t.

Second, we want to show that the convergence holds if the average actions ϖ_t^τ and th, A profile $w(\omega, \alpha)$ of wave-like strategies is characterized by (unique) frequency parameter ω and a measurable function $\alpha : \mathcal{C} \to \mathcal{A}^C$ that assigns an amplitude to each cohort. e dynamics (6.7) are restricted to $\tau \leq t$. As compared to the linearized version of the original dynamics, the unrestricted dynamics contains an error that comes from the evolution of actions ϖ_t^τ for $t \leq \tau$. It turns out if the revision opportunities are very rare, then the error has a very limited impact on the evolution of actions ϖ_t^τ for $t > \tau$, i.e., the part covered by the restricted dynamics. The idea is that

Finally, we bound the difference between the best response dynamic and its linearized version using similar arguments to those used in the analysis of the revision dynamics in the case of static games.

6.2. **Sketch of the proof of Theorem 2: necessary conditions.** We explain that if operator $K(\omega)$ has an eigenvalue with a positive real part, then there exists an initial perturbation of a particular wave-like form that initiates diverging best response dynamic. The argument relies on the approximation methods developed in the first part of the proof.

For each frequency $\omega \in R$ and measurable function $\alpha : \mathcal{C} \to \mathcal{A}^C$, define a profile of wave-like strategies so that for each cohort,

$$w_c(\omega, \alpha) = \sigma(\omega, \alpha(c)),$$

where $\sigma(\omega, \alpha(c))$ is a wave-like strategy with frequency ω and amplitude $\alpha(c)$ that is defined above in (6.2). Let $\alpha^E = \int \alpha(c)\, d\mu^\lambda(c)$ be the average amplitude in profile $w(\omega, \alpha)$.

We use (6.4) to compute an approximate best response to wave-like profiles $w\left(\omega, \alpha^{\mathcal{C}}\right)$:

$$\begin{aligned} b_t(w(\omega, \alpha), .) \approx &- \left(M_{AA}^{-1} \circ M_{AB}^*\right)\left[\mathrm{Re}\left(\alpha^E e^{-i2\pi\omega t}\right)\right] \\ &+ \int_{-\infty}^{\infty} \kappa(u)\left[\mathrm{Re}\left(\alpha^{\mathcal{C},E} e^{-i2\pi\omega(t-u)}\right)\right] du \\ =&\mathrm{Re}\left(\left(-\left(M_{AA}^{-1} \circ M_{AB}^*\right)\left[\alpha^E\right] + \mathcal{F}\kappa(\omega)\left[\alpha^{\mathcal{C},E}\right]\right) e^{-i2\pi\omega t}\right) \\ =&\sigma\left(\omega, (K(\omega) + I_{\mathcal{A}})\left[\alpha^E\right]\right). \end{aligned} \tag{6.10}$$

(The first proper equality comes from the fact that linear operators M_{AA}^{-1}, M_{AB}^*, and the operator-valued function κ have only real parts, and from the definition of the Fourier transform.) In particular, the best response has a wave-like form with the same frequency parameter as the original strategy profile. The amplitude coefficient of the best response strategy is a linear function of the average amplitude in the population. The exact value is determined by on the is equal to to the value of operator $K(\omega)$ on the amplitude of the original strategy.

We explain how to use observation (6.10) to show the instability of the best response dynamics. Suppose that the τ-period strategy in the population is given by profile $w(\omega, \alpha^\tau)$ with the average amplitude $\alpha^{E,\tau}$. The τ-period best response strategy is given by (6.10). During the next $d\tau$ periods, a fraction $\lambda d\tau$ of players receives an opportunity to revise their strategies to the best response strategy. In particular, after $d\tau$ periods, the new population profile has the same form as the original profile but with a slightly different distributions of amplitudes with the average amplitude approximately equal to

$$\alpha^{E,\tau+d\tau} = (1 - \lambda d\tau)\, \alpha^{E,\tau} + \lambda d\tau\, (K(\omega) + I_{\mathcal{A}})\left[\alpha^{E\tau}\right]. \tag{6.11}$$

Because the (approximate) best responses depend only on the average amplitude, we can trace the behavior of the best response dynamics by tracing the behavior of the average amplitudes. Heuristics (6.11) suggests that the average amplitude evolves according to the linear, time-homogenous differential equation

$$\frac{d\alpha^{E\tau}}{d\tau} = K(\omega)\left[\alpha^{E\tau}\right].$$

The long-run behavior of such equations is well-understood and it depends on the eigenvalues of operator $K(\omega)$. If α^* is a (possibly, complex) eigenvector of $K(\omega)$ with a (possibly, complex) eigenvalue ψ, then

$$\alpha^{E\tau} = e^{\psi\tau}\alpha^*$$

is a solution to the above differential equation. If ψ has a strictly positive real part, then the above solution explodes and the approximate linearized dynamic diverges. We extend

7. Stability of stationary equilibria with respect to learning dynamic

In this section, we describe the sufficient and (almost) necessary conditions for the stability of stationary equilibria with respect to the learning dynamic.

Theorem 3. *Suppose that operator M^*_{AA} is negative semi-definite and operators Γ^* and $\Gamma^*_{+\Theta}$ are stable. Take any finite set $\Omega \subseteq R$, $0 \in \Omega$ of frequencies. If family $\{K(\omega) : \omega \in \Omega\}$ is uniformly stable, then there exists $\lambda^* > 0$ such that for each $\lambda < \lambda^*$, the stationary equilibrium is (Ω, λ)-stable. If there exists $\omega \in \Omega$ such that $K(\omega)$ is unstable, then there exists $\lambda^* > 0$ such that for each $\lambda < \lambda^*$, the stationary equilibrium is (Ω, λ)-unstable.*

The Theorem says that the uniform stability of family $\{K(\omega) : \omega \in \Omega\}$ is a sufficient and almost necessary condition of the stability of stationary equilibrium.

As we explained in the discussion of Theorem 3, each operator $K(\omega)$ is a Jacobian of the best response function restricted to strategies that have wave-like form with frequency ω. The fact that it is enough consider wave-like strategies is not surprising given that the learning dynamics force the agents to predict wave-like strategies of their the opponents.

The learning dynamic encompasses variety of models that differ with respect to the range of frequencies Ω that can be detected by the agents. The Theorem implies that the larger detectable set Ω, the more stringent conditions required for the stability of equilibrium. The intuition is that if the agents can detect more patterns, this leads to more complicated dynamics, with extra possibilities for instability.

7.1. **Sketch of the proof.** As in the case of the best response dynamics, the proof of Theorem 3 relies on the linearization of the learning dynamic in the neighborhood of the stationary equilibrium. The details differ somewhat from the proof of Theorem 2. We sketch the main idea below.

Suppose that family $\{K(\omega) : \omega \in \Omega\}$ is uniformly stable. We are going to argue that for each strategy profile w in a (sufficiently small) neighborhood of the stationary equilibrium, the average coefficients $a_{\cos}^{t,E}(.)$ and $a_{\sin}^{t,E}(.)$ of regression (2.7) converge to 0. This implies the convergence of the best responses, as well as the convergence of the dynamics.

It is convenient to define "Fourier" decomposition of the average past actions: for each frequency ω, and for t,

$$a_t^E(\omega) = \frac{2}{t}\int_0^t \left(w_0^{s,E} - \alpha^*\right) e^{i2\pi\omega s} ds. \tag{7.1}$$

The complex coefficient $a_t^E(\omega)$ is closely related to the regression coefficients: for sufficiently large t, and all $\omega \neq 0$,

$$a_t^E(\omega) \approx a_{\cos}^{t,E}(\omega) + i a_{\sin}^{t,E}(\omega).$$

We also have $a_{\cos}^{t,E}(0) \approx \alpha^* + \mathrm{Re}\left(a_t^E(0)\right)$ (because $\sin(0) = 0$, the value of the coefficient $a_{\sin}^{t,E}(0)$ is not important). The approximations follow from the Euler's formula and the equations that characterize the linear regression. Due to the approximations, the agents forecast made in period t is (approximately) equal to

$$w_{s-t}^{P,t,E} = \sum_{\omega'\in\Omega} a_{\cos}^{t,E}(\omega')\cos(2\pi\omega' s) + a_{\sin}^{t,E}(\omega')\sin(2\pi\omega' s) \approx \sum_{\omega'\in\Omega} \operatorname{Re}\left(a_t^E(\omega')e^{-i2\pi\omega' s}\right). \tag{7.2}$$

In particular, the forecast has a wave-like form (6.2).

We are going to describe the dynamics of the coefficients $a_t^E(.)$. Notice that the average actions played in period s are equal to a (weighted) average of the actions planned in period 0 and the best responses chosen by players who revised the strategies between periods 0 and s:

$$w_0^{s,E} = e^{-\lambda s} w_s^{0,E} + \lambda \int_0^s b_{s-u}\left(w^{P,u}, v^u\right)e^{-\lambda(s-u)}du.$$

The weights on different periods depend on the arrival rate λ of revision opportunities. Because the first term becomes small for sufficiently large s, we are going to drop it in subsequent calculations. We can approximate the best response action using formula (6.10) as well as the approximation of the forecasted strategy (7.2):

$$b_{s-u}\left(w^{P,u}, v^u\right) - \alpha^* \approx \sum_{\omega'\in\Omega} \operatorname{Re}\left((K(\omega') + I_{\mathcal{A}})\left[a_u^E(\omega')\right]e^{-i2\pi\omega' s}\right).$$

We substitute the two above equations into (7.1): after some algebra, we obtain a sequence of approximations:

$$\begin{aligned} a_t^E(\omega) &\approx \frac{2}{t}\int_0^t \sum_{\omega'\in\Omega} \operatorname{Re}\left((K(\omega') + I_{\mathcal{A}})\left[\lambda\int_0^s \left(\left[a_u^E(\omega')\right]\right)e^{-\lambda(s-u)}du\right]e^{-i2\pi\omega' s}\right)e^{i2\pi\omega s}ds \\ &\approx \frac{2}{t}\int_0^t \sum_{\omega'\in\Omega} \operatorname{Re}\left((K(\omega') + I_{\mathcal{A}})\left[a_s^E(\omega')\right]e^{-i2\pi\omega' s}\right)e^{i2\pi\omega s}ds \\ &\approx \frac{2}{t}\int_0^t \operatorname{Re}\left((K(\omega) + I_{\mathcal{A}})\left[a_s^E(\omega')\right]e^{-i2\pi\omega s}\right)e^{i2\pi\omega s}ds \\ &\approx \frac{1}{t}\int_0^t (K(\omega) + I_{\mathcal{A}})\left[a_s^E(\omega')\right]ds \end{aligned}$$

(In the second line, we use the fact that $\lambda \int_0^s \left(\left[a_u^E(\omega')\right]\right) e^{-\lambda(s-u)} \approx a_s^E(\omega')$ for each frequency ω'. In the third line, we use the fact that an integral of a product of two wave-like functions with different frequencies $\omega' \neq \omega$ is close to 0. The last line relies on an analogous fact about the integral of the product of imaginary exponential with the real part of wave-like strategy with the same frequency ω.)

The last line of the above equation describes an approximate dynamics of the Fourier coefficients. (Notice the similarity to fictitious play formula). The approximate dynamics has a closed-form "solution"

$$a_t^E(\omega) = e^{(\log t)K(\omega)} a_0^E(\omega).$$

As we discuss above, the approximate " solution" converges if linear operator $K(\omega)$ is stable. It diverges if $K(\omega)$ is unstable, and $a_0^E(\omega)$ is chosen to be the eigenvector associated with an eigenvalue of $K(\omega)$ with a strictly positive real part.

8. Conclusions

TBA

References

Appendix A. Mathematical Preliminaries

A.1. **Terminology and notation.** For each Banach space X, each profile of X-valued paths $\chi \in D\left((C, \mathcal{C}, \mu)\overline{X}\right)$, define the real valued paths $\chi^E, \|\chi\|_. \in \overline{X}$, so that for each t,

$$\chi_t^E = \int \chi(c)\, d\mu(c), \text{ and}$$

$$\|\chi\|_t = \|\chi(.,t)\|_{L^2} = \sqrt{\int \|\chi(c,t)\|^2 d\mu(c)}.$$

Recall that $\mathcal{L}^p(R, E)$ for each $1 \leq p \leq \infty$ is the space of Lebesgue-measurable, $\mathcal{L}^p$-integrable functions $g : R \to E$. We denote the $\mathcal{L}^p$-norm of function $g \in \mathcal{L}^p(R, E)$ as $\|g\|_{\mathcal{L}^p}$.

A Lebesgue-measurable function $g : R \to E$ is *exponentially bounded* if there exists constants $P < \infty$ and $\rho > 0$ such that for each $t \in R$, $\|g(t)\| \leq Pe^{-\rho|t|}$. Let

$\mathcal{L}^{\exp}(R,E)$ be the space of exponentially bounded functions. Of course, $\mathcal{L}^{\exp}(R,E) \subseteq \mathcal{L}^{p}(R,E)$ for each $1 \leq p \leq \infty$.

It is convenient to introduce the following terminology: For any two functions $\zeta(w,v_0), \xi(w,v_0) \in X$ that depend on profiles of strategies w and initial type distributions v_0 and with the values in some Banach space X, we say that ζ is *a first (or second)-order approximation for* ξ, if there exist constants $K < \infty$ and $\epsilon > 0$ that depend only on the parameters of the model (i.e., the values and derivatives of functions g and γ), for all profiles of strategies w and initial type distributions v_0 such that $\|w - \sigma^*\| + \|v_0 - v^*\| \leq \epsilon$,

$$\|\zeta(w,v_0) - \xi(w,v_0)\|_X \leq K\left(\|w - \sigma^*\| + \|v_0 - v^*\|\right),$$
$$\left(\text{or } \|\zeta(w,v_0) - \xi(w,v_0)\|_X \leq K\left(\|w - \sigma^*\| + \|v_0 - v^*\|\right)^2\right).$$

A.2. **Convolutions and Fourier transforms.** For each $\kappa \in \mathcal{L}^1(R, B(E))$ and each $h \in \mathcal{L}^p(R,E)$, define convolution $\kappa \star h : R \to E$ as

$$(\kappa \star h)(t) = \int_{-\infty}^{\infty} \kappa(t-s)[h(s)]\,ds.$$

Notice that $\|\kappa \star h\|_{L^p} \leq \|\kappa\|_{L^1} \|h\|_{L^P}$. We can check that if $\kappa \in L^{\exp}(R, B(E', E''))$ and $h \in L^{\exp}(R, B(E, E'))$, then $\kappa \star h \in\in L^{\exp}(R,E)$.

For each $h \in \mathcal{L}^1(R,E)$, define the Fourier transform of h as $\mathcal{F}h : R \to E$,

$$\mathcal{F}h(\omega) = \hat{h}(\omega) = \int_{-\infty}^{\infty} h(t)\, e^{-2\pi i t\omega} dt.$$

Let $\mathcal{R}$ be the "flip" operator, i.e., $\mathcal{R}h(x) = h(-x)$. If functions h and $\hat{h}$ are integrable, then the Fourier transform is equal to its inverse modulo the flip:

$$\mathcal{F}^{-1}(\mathcal{R}\mathcal{F}(h)) = \mathcal{F}^{-1}(\mathcal{F}\mathcal{R}(h)) = h. \tag{A.1}$$

If $h \in L^1(R,E) \cap L^2(R,E)$, then the Plancherel's Theorem implies that

$$\|h\|_{L^2} = \left\|\hat{h}\right\|_{L^2}. \tag{A.2}$$

Also, we have $\|\mathcal{F}h\|_{\mathcal{L}^\infty} \leq \|h\|_{\mathcal{L}^1}$.

For any $\kappa \in L^{\exp}(R, B(E))$ and $h \in L^1(R, E)$, the convolution theorem says that the Fourier transform of the convolution $\kappa \star h$ is equal to the product of the Fourier transforms of κ and h,

$$\mathcal{F}(\kappa \star h)(\omega) = \mathcal{F}(\kappa)(\omega)[\mathcal{F}(h)(\omega)]. \tag{A.3}$$

A.3. **Matrix and operator exponential.** Let X be a finitely dimensional vector space. For each $A \in L(X, X)$, and each t, define linear operator $\exp(A) \in L(X, X)$ as

$$\exp(A) = \sum_{n=0}^{\infty} \frac{1}{n!}(A)^n. \tag{A.4}$$

We refer to $\exp(A)$ as the matrix exponential. It is well-known that if linear operator A is stable, then $\lim_{t\to 0} e^{tA} = 0$, and if $A$is unstable, then there exists $v_0 \in X$ such that $\lim_{t\to 0} \left\|e^{tA}v_0\right\| = \infty$. We use the following uniform version of this result.

Lemma 2. *Suppose that X is a finitely dimensional vector space. Suppose that family of operators $B \subseteq L(X, X)$ is relatively compact (i.e., its closure is a compact set) and uniformly stable. Then, there exists $P < \infty$ and $\rho > 0$ such that for each $A \in B$, each $x \in X$, each $t \geq 0$,*

$$\|\exp(At)x\| < Pe^{-\rho t}\|x\|.$$

Proof. The claim follows from the continuity of eigenvalues. □

The definition of the matrix exponential can be generalized to bounded linear operators on Banach spaces. Let X be a Banach space, and let $L(X, X)$ be the space of bounded linear operators $A : X \to X$ with the operator norm $\|A\| = \sup_{x:\|x\|\leq 1}\|Ax\|$. We can define the operator exponential using formula (A.4). Then, $\exp(A)$ is a well-defined, bounded, linear operator. Moreover, $\|\exp(A)\| \leq e^{\|A\|}$, and for each $x \in X$, $x_t = \exp(At)x$ is the unique solution to the differential equation

$$\frac{dx_t}{dt} = Ax_t$$

with initial conditions $x_0 = x$.

Lemma 3. *Suppose that A is a bounded linear operator on Banach space X. Then, there exists limit*

$$\rho_A = \lim_{t\to\infty} \frac{1}{t} \log\|\exp(At)\|.$$

Moreover, for each $\eta > 0$,

- *there is a constant $P_{\eta,A} < \infty$ such that for each $x \in X$ and t,*

$$\|\exp(At)\,x\| \leq Pe^{(\rho_A+\eta|\rho_A|)t}\|x\|, \text{ and}$$

- *for each $t \geq 0$, there exists $x \in X$ such that*

$$\|\exp(At)\,x\| \geq e^{(\rho_A-\eta|\rho_A|)t}\|x\|.$$

Remark 1. If X is finitely dimensional, then ρ_A is equal to the largest real part of eigenvalues of matrix A.

$$\rho_A = \max\{\mathrm{Re}(\lambda) : \lambda \text{ is an eigenvalue of } X\}.$$

Proof. For each t, let $\rho_t = \frac{1}{t}\log\|\exp(At)\|$. Function $\rho_.$ is continuous with t and bounded by $\|A\|$. Moreover, for each $s > t > 0$,

$$\rho_s \leq \left(1 + \frac{1}{\lfloor s/t \rfloor}t\right)\rho_t.$$

It follows that for each t, $\gamma_t \geq \limsup_{t\to\infty}\gamma_s$, and that there exist limit $\gamma_A = \lim_{t\to\infty}\gamma_t$.

Fix $\eta > 0$. Let $P_{\eta,A} = \sup_t t\left(\rho_t - (\rho_A + \eta\,|\rho_A|)\right)$. Then, $P_A < \infty$ because of continuity of $\rho_.$, its boundedness, and the existence of the limit and part (a) holds. Additionally, for each t, there exists $x \in X$ such that $\|\exp(At)\,x\| \geq e^{(\rho_A-\eta|\rho_A|)t}\|x\|$. Because $\rho_t \geq \rho_A$, part (b) follows. □

A.4. **Lyapunov functions on finitely dimensional spaces.** The next result describes Lyapunov-type functions for stable and unstable linear operators finitely dimensional space X. Recall the a set $X' \subseteq X$ is a cone if for each $x \in X'$, each$\lambda \in R$, $\lambda x \in X'$. For each cone X', say that continuous $y : X' \to R_+$ is a cone function, if $y(x) = 0$ for $x \in X'$ iff $x = 0$ and such that $y(\lambda x) = |\lambda|\,y(x)$.

Lemma 4. *Suppose that X is a finitely dimensional (complex) vector space and $A \in L(X, X)$.*

(1) *If A is stable, then there exists a cone function $y_A : X \to R$ such that for each $x \neq 0$, $y_A(x) > 0$, $y_A(x)$ is differentiable, and $\nabla y_A(x) \cdot Ax < 0$.*

(2) *If A is unstable, then there exists a cone $X_A \subseteq X$, a cone function $y_A : X_A \to R_+$, and constants $\eta, \rho > 0$ such that for each $x \in X_A$, each $\epsilon \in X$, if $\|\epsilon\| \leq \eta \|x\|$, then for each $d > 0$*

$$Ax + \epsilon \in X_A \text{ and } y_A (x + d (Ax + \epsilon)) - y (x) \geq \rho d \|x\| .$$

Proof. Part 1. Let

$$y_A (x) = \int_0^\infty \|\exp (At) x\| \, dt.$$

Part 2. Let

$$A = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ldots & \\ & & & J_n \end{bmatrix}$$

be the Jordan decomposition of operator A, let λ_i be the eigenvalue associated with block matrix J_i, and let $\mathbf{e} = \left(e^1_1, ..., e^1_{m_1}, ..., e^n_{m_n}\right)$ be the associated basis. For each x, let $\left(x^1_1, ..., x^1_{m_1}, ..., x^n_{m_n}\right)$ denote the representation of vector x in the basis. Then,

$$J_i \begin{bmatrix} x^i_1 \\ x^i_2 \\ \ldots \\ x^i_{n_i} \end{bmatrix} = \begin{bmatrix} \lambda_i x^i_1 + x^i_2 \\ \lambda_i x^i_2 + x^i_3 \\ \ldots \\ \lambda_i x^i_n \end{bmatrix}.$$

Let I be the set of eigenvalues such that $\mathrm{Re}(\lambda_i) > 0$. Because operator A is unstable, set I is nonempty. For each $i \in I$, let $w^i_1 = 1$, and, recursively, $w^i_{k+1} = \frac{2}{\mathrm{Re}(\lambda_i)} w^i_k$ for each $k = 1, ..., m_i - 1$.

For each x, define

$$y_A (x) = \max_{i \in I} \max_{m \leq m_i} w^i \left| x^i_m \right|,$$
$$y' (x) = \sum_{j \notin I, m} \left| x^j_{m_j} \right|.$$

Define cone

$$X_A = \left\{ x : \left(\min_{i \in I} \mathrm{Re} (\lambda_i) \right) y_A (x) \geq 8 y' (x) \right\}.$$

Then, for each $x \in X_A$, $y_A (x) \geq \mu \|x\|$ for some constant $\mu > 0$.

Notice that for each $i \in I$,

$$\begin{aligned} w_m^i \left|(1+d\lambda_i)\, x_m^i + dx_{m+1}^i\right| \geq & w_m^i \left|x_m^i\right| + dw_m^i \left(\mathrm{Re}\,(\lambda_i) \left|x_m^i\right| - \left|x_{m+1}^i\right|\right) \\ \geq & w_m^i \left|x_m^i\right| + \frac{1}{2} d\mathrm{Re}\,(\lambda_i) \left(w_m^i \left|x_m^i\right| - w_{m+1}^i \left|x_{m+1}^i\right|\right) \\ & + \frac{1}{2} d\mathrm{Re}\,(\lambda_i)\, w_m^i \left|x_m^i\right|. \end{aligned}$$

If $w_m^i \left|x_m^i\right| \geq w_{m+1}^i \left|x_{m+1}^i\right|$, then

$$w_m^i \left|(1+d\lambda_i)\, x_m^i + dx_{m+1}^i\right| \geq \left(1 + \frac{1}{2} d\mathrm{Re}\,(\lambda_i)\right) w_m^i \left|x_m^i\right|.$$

Thus,

$$y_A\,(x + dAx) - y_A\,(x) \geq \frac{1}{2} d \left(\min_{i \in I} \mathrm{Re}\,(\lambda_i)\right) y_A\,(x)\,.$$

Moreover,

$$y_A\,(x + dAx + \epsilon) \geq y_A\,(x + dAx) - 2 \max_{i \in I} \left(\frac{2}{\mathrm{Re}\,(\lambda_i)}\right)^{m_i} \|\epsilon\|\,.$$

Thus, there exist constants $\eta', \rho > 0$ such that for each $x \in X_A$ and $\epsilon \in X$, if $\|\epsilon\| \leq \eta' \|x\|$, then

$$y_A\,(x + dAx + \epsilon) - y_A\,(x + dAx) \leq \frac{1}{4} d \left(\min_{i \in I} \mathrm{Re}\,(\lambda_i)\right) y_A\,(x)\,,$$

and

$$y_A\,(x + d\,(Ax + \epsilon)) - y\,(x) \geq \rho d \|x\|\,.$$

A similar argument shows that

$$y_A\,(Ax + \epsilon) \geq \frac{1}{4} d \left(\min_{i \in I} \mathrm{Re}\,(\lambda_i)\right) y_A\,(x)$$

if $\|\epsilon\| \leq \eta' \|x\|$ for each $x \in X_A$ and $\epsilon \in X$. Because

$$y'\,(Ax) \geq 2y'\,(x)\,,$$

we get that $A'x + \epsilon \in X_A$. □

Appendix B. Approximately linear differential equations

In this section, we analyze differential process that are approximately linear. The main purpose is to provide various bounds on the quality of approximation.

B.1. **Banach-valued differential processes.** Let X be a Banach space, and let $A : X \to X$ be a bounded linear operator. We consider an (in)stability of a differential processes $x_t \in X$ such that

$$\frac{dx_t}{dt} = Ax_t + q_t, \tag{B.1}$$

where $q_t \in X$ is small relative to x_t.

Lemma 5. *Let X be a finitely dimensional (complex) vector space and suppose that operator A is unstable. Then, there exists $\eta > 0$, constants $P, \rho > 0$, and $x^* \in X$ with the following property. Take any differential process x_t such that $x_0 = x^*$, and for each t, (B.1) holds (or, $\frac{d}{dt}x_t = \frac{1}{t}(Ax_t + q_t))$) and where $q_t \leq \eta \max_{s \leq t} \|x_s\|$. Then, for each t*

$$\|x_t\| \geq Pe^{\rho t} \|x_0\| \quad (or,\ \|x_t\| \geq Pt^{\rho} \|x_0\|).$$

Proof. The result follows from Lemma 4, which implies that if $x_0 \in X_A$, then for each t,

$$\frac{d}{dt} y_A(x_t) \geq \rho > 0.$$

□

B.2. **Profiles of differential processes.** In this section, we characterize upper bounds on the behavior of certain differential processes.

Real-valued process. First, consider a class of processes $y \in D\overline{R_+}$ and $z \in \overline{R_+}$ such that y_{ct} and z_t are differentiable in t for each c, and such that there exists constants γ_A, γ_B, K, and processes $p_A, q_A \in D\overline{R_+}$ and $p_B, q_B \in \overline{R_+}$ such that

$$\begin{aligned} \frac{dy_{ct}}{dt} &\leq -\gamma_A y_{ct} + Kz_t + q_{A,ct} + p_{A,ct}, \\ \frac{dz_t}{dt} &\leq -\gamma_B z_t + q_{B,t} + p_{B,t}. \end{aligned} \tag{B.2}$$

Lemma 6. *For each $\gamma_A, \gamma_B > 0$ and $K < \infty$, there exist constants $P, Q < \infty$ and $\gamma, \epsilon > 0$ such that if processes $y_{ct} \geq 0$ and $z_{ct} \geq 0$ satisfy (B.2), and if$\|q_A\|_t, \|q_B\|_t \leq K\epsilon^{1/2} \|y\|_t$, then, for each $t \geq 0$,*

$$\|y\|_t \leq Pe^{-\gamma t} \|y\|_0 + Q \max_{s \leq t} (\|p_A\|_s + \|p_B\|_s) e^{-\gamma(t-s)}.$$

Proof. Let

$$P = \frac{4}{\gamma_A} K, \; Q = \max\left(\frac{4}{\gamma_A}, \frac{16}{\gamma_A \gamma_B} K\right), \epsilon \leq \min\left(\left(\frac{1}{4K}\gamma_A\right)^2, \left(\frac{1}{4KP}\gamma_B\right)^2\right).$$

Define

$$\psi_t := \max\left(\frac{1}{P}\|y\|_t, z_t\right).$$

We will show that for each period t,

$$\text{if } \psi_t \geq \frac{Q}{P}\left(\|p_A\|_t + \|p_B\|_t\right), \text{ then } \frac{d\psi_t}{dt} \leq -\gamma\psi_t. \tag{B.3}$$

We consider separately two cases:

- First, suppose that $\|y\|_t > P z_t$. Then, $\|y\|_t = P\psi_t \geq Q\|p_A\|_t$, and

$$\begin{aligned}
\frac{d\psi_t}{dt} =& \frac{1}{P}\frac{d\|y\|_t}{dt} \\
=& \frac{1}{P}\frac{1}{\|y\|_t}\int y_{ct}\left(\frac{dy_{ct}}{dt}\right)dc \\
\leq& \frac{1}{P}\frac{1}{\|y\|_t}\int y_{ct}\left(-\gamma_A y_{ct} + K z_t + q_{A,ct} + p_{A,ct}\right)dc \\
\leq& -\frac{1}{P}\gamma_A\|y\|_t + \frac{1}{P}\left(\frac{K}{P} + K\epsilon^{1/2} + \frac{1}{Q}\right)\left(\int y_{ct}dc\right) \\
\leq& -\frac{1}{P}\gamma_A\|y\|_t + \frac{1}{P}\left(\frac{K}{P} + K\epsilon^{1/2} + \frac{1}{Q}\right)\|y\|_t \\
=& -\frac{1}{4P}\gamma_A\|y\|_t = -\frac{1}{4}\gamma_A\psi_t,
\end{aligned}$$

  where the last inequality follows from the Cauchy-Schwartz inequality.
- Next, suppose that $\|y\|_t < P z_t$. Then, $z_t = \psi_t \geq \frac{Q}{P}\|p_B\|_t$, and

$$\begin{aligned}
\frac{d\psi_t}{dt} =& \frac{dz_t}{dt} \leq -\gamma_B z_t + q_{A,t} + p_{B,t} \\
&\leq -\left(\gamma_B z_t - PK\epsilon^{1/2} z_t - \frac{P}{Q} z_t\right) \leq -\frac{1}{4}\gamma_B\psi_t.
\end{aligned}$$

Proof. Claim (B.3) implies that

$$\psi_t \leq e^{-\gamma t}\psi_0 + \frac{Q}{P}\max_{s\leq t} e^{-\gamma(t-s)}\left(\|p_A\|_s + \|p_B\|_s\right).$$

The Lemma follows from the fact that $\|y\|_t \leq P\psi_t$. □

□

Vector-valued processes. Let X be a finitely dimensional space. We analyze the behavior of processes $\chi \in D\overline{X}$ that satisfy the differential equation

$$\frac{d\chi_{ct}}{dt} = A\chi_{ct} + (B - A)\,\chi_t^E + p_{ct} + q_{ct}, \tag{B.4}$$

where A and B are linear operators on X, and $p, q \in D\overline{X}$ are some differential processes.

Lemma 7. *Suppose that operators A and B are stable. There exist constants $P, Q < \infty$ and $\gamma, \epsilon > 0$ such that if (i) process $\chi_{ct} \in X$ satisfy (B.4) for each c and t, (ii)$\|\chi\|_0 \leq \epsilon$, $\|p\| \leq \epsilon$, and (iii) for each t,$\|q\|_t \leq \epsilon^{1/2}\|\chi\|_t$, then*

$$\|\chi\|_t \leq Pe^{-\gamma t}\|\chi\|_0 + Q\max_{s\leq t} e^{-\gamma(t-s)}\|p\|_s\,.$$

Proof. Let y_A be the function from Lemma 4 chosen for operator A. Define $m_A, M_A > 0$ as, respectively, the minimum and the maximum value of function $y_A(x)$ on the unit sphere $\{x \in X : \|x\| = 1\}$. Let

$$\gamma_A = -\max_{x:\|x\|=1} \nabla y_A(x)\cdot Ax > 0.$$

Similarly, define $y_B(.)\,, m_B, M_B$, and γ_B.

Because function $y_A(.)$ is homogeneous of degree 1 and differentiable everywhere except for $x = 0$, the Euler's theorem implies that for each $x \neq 0$,

$$\nabla y_A(x)\cdot Ax = \nabla y_A\left(\frac{x}{\|x\|}\right)\cdot A\frac{x}{\|x\|}\|x\| \geq -\gamma_A\|x\| \geq -\gamma_A m_A y_A(x)\,.$$

An analogous property holds for function $y_B(.)$.

For each c,

$$\begin{aligned}\frac{dy_A(\chi_{ct})}{dt} =& \nabla y_A(\chi_{ct})\cdot\left(A\chi_{ct} + (B-A)\,\chi_t^E + q_{ct} + p_{ct}\right)\\ \leq& -\gamma_A m_A y_A(\chi_{ct}) + \|B-A\|\,M_A^2 y_A\left(\chi_t^E\right) + M_A\epsilon^{1/2}\|\chi\|_t + M_A p_{ct}.\end{aligned}$$

Because

$$\frac{d\chi_t^E}{dt} = B\chi_t^E + p_t^E + q_t^E,$$

we have

$$\begin{aligned}\frac{dy_B\left(\chi_t^E\right)}{dt} =& \nabla y_B\left(\chi_t^E\right)\cdot\left(\frac{d\chi_t^E}{dt}\right)\\ =& \nabla y_B\left(\chi_t^E\right)\cdot\left(B\chi_t^E+q_t^E+p_t^E\right)\\ \leq& -\gamma_B m_B y\left(\chi_t^E\right)+M_B\epsilon^{1/2}\left\|q\right\|_t+M_B\left\|p\right\|_t.\end{aligned}$$

The result follows from an application of Lemma 6 to $y_{ct}=y_A\left(\chi_{ct}\right)$ and $z_t=y_B\left(\chi_t^E\right)$.

□

The above Lemma has a simple extension.

Corollary 1. *Suppose that operators A and B are stable. There exists constants $P,Q<\infty$ and $\gamma,\epsilon>0$ such that for each $\chi\in D\overline{X}$, if (i) process $\chi_{ct}\in X$ satisfy (B.4) for each c and t, (ii)$\left\|\chi\right\|_0\leq\epsilon$, $\left\|p\right\|\leq\epsilon$, and (iii) for each t, $\left\|q\right\|_t\leq\frac{1}{2}\epsilon^{1/2}\left\|\chi\right\|_t+\zeta^*\left\|\chi\right\|_t^2$, then*

$$\left\|\chi\right\|_t\leq Pe^{-\gamma t}\left\|\chi\right\|_0+Q\max_{s\leq t}e^{-\gamma(t-s)}\left\|p\right\|_s.$$

Proof. Notice that as long as $\zeta^*\left\|\chi\right\|_t\leq\frac{1}{2}\epsilon^{1/2}$, then for each c, $\left\|q_{ct}'\right\|\leq\epsilon^{1/2}\left\|\chi\right\|_t$ and Lemma 1 applies. In particular, there are constants $P,Q<\infty$ and $\epsilon'>0$ such that

$$\zeta^*\left\|\chi\right\|_t\leq P\zeta^*\epsilon'+Q\zeta^*\epsilon'.$$

Thus, in order to ensure that $\zeta^*\left\|\chi\right\|_t\leq\frac{1}{2}\epsilon^{1/2}$, it is enough to require that,

$$\epsilon\leq\min\left(\epsilon',\left(2\left(P+Q\right)\zeta^*\right)^{-2}\right).$$

□

B.3. **Linear and time-varying differential equations.** Next, we describe another consequence of Lemma 7. Let $B\in\overline{L\left(X,X\right)}$ be a path of linear operators on finitely dimensional X. For each $x\in X$, let $y\left(B,x\right)\in\overline{X}$ be the solution to the differential equation

$$\frac{dy_t}{dt}=B_ty_t.$$

Lemma 8. *Suppose that linear operator $A\in L\left(X,X\right)$ is stable. There exists a constant $K<\infty$ and $\epsilon>0$ such that for each $x\in X$,*

(1) *for any two paths* $B, C \in \overline{L(X,X)}$ *st.* $\|B-A\|, \|C-A\| \leq \epsilon$,

$$\|y(B,x) - y(C,x)\| \leq K \|B-C\| \|x\|,$$

(2) *for any paths* $B, C \in \overline{L(X,X)}$ *st.* $\|B-A\|, \|B-C\| \leq \epsilon$,

$$\left\|\frac{1}{2}(y(B,x) + y(C,x)) - y\left(\frac{1}{2}B + \frac{1}{2}C, x\right)\right\| \leq K \|B-C\|^2.$$

Proof. It is enough to assume that $\|x\| = 1$. *Part* 1. Let $b_t = y_t(B,x)$, $c_t = y_t(C,x)$, and$\chi_t = b_t - c_t$. Then,

$$\begin{aligned}\frac{d\chi_t}{dt} &= B_t b_t - C_t c_t = A\chi_t + (B_t - A_t)\chi_t + (B_t - C_t)c_t.\\ &= A\chi_t + p_t + q_t,\end{aligned}$$

where $p_t = (B_t - A_t)\chi_t$, and $q_t = (B_t - C_t)c_t$. We have,$\|p\| \leq \epsilon \|\chi\|$, and, by part 1, $\|q\| \leq K\|B-C\|\|x\|$. The result follows from Lemma 7.

Part 2. Let $b_t = y_t(B,x)$, $c_t = y_t(C,x)$, $u_t = y_t\left(\frac{1}{2}B + \frac{1}{2}C, x\right)$, and

$$\chi_t = \frac{1}{2}(b_t + c_t) - u_t.$$

Then,

$$\begin{aligned}\frac{d\chi_t}{dt} =& \frac{1}{2}B_t b_t + \frac{1}{2}C_t c_t - \frac{1}{2}(B_t + C_t)u_t\\ =& A\chi_t + (B_t - A)\chi_t + (C_t - B_t)(c_t - u_t) = A\chi_t + p_t + q_t,\end{aligned}$$

where $p_t = (B_t - A)\chi_t$, $q_t = (C_t - B_t)(c_t - u_t)$. We have,$\|p\| \leq \epsilon\|\chi\|$, and, by part 2, $\|q\| \leq K\|B-C\|^2\|x\|$. The result follows from Lemma 7. □

Appendix C. Evolution of type distributions

In this Appendix, we analyze the evolution of type distributions.

C.1. **Evolution of population type distribution.** In this section, we compute a linear approximation to the evolution of the type distribution in population. Recall that $v(w, v_0) \in D\left(\overline{\Phi(\Theta)}\right)$ is the profile of the paths of type distributions in the cohorts given profile of strategies w and initial conditions v_0. Let $u(w, v_0) \in D\left(\overline{\Phi(\Theta)}\right)$

be a profile of solutions to a profile of linear equations

$$\frac{d}{dt}u_{ct} = \Gamma^*\left[u_{ct}\right] + \Gamma^{**}_{\Theta}\left[u^E_t\right] + \Gamma^{**}_{A}\left[w_{ct} - \alpha^*\right] + \Gamma^{**}_{B}\left[w^E_t - \alpha^*\right]. \tag{C.1}$$

(we use notation from Section 4 and letting $u_{ct} \approx v_{ct} - v^*$) with initial conditions $u_0(w, v_0) = v_0$. Also, define measurable function $\kappa^E : R \to L(\mathcal{A}, \Phi(\Theta))$

$$\left(\kappa^E(t)\right)[.] = \begin{cases} \exp\left(t\Gamma^*_{+\Theta}\right) \circ \left(\Gamma^*_{A+B}[.]\right), & \text{if } t \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

If operator $\Gamma^*_{+\Theta}$ is stable, then κ^E is exponentially bounded.

Lemma 9. *Suppose that the linear operators Γ^* and $\Gamma^*_{+\Theta}$ are stable on $\Phi(\Theta)$.*

(1) *There exist constants $\gamma, \epsilon > 0$ and $P, Q < \infty$ that depend only on the fundamentals of the model such that if $\|w - \sigma^*\| + \|v_0 - v^*\| \leq \epsilon$, then*

$$\|v_t(w, v_0) - v^*\| \leq Pe^{-\gamma t}\|v_0 - v^*\| + Q\max_{s \leq t} e^{-\gamma(t-s)}\|w - \sigma^*\|_s.$$

(2) *The solution u_{ct} to the system of differential linear equations (C.1) with the initial conditions $u_{c0} = v_{c0} - v^*$ is a second order approximation of $v(w, v_0)$. Moreover, $u^E(w, v_0)$ is a second order approximation of $v^E(w, v_0)$, where*

$$u^E_t(w, v_0) = \left(\kappa^E \star \left(w^E - \sigma^*\right)\right)(t) + \exp\left(t\Gamma^*_{+\Theta}\right)v_0,$$

and we take that $\left(w^E - \sigma^*\right)_t = 0$ for all $t < 0$.

Proof. To shorten the notation, write $v_{ct} = v_{ct}(w, v_0)$ for the induced evolution of type distributions. We use the following decomposition:

$$\begin{aligned}&\left(\int_C \sum_{\theta,\phi} v_{\xi t}(\phi)\,\gamma\left(w_{ct}(\theta), \theta, w_{\xi t}(\phi), \phi\right) d\mu^\lambda(\xi)\right) v_{ct}(\theta)\\ =&\Gamma^*\left[v_{ct} - v^*\right] + \Gamma^{**}_{\Theta}\left[v^E_t\right] + p^v_{ct} + q_{ct}\end{aligned} \tag{C.2}$$

where

$$p^v_t = \sum_{\theta,\phi}\left(\int_C \left(\gamma\left(w_{ct}(\theta), \theta, w_{\xi t}(\phi), \phi\right) - \gamma\left(\alpha^*(\theta), \theta, \alpha^*(\phi), \phi\right)\right) d\mu^\lambda(\xi)\right) v^*(\theta)\, v^*(\phi),$$

and

$$q_t^v = \left(\sum_{\theta,\phi} \gamma\left(\alpha^*\left(\theta\right), \theta, \alpha^*\left(\phi\right), \phi\right) d\mu^\lambda\left(\xi\right) \right) \left(v_{ct}\left(\theta\right) - v^*\left(\theta\right)\right) \left(v_t^E\left(\phi\right) - v^*\left(\phi\right)\right)$$

$$+ \sum_{\theta,\phi} \left(\int_C \left(v_{\xi t}\left(\phi\right) - v^*\left(\phi\right)\right) \left(\gamma\left(w_{ct}\left(\theta\right), \theta, w_{\xi t}\left(\phi\right), \phi\right) - \gamma\left(\alpha^*\left(\theta\right), \theta, \alpha^*\left(\phi\right), \phi\right)\right) d\mu^\lambda\left(\xi\right) \right) v_{ct}\left(\theta\right).$$

By the Taylor's formula and the Cauchy-Schwartz inequality, there exists constants $K_1, K_2, K_3 < \infty$ that depend only on the derivatives of function γ computed at the stationary equilibrium such that

$$\left\| p_t^v \right\| \leq K_1 \left\| w - \sigma \right\|, \tag{C.3}$$

and

$$\left\| q_t \right\| \leq K_2 \left\| v_t\left(w, v_0\right) - v^* \right\| \left\| w - \sigma^* \right\| + K_3 \left\| v_t\left(w, v_0\right) - v^* \right\|^2. \tag{C.4}$$

We prove each part of the Lemma separately.

(1) Define $\chi^v = v\left(w, v_0\right) - v^* \in D\left(\overline{\Phi\left(\Theta\right)}\right)$. Because of the decomposition (C.2),

$$\begin{aligned} \frac{d\chi_{ct}^v}{dt} &= \left(\int_C \sum_{\theta,\phi} v_{st}\left(\phi\right) \gamma\left(w_{ct}\left(\theta\right), \theta, w_{\xi t}\left(\phi\right), \phi\right) d\mu^\lambda\left(\xi\right) \right) v_{ct}\left(\theta\right) \\ &= \Gamma^*\left[\chi_{ct}^v\right] + \Gamma_\Theta^{**}\left[\chi_t^{v,E}\right] + p_{ct}^v + q_{ct}. \end{aligned}$$

By Corollary 1 and due to bounds (C.3) and (C.4), there exists constants $P < \infty$ and $\gamma, \epsilon_1 > 0$ such that $\left\| w - \sigma^* \right\| + \left\| v_0 - v^* \right\| \leq \epsilon_1$, then

$$\left\| v\left(w, v_0\right) - v \right\| \leq P e^{-\gamma t} \left\| v_0 - v^* \right\| + Q \max_{s \leq t} e^{-\gamma(t-s)} \left\| w - \sigma^* \right\|_s. \tag{C.5}$$

(2) Define $\chi = v\left(w, v_0\right) - u \in D\left(\overline{\Phi\left(\Theta\right)}\right)$. Then,

$$\begin{aligned} \frac{d\chi_{ct}}{dt} &= \left(\int_C \sum_{\theta,\phi} v_{st}\left(\phi\right) \gamma\left(w_{ct}\left(\theta\right), \theta, w_{\xi t}\left(\phi\right), \phi\right) d\mu^\lambda\left(\xi\right) \right) v_{ct}\left(\theta\right) \\ &- \left(\Gamma^*\left[u_{ct}\right] + \Gamma_\Theta^{**}\left[u_t^E\right] + \Gamma_A^{**}\left[w_{ct} - \alpha^*\right] + \Gamma_B^{**}\left[w_t^E - \alpha^*\right]\right). \end{aligned} \tag{C.6}$$

Because of the decomposition (C.2),

$$\frac{d\chi_{ct}}{dt} = \Gamma^*\left[\chi_{ct}\right] + \Gamma_\Theta^{**}\left[\chi_t^E\right] + p_t + q_t,$$

where

$$p_t = \sum_{\theta,\phi} \int_C \left(\gamma\left(w_{ct}\left(\theta\right), \theta, w_{\xi t}\left(\phi\right), \phi\right) - \gamma\left(\alpha^*\left(\theta\right), \theta, \alpha^*\left(\phi\right), \phi\right)\right) d\mu^\lambda\left(\xi\right) v^*\left(\theta\right) v^*\left(\phi\right)$$

$$- \gamma_{a;\theta,\phi}\left[w_{ct} - \alpha^*\right] - \gamma_{b,\theta,\phi}\left[w_t^E - \alpha^*\right].$$

By the Taylor's formula and the Cauchy-Schwartz inequality, there exists constant $K_5 < \infty$ that depend only on the derivatives of function γ computed at the stationary equilibrium such that

$$\|p_t\| \leq K_5 \|w - \sigma^*\|^2. \tag{C.7}$$

The claim follows from Corollary 1 and bounds (C.7) , (C.4), and (C.5).

□

C.2. **Evolution of private type distribution.** In this section, we analyze the evolution of the type distributions of a single player. Suppose that the population is characterized by a profile of strategies w and initial distributions v_0. Let $v\left(\theta, \sigma; w, v_0\right) \in \overline{\Delta\Theta}$ denote the path of the expected distributions over types of a player whose initial type is θ and who uses strategy σ. The path $v\left(\theta, \sigma; w, v_0\right)$ obeys the equation

$$\frac{d}{dt} v_t\left(\theta, \sigma; w, v_0\right) = \sum_{\theta',\phi} \int_C \gamma\left(\sigma\left(\theta'\right), \theta, w_{ct}\left(\phi\right), \phi\right) v_{ct}\left(\phi; w, v_0\right) v_t\left(\theta', \theta, \sigma; w, v_0\right) d\mu^\lambda\left(c\right). \tag{C.8}$$

with the initial condition $v_0\left(\theta, \sigma; w, v_0\right) = \delta_\theta$. The next Lemma describes some regularity properties of $v\left(\sigma; w, v_0\right)$.

Lemma 10. *Suppose that the linear operator Γ^* and $\Gamma^*_{+\Theta}$ are stable on $\Phi\left(\Theta\right)$. Then, there exists $K < \infty$ and $\epsilon > 0$ such that for any strategies σ, σ', profiles of strategies w, w', and initial type distributions v_0, v_0' such that $\|\sigma - \sigma^*\|, \|w - \sigma^*\|, \|w' - \sigma^*\|, \|v_0 - v^*\|, \|v_0' - v^*\|$ ϵ*

$$\|v\left(\sigma; w, v_0\right) - v\left(\sigma'; w', v_0'\right)\| \leq K\left(\|\sigma - \sigma'\| + \|w - w'\| + \|v - v'\|\right), \text{ and}$$

$$\left\|\frac{1}{2} v\left(\theta, \sigma; w, v_0\right) + \frac{1}{2} v\left(\theta, \sigma'; w, v_0\right) - v\left(\theta, \frac{1}{2}\sigma + \frac{1}{2}\sigma'; w, v_0\right)\right\| \leq K \|\sigma - \sigma'\|^2.$$

Proof. The result follows from Lemma 8. □

Next, we derive an approximation to the private type evolution $v(\theta,\sigma^*;w,v_0)$, i.e. the evolution when the player uses the stationary strategy σ^*. Define

$$u(\theta;w,v_0)=\exp(\Gamma^* t)[\delta_\theta] + \left(\int_0^t \left(\exp(\Gamma^*(t-s))\circ\left(\Gamma_B^*\left[w_s^E-\alpha^*\right]+\Gamma_\Theta^{**}\left[v_s^E(w,v_0)-v^*\right]\right)\right)\circ\exp(\Gamma^* s)\,ds\right)[\delta_\theta].$$

Lemma 11. *Suppose that the linear operator Γ^* and $\Gamma^*_{+\Theta}$ are stable on $\Phi(\Theta)$. Then, $u(\theta;w,v_0)$ is a second-order approximation to $v(\theta,\sigma^*;w,v^*)$.*

Proof. Notice that $u(\theta,\sigma;w,v_0)$ is a solution to equation:

$$\frac{d}{dt}u_t=\Gamma^*[u_t]+\left(\Gamma_B^*\left[w_t^E-\alpha^*\right]+\Gamma_\Theta^{**}\left[v_t^E(w,v_0)-v^*\right]\right)[v_t^*(\theta)]. \tag{C.9}$$

with the initial conditions $u_0=\delta_\theta$. Let $\chi=v(\theta,\sigma^*;w,v^*)-u(\theta;w,v_0)$. Then,

$$\begin{aligned}\frac{d\chi_t}{dt}=&\sum_{\theta',\phi}\int_C \gamma(\alpha^*(\theta'),\theta',w_{ct}(\phi),\phi)\,v_{ct}(\phi;w,v_0)\,v_t(\theta',\theta,\sigma^*;w,v_0)\,d\mu^\lambda(c)\\ &-\Gamma^*[u_t]-\left(\Gamma_B^*\left[w_t^E-\alpha^*\right]+\Gamma_\Theta^{**}\left[v_t^E(w,v_0)-v^*\right]\right)[v_t^*(\theta)]\\ =&\Gamma^*\chi_t+p_t.\end{aligned}$$

where

$$\begin{aligned}p_t=&\sum_{\theta',\phi}\int_C(\gamma(\alpha^*(\theta'),\theta',w_{ct}(\phi),\phi)-\gamma(\alpha^*(\theta'),\theta',\alpha^*(\phi),\phi))\\ &\qquad\cdot v_{ct}(\phi;w,v_0)\,v_t(\theta',\theta,\sigma^*;w,v_0)\,d\mu^\lambda(c)\\ &-\Gamma_B^*\left[w_t^E-\alpha^*\right]\\ =&\sum_{\theta',\phi}\int_C\left(\gamma(\alpha^*(\theta'),\theta',w_{ct}(\phi),\phi)-\gamma(\alpha^*(\theta'),\theta',\alpha^*(\phi),\phi)-\gamma_{b;\theta',\phi}\left[w_t^E(\phi)-\alpha^*(\phi)\right]\right)\\ &\qquad\cdot v^*(\phi)\,v_t(\theta',\theta,\sigma^*;w,v_0)\,d\mu^\lambda(c)\\ &+\sum_{\theta',\phi}\int_C(\gamma(\alpha^*(\theta'),\theta',w_{ct}(\phi),\phi)-\gamma(\alpha^*(\theta'),\theta',\alpha^*(\phi),\phi))\,v\\ &\qquad\cdot(v_{ct}(\phi;w,v_0)-v^*(\phi))\,v_t(\theta',\theta,\sigma^*;w,v_0)\,d\mu^\lambda(c)\end{aligned}$$

Because function γ is twice differentiable, the Cauchy-Schwartz inequality implies that there exists a constant $K' < \infty$ such that

$$\|p\| \leq K\left(\|w-\sigma^*\| + \|v\left(w, v_0\right) - v^*\|\right)^2.$$

The result follows from Lemma 9 and Lemma 7. □

Appendix D. Linear approximation to the best response function

In this part of the Appendix, we we derive an approximation to the best response function. Suppose that linear operator M^*_{AA} has an inverse and that linear operators Γ^* and $\Gamma^*_{+\Theta}$ are stable on $\Phi\left(\Theta\right)$. Define measurable functions $\kappa^V_\Theta : R \to L\left(\Phi\left(\Theta\right), \Phi\left(\Theta\right)\right)$, and $\kappa^V_B : R \to L\left(\mathcal{A}, \Phi\left(\Theta\right)\right)$

$$\left(\kappa^V_\Theta\left(t\right)\right)[.] = \begin{cases} M^*_\Theta[.] \circ \exp\left(\left(rI_{\Phi(\Theta)} - \Gamma^*\right)t\right), & \text{if } t \leq 0 \\ 0, & \text{otherwise.} \end{cases}$$

$$\left(\kappa^V_B\left(t\right)\right)[.] = \begin{cases} M^*_B[.] \circ \exp\left(\left(rI_{\Phi(\Theta)} - \Gamma^*\right)t\right), & \text{if } t \leq 0 \\ 0, & \text{otherwise.} \end{cases}$$

Define linear operator $A \in L\left(\mathcal{A}, \mathcal{A}\right)$

$$A = -M^{-1}_{AA} \circ M^*_{AB}.$$

Define measurable function $\kappa : R \to L\left(\mathcal{A}, \mathcal{A}\right)$

$$\left(\kappa\left(t\right)\right)[.] = -M^{-1}_{AA} \circ \hat{\Gamma}^*_A\left[\left(\left(\kappa^V_\Theta \star \kappa^E + \kappa^V_B\right)\left(t\right)\right)[.]\right] - M^{-1}_{AA} \circ M^*_{A\Theta}\left[\left(\kappa^E\left(t\right)\right)[.]\right].$$

Because κ is a (sum of) convolutions of exponentially bounded functions, it is exponentially bounded. Finally, define measurable function $c : R \to L\left(\Phi\left(\Theta\right), \mathcal{A}\right)$ as

$$\left(c\left(t\right)\right)[.] = -M^{-1}_{AA} \circ M^*_{A\Theta}\left[\exp\left(t\Gamma^*_{+\Theta}\right)[.]\right] - M^{-1}_{AA} \circ \hat{\Gamma}^*_A\left[\kappa^V_\Theta \star \exp\left(t\Gamma^*_{+\Theta}\right)[.]\right].$$

Function c is exponentially bounded. Given the assumptions, the operators and functions are well-defined and all the functions are exponentially bounded.

Lemma 12. *Suppose that linear operator M^*_{AA} has an inverse and that linear operators Γ^* and $\Gamma^*_{+\Theta}$ are stable on $\Phi\left(\Theta\right)$. Then, strategy $b^A\left(w, v_0\right) \in \overline{\mathcal{A}}$, where*

$$b^A_t\left(w, v_0\right) = A\left(w^E_t - \alpha^*\right) + \left(\kappa \star \left(w^E - \sigma^*\right)\right)_t + c_t\left[v_0 - v^*\right]$$

is a second-order approximation to the best response function $(b(w, v_0) - \sigma^*)$*. Moreover,* $\mathcal{F}\kappa(\omega) = K(\omega) + I_{\mathcal{A}} - A$

The proof is divided into 5 steps.

D.1. **Step 1: First-order approximation to the value function.** We are going to show that constant V^* is a first-order approximation to $V(w, v_0)$.

It is enough to show there exist constant $K < \infty$ and $\epsilon > 0$ such that for all profiles of strategies w and initial distributions v_0 st. $\|\sigma - \sigma^*\|, \|w - \sigma^*\|, \|v_0 - v^*\| \leq \epsilon$, any strategy σ,

$$\|G(\theta, \sigma, w, v_0) - G(\theta, \sigma, \sigma^*, v^*)\| \leq K(\|w - \sigma^*\| + \|v_0 - v_0^*\|).$$

Notice that

$$\begin{aligned}
&G(\theta, \sigma; w, v_0) - G(\theta, \sigma, \sigma^*, v^*) \\
=& \int_0^\infty e^{-rt}\left(\int_C \sum_{\phi,\theta_t} g(\sigma_t(\theta_t), \theta_t, \alpha^*(\phi), \phi)(v(\theta_t, \theta, \sigma; w, v_0) - v^*(\theta_t, \sigma, \sigma^*, v^*))\, v^*(\phi)\right) dt \\
&+ \int_0^\infty e^{-rt}\left(\int_C \sum_{\phi,\theta_t} g(\sigma_t(\theta_t), \theta_t, \alpha^*(\phi), \phi)\, v(\theta_t, \theta, \sigma; w, v_0)(v_{ct}(\phi; w, v_0) - v^*(\phi))\right) dt \\
&+ \int_0^\infty e^{-rt}\left(\int_C \sum_{\phi,\theta_t} (g(\sigma_t(\theta_t), \theta_t, w_{ct}, \phi) - g(\sigma_t(\theta_t), \theta_t, \alpha^*(\phi), \phi))\, v(\theta_t, \theta, \sigma; w, v_0)\, v_{ct}(\phi; w, v_0)\right) dt.
\end{aligned}$$

The result follows from Lemma 10 , Lemma 9, and the differentiability of function g.

D.2. **Step 2: An intermediary approximation to best response.** We are going to show that strategy $b^{A0}(w, v_0) \in \overline{\mathcal{A}}$, where

$$\begin{aligned}
&b_t^{A0}(w, v_0) - \sigma_t^* \\
=& - M_{AA}^{-1} \circ M_{AB}^*\left[w_t^E - \alpha^*\right] \\
&- M_{AA}^{-1} \circ M_{A\Theta}^*\left[v_t^E(w, v_0) - v^*\right] \\
&- M_{AA}^{-1} \circ \hat{\Gamma}_A^*\left[V_.(w, v_0) - V\right]
\end{aligned}$$

is a second-order approximation to the best response function $(b_t(w, v_0) - \alpha^*)$.

Define

$$M_t(a,\theta,w,v_0) = \int \sum_{\phi} \Big(g(a,\theta,w_{ct}(\phi),\phi) + (V_t(w,v_0))^T \gamma(a,\theta,w_{ct}(\phi),\phi) \Big) v_{ct}(\phi;w,v_0)\, d\mu^{\lambda}(c).$$

By Lemma 1, the period t best response action a of player with type θstrategy must maximize $M_t(a,\theta,w,v_0)$. The assumption that linear operator M^*_{AA} is negative definite implies that $M_t(.,\theta,\sigma^*,v^*)$ is strictly concave in a neighborhood of $\alpha^*(\theta)$ and, by the definition of equilibrium, it has a $M_t(.,\theta,\sigma^*,v^*)$ has a maximum at $\alpha^*(\theta)$. It follows the form of function M_t and Step 1 that $M_t(a,\dot{\theta},w,v_0)$is strictly concave for each t and all profiles of strategies w and type distributions v_0 in some neighborhood of the stationary equilibrium. Thus, $b_t \in \mathcal{A}$ is the period t best response action of player with type θ given w, v_0 if and only if it satisfies the first-order conditions

$$\int \sum_{\phi} \Big[M_a(b_t,\theta,w_{ct}(\phi),\phi) + (V_t(w,v) - V^*)^T \gamma_a(b_t,\theta,w_{ct}(\phi),\phi) \Big] v_{ct}(\phi)\, d\mu^{\lambda}(c) = 0, \tag{D.1}$$

where we use the notation of function M from Section 4.

In order to find an approximation to the best response, we compute a first-order approximation to equation (D.1) (to shorten the notation, we write $v_{ct}(\phi) = v_{ct}(\phi;w,v_0)$)

$$\begin{aligned}
&\int \sum_{\phi} M_a(b_t,\theta,w_{ct}(\phi),\phi)\, v_{ct}(\phi)\, d\mu^{\lambda}(c) \\
=& \sum_{\phi} M_a(\alpha^*(\theta),\theta,\alpha^*(\phi),\phi)\, v^*(\phi) \\
&+ \sum_{\phi} \Big(M_a(\alpha^*(\theta),\theta,\alpha^*(\theta),\phi) \Big(v^E_t(\phi) - v^*(\phi) \Big) \Big) \\
&+ \sum_{\phi} \int \left(M_a(b_t,\theta,w_{ct}(\phi),\phi)\, v_{ct}(\phi) - M_a(\alpha^*(\theta),\theta,\alpha^*(\phi),\phi)\, v_{ct}(\phi) \right) d\mu^{\lambda}(c) \\
&+ \sum_{\phi} \left(M_{aa,\theta,\phi}\left[b_t - \alpha^*(\theta) \right] \right) v^*(\phi) \\
&+ \sum_{\phi} \Big(M_{b;\theta,\phi}\Big[w^E_t - \alpha^* \Big] \Big) v^*(\phi) + p^M_t,
\end{aligned}$$

where

$$
\begin{aligned}
p_t^M =& \sum_\phi \int \left(\left(M_a \left(b_t, \theta, w_{ct}\left(\phi\right), \phi \right) - M_a \left(\alpha^*\left(\theta\right), \theta, \alpha^*\left(\phi\right), \phi \right) \right) \left(v_{ct}\left(\phi\right) - v^*\left(\phi\right) \right) \right) d\mu^\lambda\left(c\right) \\
&+ \sum_\phi \int \left(M_a \left(b_t, \theta, \alpha^*\left(\phi\right), \phi \right) - M_a \left(\alpha^*\left(\theta\right), \theta, \alpha^*\left(\phi\right), \phi \right) - M_{aa,\theta,\phi} \left[b_t - \alpha^*\left(\theta\right) \right] \right) d\mu^\lambda\left(c\right) v^*\left(\phi\right) \\
&+ \sum_\phi \left(\int \left(M_a \left(b_t, \theta, w_{ct}\left(\phi\right), \phi \right) - M_a \left(b_t, \theta, \alpha^*\left(\phi\right), \phi \right) \right) d\mu^\lambda\left(c\right) - M_{b;\theta,\phi} \left[w_t^E - \alpha^* \right] \right) v^*\left(\phi\right).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\int \sum_\phi \left[\left(V_t\left(w,v\right) - V^* \right)^T \gamma_a \left(b_t, \theta, w_{ct}\left(\phi\right), \phi \right) \right] v_{ct}\left(\phi\right) d\mu^\lambda\left(c\right) \\
=& \sum_\phi \left(V_t\left(w,v\right) - V^* \right)^T \gamma_a \left(\alpha^*\left(\theta\right), \theta, \alpha^*\left(\phi\right), \phi \right) v^*\left(\phi\right) + p_t^\gamma,
\end{aligned}
$$

where

$$
\begin{aligned}
p_t^\gamma =& \int \sum_\phi \left[\left(V_t\left(w,v\right) - V^* \right)^T \left(\gamma_a \left(\alpha^*\left(\theta\right), \theta, \alpha^*\left(\phi\right), \phi \right) - \gamma_a \left(b_t, \theta, w_{ct}\left(\phi\right), \phi \right) \right) \right] v^*\left(\phi\right) d\mu^\lambda\left(c\right) \\
&+ \int \sum_\phi \left[\left(V_t\left(w,v\right) - V^* \right)^T \gamma_a \left(b_t, \theta, w_{ct}\left(\phi\right), \phi \right) \right] \left(v_{ct}\left(\phi\right) - v^*\left(\phi\right) \right) d\mu^\lambda\left(c\right).
\end{aligned}
$$

By the Cauchy-Schwartz's inequality, Lemma 9, and Step 1 of the proof, there exist constant $K' < \infty$ that depend only on functions g and γ and such that

$$
\begin{aligned}
\left\| p_t^M \right\| \leq& K' \left(\|b - \sigma^*\| + \|w - \sigma^*\| \right) \left(\|w - \sigma^*\| + \|v_0 - v^*\| \right) \\
&+ K' \|b - \sigma^*\|^2 + K' \|w - \sigma^*\|^2, \\
\|p_t^\gamma\| \leq& K' \|w - \sigma^*\| \left(\|b - \sigma^*\| + \|w - \sigma^*\| \right) \\
&+ K' \left(\|b - \sigma^*\| + \|w - \sigma^*\| \right) \left(\|w - \sigma^*\| + \|v_0 - v^*\| \right).
\end{aligned}
\tag{D.2}
$$

We can rewrite the first order conditions (D.1) using the approximations together and the notation from Section 4 yields

$$
\begin{aligned}
M_{AA} \left[b_t\left(w, v_0\right) - \alpha^* \right] =& M_{AB}^* \left[w_t^E - \alpha^* \right] + M_{A\Theta}^* \left[v_t^E\left(w, v_0\right) - v^* \right] + \hat{\Gamma}_A^* \left[V_t\left(w, v_0\right) - V \right] \\
&+ p_t^M + p_t^\gamma.
\end{aligned}
$$

The above argument implies that $\|b(w, v_0) - \alpha^*\|$ can be made arbitrarily small. The Lemma follows from the above equality, approximations (D.2), and the fact that linear operator M^*_{AA} is invertible.

D.3. **Step 3: Envelope theorem.** We are going to show that $G(\sigma^*; w, v_0)$ is a second-order approximation to $V(w, v_0)$.

In order to shorten the notation, let

$$\sigma = \sigma^*, \sigma' = 2b(w, v_0) - \sigma^*, b = b(w, v_0),$$

for each strategy $s = \sigma, \sigma', b$, let

$$\begin{aligned} v^s &= v(\theta, s; w, v_0), \\ g^s_t(\theta) &= \int \sum_{\phi} g(s(\theta), \theta_t, \alpha^*(\phi), \phi) v_{ct}(w, v_0) d\mu^\lambda(c). \end{aligned}$$

We start with a preliminary observation. For each t, each type θ, the triangle inequality implies that

$$\begin{aligned} &\left\| \frac{1}{2} g^\sigma_t(\theta) v^\sigma(\theta) + \frac{1}{2} g^{\sigma'}_t(\theta) v^{\sigma'}(\theta) - g^b_t(\theta) v^b(\theta) \right\| \\ = &\left\| \frac{1}{2} g^\sigma_t(\theta) + \frac{1}{2} g^{\sigma'}_t(\theta) - g^b_t(\theta) \right\| \|v^*(\theta)\| \\ &+ \|g^*(\theta)\| \left\| \left(\frac{1}{2} v^\sigma(\theta) + \frac{1}{2} v^{\sigma'}(\theta) - v^b(\theta) \right) \right\| \\ &+ \frac{1}{2} \|g^\sigma_t(\theta) - g^*(\theta)\| \|v^\sigma(\theta) - v^*(\theta)\| + \frac{1}{2} \left\| g^{\sigma'}_t(\theta) - g^*(\theta) \right\| \left\| v^{\sigma'}(\theta) - v^*(\theta) \right\| \\ &- \left\| g^b_t(\theta) - g^*(\theta) \right\| \left\| v^b(\theta) - v^*(\theta) \right\|. \end{aligned}$$

Because function g is twice differentiable, and due to Lemma 10, there exists constant K' such that if $\|w - \sigma^*\| + \|v_0 - v^*\|$ is sufficiently small, then

$$\left\| \frac{1}{2} g^\sigma_t(\theta) v^\sigma(\theta) + \frac{1}{2} g^{\sigma'}_t(\theta) v^{\sigma'}(\theta) - g^b_t(\theta) v^b(\theta) \right\| \leq K' (2 \|b - \sigma\|)^2,$$

which implies that

$$\begin{aligned}
&\left\|\frac{1}{2}G\left(\sigma^{*};w,v_{0}\right)+\frac{1}{2}G\left(\sigma'\right)-G\left(\frac{1}{2}\sigma+\frac{1}{2}\sigma';w,v_{0}\right)\right\| \\
\leq&\int_{0}^{\infty}e^{-rt}\left(\sum_{\theta_{t}}\left\|\frac{1}{2}g_{t}^{\sigma}\left(\theta_{t}\right)v^{\sigma}\left(\theta_{t}\right)+\frac{1}{2}g_{t}^{\sigma'}\left(\theta_{t}\right)v^{\sigma'}\left(\theta_{t}\right)-g_{t}^{b}\left(\theta_{t}\right)v^{b}\left(\theta_{t}\right)\right\|\right)dt \\
\leq&\frac{1}{r}\left|\Theta\right|\left(K''\left(\left\|w-\sigma^{*}\right\|+\left\|v_{0}-v_{0}^{*}\right\|\right)^{2}\right). \qquad \text{(D.3)}
\end{aligned}$$

We can move to the proof of the Lemma. By (D.3),

$$\begin{aligned}
&\frac{1}{2}\left(G\left(b\left(w,v_{0}\right);w,v_{0}\right)-G\left(\sigma^{*};w,v_{0}\right)\right) \\
\leq&\frac{1}{2}\left(G\left(2b\left(w,v\right)-\sigma^{*};w,v_{0}\right)-G\left(b\left(w,v_{0}\right);w,v_{0}\right)\right) \\
&+\frac{1}{2}K\left(\left\|w-\sigma^{*}\right\|+\left\|v_{0}-v_{0}^{*}\right\|\right)^{2}.
\end{aligned}$$

Because $b(w,v)$ maximizes $G_t(.;w,v_0)$, it follows that

$$G\left(b\left(w,v_{0}\right);w,v_{0}\right)-G\left(\sigma^{*};w,v_{0}\right)\leq K\left(\left\|w-\sigma^{*}\right\|+\left\|v_{0}-v_{0}^{*}\right\|\right)^{2}.$$

Finally, notice that

$$V\left(w,v_{0}\right)-G\left(\sigma^{*};w,v_{0}\right)=G\left(b\left(w,v\right);w,v_{0}\right)-G\left(\sigma^{*};w,v_{0}\right).$$

D.4. **Step 4: (Second-order) approximation to value function.** We are going to show that function

$$\kappa_{\Theta}^{V}\star u^{E}\left(w,v_{0}\right)\left(t\right)+\kappa_{B}^{V}\star\left(w^{E}-\sigma^{*}\right)\left(t\right)$$

is a second-order approximation to $V_t(w,v_0)$.

Because of Step 3 of the proof, it is enough to derive the approximation to

$$G_{t}\left(\sigma^{*},w,v\right)-G_{t}\left(\sigma^{*},\sigma^{*},v^{*}\right)=\int_{t}^{\infty}e^{-r(s-t)}\Delta_{s}ds, \qquad \text{(D.4)}$$

where, for each period s,

$$\begin{aligned}
\Delta_{s}=&\int\sum_{\theta_{s},\phi}\left(g\left(\alpha^{*}\left(\theta_{s}\right),\theta_{s},w_{cs}\left(\phi\right),\phi\right)v\left(\theta_{s},\theta,\sigma^{*};w,v_{0}\right)v_{cs}\left(\phi;w,v_{0}\right)\right)d\mu^{\lambda}\left(c\right) \\
&-\sum_{\theta_{s},\phi}\left(g\left(\alpha^{*}\left(\theta_{s}\right),\theta_{s},\alpha^{*}\left(\phi\right),\phi\right)v_{s}^{*}\left(\theta_{s},\theta\right)v^{*}\left(\phi\right)\right).
\end{aligned}$$

In order to shorten the notation, let

$$\begin{aligned} v_{cs}(\phi) &= v_{cs}(\phi; w, v_0), \\ v_s(\theta_s, \theta) &= v_s(\theta_s, \theta, \sigma^*; w, v_0), \\ g^*_\phi(\theta) &= g(\alpha^*(\theta_s), \theta_s, \alpha^*(\phi), \phi) \end{aligned}$$

Notice that

$$\begin{aligned} \Delta_s =& \sum_{\theta_s,\phi} g^*_\phi(\theta_s) v^*_s(\theta_s, \theta) \left(v^E_s(\phi) - v^*(\phi)\right) \\ &+ \int \sum_{\theta_s,\phi} \left(g^*_\phi(\theta_s)(v_s(\theta_s, \theta) - v^*_s(\theta_s, \theta)) v^*(\phi)\right) d\mu^\lambda(c) \\ &+ \int \sum_{\theta_s,\phi} \left(\left(g(\alpha^*(\theta_s), \theta_s, w_{cs}(\phi), \phi) - g^*_\phi(\theta_s)\right) v^*_s(\theta_s, \theta) v^*(\phi)\right) d\mu^\lambda(c) \\ &+ p_s, \end{aligned}$$

where

$$\begin{aligned} p_s =& \int \sum_{\theta_s,\phi} \left(\left(g(\alpha^*(\theta_s), \theta_s, w_{cs}(\phi), \phi) v(\theta_s, \theta) - g^*_\phi(\theta_s) v^*_s(\theta_s, \theta)\right)(v_{cs}(\phi) - v^*(\phi))\right) d\mu^\lambda(c) \\ &+ \int \sum_{\theta_s,\phi} \left(\left(g(\alpha^*(\theta_s), \theta_s, w_{cs}(\phi), \phi) - g^*_\phi(\theta_s)\right)(v_s(\theta_s, \theta) - v^*(\theta_s, \theta)) v^*(\phi)\right) d\mu^\lambda(c). \end{aligned}$$

Using the Cauchy-Schwartz's inequality, we can show that there exists $K' < \infty$ such that for all profiles of strategies w and initial distributions v_0, if $\|w - \sigma^*\| + \|v_0 - v^*\| \leq \epsilon'$, then

$$\begin{aligned} \|p_s\| \leq& K' (\|w - \sigma^*\| + \|v(., \theta) - v^*(., \theta)\|) \|v(.) - v^*(.)\| \\ &+ K' \|w - \sigma^*\| \|v(., \theta) - v^*(., \theta)\|. \end{aligned}$$

Due to Lemmas 9 and 10, there exist constants $K'' < \infty$ and $\epsilon'' > 0$ such that if $\|w - \sigma^*\| + \|v_0 - v^*\| \leq \epsilon''$,

$$\|p_s\| \leq K'' (\|w - \sigma^*\| + \|v_0 - v^*\|)^2. \tag{D.5}$$

We use (D.5), the approximations derived in Lemmas 9, 11, and the fact that function g is twice differentiable with bounded derivatives to derive an approximation to

$$
\begin{aligned}
&\Delta_s \\
=&\left(\sum_{\phi} g_{\phi}^{*}\left(v_s^E(\phi)-v^*(\phi)\right)\right)\circ\exp\left(\Gamma^*(s-t)\right)[\delta_\theta] \\
&+g^*\circ\left(\left(\int_0^{s-t}\left(\exp\left(\Gamma^*(s-t-u)\right)\circ\Gamma_B^*\left[w_{t+u}^E-\alpha^*\right]\right)\circ\exp\left(\Gamma^* u\right)du\right)[\delta_\theta]\right) \\
&+g^*\circ\left(\left(\int_0^{s-t}\left(\exp\left(\Gamma^*(s-t-u)\right)\circ\Gamma_\Theta^{**}\left[v_{t+u}^E-v^*\right]\right)\circ\exp\left(\Gamma^* u\right)du\right)[\delta_\theta]\right) \\
&+\sum_{\theta_s,\phi}\left(v^*(\phi)\,g_{b;\theta_s,\phi}^*\left[w_s^E(\phi)-\alpha^*(\phi)\right]\right)v_s^*(\theta_s,\theta)+q_s,
\end{aligned}
$$

where, for some constant $K''' < \infty$,

$$
\|q_s\|\le K'''\left(\|w-\sigma^*\|+\|v_0-v^*\|\right)^2.
$$

We can proceed now with an approximation to the derive an approximation to (D.4). It is convenient to first consider only the terms with the average type distribution $v_s^E - v^*$. The Fubini's theorem and some algebra imply that

$$
\begin{aligned}
&\int_t^{\infty} e^{-r(s-t)}\left(\sum_{\phi} g_{\phi}^{*}\left(v_s^E(\phi)-v^*(\phi)\right)\right)\circ\exp\left(\Gamma^*(s-t)\right)[\delta_\theta]\,ds \\
+&\int_t^{\infty} e^{-r(s-t)} g^*\circ\left(\left(\int_0^{s-t}\left(\exp\left(\Gamma^*(s-t-u)\right)\circ\Gamma_\Theta^{**}\left[v_{t+u}^E-v^*\right]\right)\circ\exp\left(\Gamma^* u\right)du\right)[\delta_\theta]\right)ds \\
=&\int_0^{\infty} e^{-rs}\left(M_\Theta^*\left[v_{t+s}^E-v^*\right]\right)\circ\exp\left(\Gamma^* s\right)[\delta_\theta]\,ds,
\end{aligned}
$$

and

$$\int_t^\infty e^{-r(s-t)} g^* \circ \left(\left(\left(\int_0^{s-t} \left(\exp\left(\Gamma^* \left(s-t-u\right)\right) \circ \Gamma_B^* \left[w_{t+u}^E - \alpha^* \right] \right) \circ \exp\left(\Gamma^* u\right) du \right) [\delta_\theta] \right) ds$$
$$+ \int_t^\infty e^{-r(s-t)} \sum_\phi \left(v^* \left(\phi\right) g_{b;.,\phi}^* \left[w_s^E \left(\phi\right) - \alpha^* \left(\phi\right) \right] \right) \circ \exp\left(\Gamma^* \left(s-t\right)\right) [\delta_\theta] \left(\theta_s\right) ds$$
$$= \int_0^\infty e^{-rs} \left(M_B^* \left[v_{t+s}^E - v^* \right] \right) \circ \exp\left(\Gamma^* s\right) [\delta_\theta] \, ds.$$

This concludes the proof of the Lemma.

D.5. **Proof of Lemma 12.** By Lemma 9, and steps 2 and 4 of the proof,

$$\begin{aligned}
&- M_{AA}^{-1} \circ M_{AB}^* \left[w_t^E - \alpha^* \right] \\
&- M_{AA}^{-1} \circ M_{A\Theta}^* \left[\exp\left(t \Gamma_{+\Theta}^* \right) v_0 \right] - M_{AA}^{-1} \circ \hat{\Gamma}_A^* \left[\left(\kappa_\Theta^V \star \exp\left((.)\, \Gamma_{+\Theta}^* \right) (t) \right) [v_0] \right] \\
&- M_{AA}^{-1} \circ M_{A\Theta}^* \left[\kappa^E \star \left(w^E - \sigma^* \right) (t) \right] \\
&- M_{AA}^{-1} \circ \hat{\Gamma}_A^* \left[\kappa_\Theta^V \star \left(\kappa^E \star \left(w^E - \sigma^* \right) \right) (t) \right] \\
&- M_{AA}^{-1} \circ \hat{\Gamma}_A^* \left[\kappa_B^V \star \left(w^E - \sigma^* \right) (t) \right]
\end{aligned}$$

is a second-order approximation of $b_t\left(w, v_0\right) - \sigma_t^*$. The properties of convolution and Fourier transform imply that for each $\omega \in R$,

$$\begin{aligned}
\left(\mathcal{F}\kappa\left(\omega\right)\right)[.] = &- M_{AA}^{-1} \circ \hat{\Gamma}_A^* \left[\left(\mathcal{F}\kappa_\Theta^V \left(\omega\right) \right) \left(\mathcal{F}\kappa^E \left(\omega\right) \right) [.] + \left(\mathcal{F}\kappa_B^V \left(\omega\right) \right) [] \right] \\
&- M_{AA}^{-1} \circ M_{A\Theta}^* \left[\left(\mathcal{F}\kappa^E \left(\omega\right) \right) [] \right].
\end{aligned}$$

we compute

$$\begin{aligned}
\mathcal{F}\kappa_{\Theta}^{V}(\omega) =& M_{\Theta}^{*}[.] \circ \left(\int_{0}^{\infty} \exp\left(- \left(r I_{\Phi(\Theta)} + 2\pi i \omega I_{\Phi(\Theta)} - \Gamma^{*} \right) t \right) dt \right) \\
=& M_{\Theta}^{*}[.] \circ \left(r I_{\Phi(\Theta)} + 2\pi i \omega I_{\Phi(\Theta)} - \Gamma^{*} \right)^{-1}, \\
\mathcal{F}\kappa^{E}(\omega)[.] =& \left(\int_{0}^{\infty} \exp\left(- \left(2\pi i \omega I_{\Phi(\Theta)} - \Gamma_{+\Theta}^{*} \right) \right) dt \right) \circ \left(\Gamma_{A+B}^{*}[.] \right) \\
=& \left(2\pi i \omega I_{\Phi(\Theta)} - \Gamma_{+\Theta}^{*} \right)^{-1} \circ \left(\Gamma_{A+B}^{*}[.] \right), \\
\mathcal{F}\kappa_{B}^{V}(\omega) =& M_{B}^{*}[.] \circ \left(r I_{\Phi(\Theta)} + 2\pi i \omega I_{\Phi(\Theta)} - \Gamma^{*} \right)^{-1}.
\end{aligned}$$

Substitutions yield

$$\begin{aligned}
& \left(\mathcal{F}\kappa(\omega) \right)[.] \\
=& - M_{AA}^{-1} \circ \hat{\Gamma}_{A}^{*} \circ \left(M_{\Theta}^{*} \left[\left(2\pi i \omega I_{\Phi(\Theta)} - \Gamma_{+\Theta}^{*} \right)^{-1} \circ \left(\Gamma_{A+B}^{*}[.] \right) \right] \right) \circ \left(r I_{\Phi(\Theta)} + 2\pi i \omega I_{\Phi(\Theta)} - \Gamma^{*} \right)^{-1} \\
& - M_{AA}^{-1} \circ \hat{\Gamma}_{A}^{*} \circ \left(M_{B}^{*}[.] \right) \circ \left(r I_{\Phi(\Theta)} + 2\pi i \omega I_{\Phi(\Theta)} - \Gamma^{*} \right)^{-1} \\
& - M_{AA}^{-1} \circ M_{A\Theta}^{*} \circ \left(2\pi i \omega I_{\Phi(\Theta)} - \Gamma_{+\Theta}^{*} \right)^{-1} \circ \left(\Gamma_{A+B}^{*}[.] \right).
\end{aligned}$$

Appendix E. Proof of Theorem 1

Suppose that Γ^{*} is stable. If $\Gamma_{\Theta+}^{*}$ is stable, then the stability of the type distribution follows from Lemma 7 and the fact that $\Gamma_{+\Theta}^{*}[\upsilon]$ is a second-order approximation to

$$B(\upsilon) = \sum_{\theta,\phi} \gamma\left(\alpha^{*}(\theta), \theta, \alpha^{*}(\phi), \phi \right) \left(v^{*}(\phi) + \upsilon(\phi) \right) \left(v^{*}(\theta) + \upsilon(\theta) \right).$$

For the second part of the Theorem, suppose that $\Gamma_{\Theta+}^{*}$ is unstable. Choose any $\upsilon \in \Phi(\Theta)$ and consider an initial perturbation $v_{c}^{0} = v^{*} + \upsilon_{0}$ for each cohort c. Let $\upsilon_{t} \in \Phi(\Theta)$ be a process such that $v_{ct}(\sigma^{*}, v^{0}) = v^{*} + \upsilon_{t}$ for each t. (Notice that the type distribution in each cohort evolves in the same way.) Then, υ_{t} is a Markov process described by a differential equation

$$\frac{d\upsilon_{t}}{dt} = B(\upsilon_{t})$$

and $\Gamma^*_{\Theta+}$ is the Jacobian of function $B(.)$ computed at the stationary point $\upsilon^* = 0$. It is well-known that if υ^*_0 is an eigenvector that corresponds to an eigenvalue of $\Gamma^*_{\Theta+}$ with a strictly positive real part, then there exists $\eta > 0$ such that for each $\epsilon > 0$, if $\upsilon_0 = \epsilon \upsilon^*_0$, then there exists t such that $\|\upsilon_t\| \geq \eta$.

Appendix F. Proof of Theorem 2: Best response dynamics

Explain. The instability on in the last part.

F.1. **Notation.** Let $\gamma_1, \epsilon_1 > 0$ and $P_1, Q_1 < \infty$ be the constants from the first part of Lemma 9.

Let κ and c be the exponentially bounded functions let A be the operator from Lemma 12. Let $P_\kappa, P_c < \infty$ and $\rho_\kappa, \rho_c > 0$ be constants such that for each t,

$$\|\kappa(t)\| \leq P_\kappa e^{-\rho_\kappa |t|} \text{ and } \|c(t)\| \leq P_c e^{-\rho_c |t|}.$$

Let $\|\kappa\| = \sup_t \|\kappa(t)\|$. Let $K < \infty$ be the constant that determines the quality of the second-order approximation in part **??** of Lemma 12. Also, Lemma 12 implies that σ^* is a first-order approximation to $b(w, v)$ and that there exists a constant $K_1 < \infty$ such that for each w and v,

$$\|b(w, v) - \sigma^*\| \leq K_1 (\|w - \sigma^*\| + \|v - v^*\|).$$

Let

$$\Gamma_\kappa = \frac{2\|A\|}{\rho_k} + \frac{8P_\kappa}{\rho_\kappa^2}. \tag{F.1}$$

Define a functional linear operators $\mathcal{K}, \mathcal{K} + I_A$ on $\overleftrightarrow{\mathcal{A}}$: for each $h \in \overleftrightarrow{\mathcal{A}}$, let

$$\mathcal{K}[h] = A[h] + \kappa \star h,$$
$$(\mathcal{K} + I_A)[h] = (A + I_A)[h] + \kappa \star h.$$

Then, part **??** of Lemma 12 implies that

$$(\mathcal{K} + I_A)[h]$$

is an approximation to the best response strategy given that the opponents play unbounded strategy h. (The approximation drops the second-order terms, as well as the term $c[v]$ that accounts for the effect of the initial type distribution.)

F.2. **Stability of the approximate linearized dynamics.** It is convenient to consider first an approximate dynamics on the paths of unbounded strategies $\overleftrightarrow{\mathcal{A}}$. The approximation is based on the linearized version of the best response established in Lemma 12.

Consider a path of unbounded strategies $h^t \in \overleftrightarrow{\mathcal{A}}$ that satisfies the following functional integral equations:

$$h^t = \lambda \int_0^t e^{-\lambda(t-s)} \left(\mathcal{K} + I_A\right)[h^s]\, ds \text{ for each } t \geq 0. \tag{F.2}$$

One can show that for each h^0, there exists a unique path h^t that satisfies (F.2). Moreover, if $e^{\lambda\mathcal{K}t}$ is the operator exponential defined in Appendix A.3, then the solution to the unique solution to the integral equations is given by

$$h^t = e^{\lambda\mathcal{K}t} h^0.$$

The next results characterizes the conditions under which $e^{\lambda\mathcal{K}t} \to 0$ or $e^{\lambda\mathcal{K}t} \to \infty$ as $t \to \infty$.

Lemma 13. *Suppose that the family of linear operators $\{A + \hat{\kappa}(\omega) : \omega \in R\}$ is uniformly stable. Then, there exists $P < \infty$ and $\rho > 0$ such that for each $h \in L^\infty(\mathcal{A})$, each $t \geq 0$,*

$$\left\| e^{\lambda\mathcal{K}t} h \right\|_{L^\infty} \leq P e^{-\lambda\rho t} \left\| h \right\|_{L^\infty}. \tag{F.3}$$

Moreover, for each $h \in L^\infty(\mathcal{A})$, and each s_0, if we define function $h_{f,s_0}(s) = \begin{cases} h(s), & \text{if } s \leq s_0 \\ 0, & \text{if } s > s_0 \end{cases}$, then for each t, s,

$$\left\| \left(e^{\lambda\mathcal{K}t} h_{f,s_0} \right)(s) \right\| \leq e^{\lambda \frac{1}{2}\rho_\kappa \Gamma_\kappa t - \frac{1}{2}\rho_\kappa (s - s_0)} \left\| h \right\|_{\mathcal{L}^\infty}.$$

F.2.1. *Square-integrable version of Lemma 13.* We divide the proof of Lemma 13 into steps. In the first step, we establish a square-integrable version of the first part of the Lemma. (Notice that the existence of square-integrable solution follows from the same fact as the existence of the $\mathcal{L}^\infty$-solution.) We show that there exists $P_2 < \infty$ and $\rho_2 > 0$ such that for each initial conditions $h \in \mathcal{L}^2(R, \mathcal{A})$ each $t \geq 0$,

$$\left\| e^{\lambda\mathcal{K}t} h \right\|_{\mathcal{L}^2} \leq P_2 e^{-\lambda\rho_2 \tau} \left\| h \right\|_{\mathcal{L}^2}.$$

Notice that (up to a constant), $\left\|e^{\lambda\mathcal{K}t}h\right\|_{\mathcal{L}^2} = \left\|\widehat{e^{\lambda\mathcal{K}t}h}\right\|_{\mathcal{L}^2}$. Thus, it is enough to show that there exists $P_2 < \infty$ and $\gamma_2 > 0$ such that for each initial conditions $h \in \mathcal{L}^p(\mathcal{A})$, each $\tau \geq 0$,

$$\left\|\widehat{e^{\lambda\mathcal{K}t}h}\right\|_{\mathcal{L}^2} \leq P_2 e^{-\lambda\rho_2\tau}\left\|\widehat{h}\right\|_{l^2}.$$

By the properties of the Fourier transform, we get for each $\omega \in R$, each $t \geq 0$,

$$\left(\widehat{e^{\lambda\mathcal{K}t}h}\right)(\omega) = (A + I_A + \hat{\kappa}(\omega))\left[\lambda\int_0^t e^{-\lambda(t-s)}\left(\widehat{e^{\lambda\mathcal{K}s}h}\right)(\omega)\,ds\right],$$

and with the initial For each ω, the above equation is a matrix-valued integral equation with a simple solution

$$\left(\widehat{e^{\lambda\mathcal{K}t}h}\right)(\omega) = e^{\lambda(A+\hat{\kappa}(\omega))t}\left[\hat{h}(\omega)\right],$$

where $e^{\lambda(A+\hat{\kappa}(\omega))t}$ is matrix exponential defined in Appendix A.3.

Because $\hat{\kappa}(\omega) \to 0$ when $|\omega| \to \infty$, family $\{(A + \hat{\kappa}(\omega)) : \omega \in R\}$ is relatively compact. By the assumption, it is also uniformly stable. By Lemma 2, there exists $P_2 < \infty$ and $\rho_2 > 0$ such that for each ω,

$$\left\|\left(\widehat{e^{\lambda\mathcal{K}t}h}\right)(\omega)\right\|_{\mathcal{L}^2} \leq P_2 e^{-\rho_2\tau}\left\|\hat{h}(\omega)\right\|.$$

Thus,

$$\begin{aligned}\left\|\left(\widehat{e^{\lambda\mathcal{K}t}h}\right)\right\| &= \left(\int\left\|\left(\widehat{e^{\lambda\mathcal{K}t}h}\right)(\omega)\right\|^2 d\omega\right)^{1/2} \\ &\leq P_2 e^{-\gamma\rho_2\tau}\left(\int\left\|\hat{h}(\omega)\right\|^2 d\omega\right)^{1/2} = P_2 e^{-\rho_2\tau}\left\|\hat{h}\right\|.\end{aligned}$$

This completes the proof of the first part of the proof.

F.2.2. *Relation between $\mathcal{L}^\infty$-convergence to $\mathcal{L}^2$-convergence.* The proof in the case $\mathcal{L}^\infty$ is a bit more complicated. The idea is to reduce it to the $\mathcal{L}^2$ case. We start with two preliminary results. The first result relates $\mathcal{L}^\infty$-convergence to $\mathcal{L}^2$-convergence for square integrable and bounded functions.

Lemma 14. *There exist constants $P' < \infty$ and $\rho' > 0$, such that if $h \in \mathcal{L}^2(\mathcal{A}) \cap \mathcal{L}^\infty(\mathcal{A})$, then for each $\tau \geq 0$,*

$$\left\| e^{\lambda \mathcal{K} \tau} h \right\|_{\mathcal{L}^\infty} \leq P' e^{-\lambda \rho' \tau} \left(\|h\|_{\mathcal{L}^2} + \|h\|_{\mathcal{L}^\infty} \right).$$

Proof. Let $y_A : \mathcal{A} \to R$ be the function from Lemma 4 chosen for operator A (note that operator A is stable, because it belongs to the closure of family $\{A + \hat{\kappa}(\omega) : \omega \in R\}$). Define $m_A, M_A > 0$ as, respectively, the minimum and the maximum value of function $y_A(v)$ on the unit sphere $v \in \{v \in \mathcal{A} : \|v\| = 1\}$. Let

$$\gamma = -\max_{v:\|v\|=1} \nabla y_A(v) \cdot Av > 0.$$

Fix $h \in \mathcal{L}^2(\mathcal{A}) \cap \mathcal{L}^\infty(\mathcal{A})$. The first part of the proof of the Lemma implies that for each $h \in \mathcal{L}^2(\mathcal{A})$, each $\tau \geq 0$

$$\left\| e^{\lambda \mathcal{K} \tau} h \right\|_{\mathcal{L}^2} \leq P_2 e^{-\gamma_2 \tau} \|h\|_{\mathcal{L}^2}.$$

Let

$$y^\tau = \left\| y_A \circ e^{\lambda \mathcal{K} \tau} h \right\|_{\mathcal{L}^\infty}.$$

Then, $m_A \left\| e^{\lambda \mathcal{K} \tau} h \right\|_{\mathcal{L}^\infty} \leq y^\tau \leq M_A \left\| e^{\lambda \mathcal{K} \tau} h \right\|_{\mathcal{L}^\infty}$.

Let $\rho' = \min(\gamma m_A, \rho_2)$. By Cauchy-Schwartz inequality, for each t

$$\begin{aligned}
\frac{dy^\tau}{d\tau} =& \frac{dy_A\left(\left(e^{\lambda \mathcal{K} \tau} h\right)(t)\right)}{d\tau} \\
=& \lambda \nabla y_A(h^\tau(t)) \cdot \left(A\left(e^{\lambda \mathcal{K} \tau} h\right)(t) + \left(\kappa \star \left(e^{\lambda \mathcal{K} \tau} h\right)\right)(t) \right) \\
\leq& -\lambda \gamma \|h^\tau(t)\| + \lambda \|\kappa\|_2 \left\| e^{\lambda \mathcal{K} \tau} h \right\|_2 \\
\leq& -\lambda \gamma m_A y^\tau + P_2 \|\kappa\|_2 e^{-\lambda \rho_2 \tau} \|h\|_{\mathcal{L}^2} \\
\leq& -\rho' y^\tau + P \|\kappa\|_2 e^{-\rho' \tau} \|h\|_{\mathcal{L}^2}
\end{aligned}$$

It follows that

$$\begin{aligned}
y^\tau &\leq e^{-\rho' \tau} \left(y^0 + P_2 \|\kappa\|_2 \|h\|_{\mathcal{L}^2} \right) \\
&\leq e^{-\rho' \tau} \left(P_2 \|\kappa\|_2 + M_A \right) \left(\|h\|_{\mathcal{L}^2} + \|h\|_{\mathcal{L}^\infty} \right).
\end{aligned}$$

The result follows from the fact $\left\| e^{\lambda \mathcal{K} \tau} \right\|_{\mathcal{L}^\infty} \leq \frac{1}{m_A} y^\tau$. ∎

F.2.3. *Bound on the long-distance impact.* The next result shows that the impact of distant regions remains limited through initial period of dynamics.

Lemma 15. *For each $h \in \mathcal{L}^{\infty}(\mathcal{A})$, if $h(t) = 0$ for each $t \geq 0$, then, for each $\tau \geq 0$, each t,*

$$\left\| \left(e^{\lambda \mathcal{K} \tau} h \right)(t) \right\| < e^{-\frac{1}{2}\rho_{\kappa}(t - \lambda \Gamma_{\kappa} \tau)} \|h\|_{\mathcal{L}^{\infty}}.$$

(Note that the constant Γ_{κ} is defined in (F.1).)

Proof. Fix $h \in \mathcal{L}^{\infty}(\mathcal{A})$. For each $\tau \geq 0$, and each t, define

$$y^{\tau}(t) = e^{\frac{1}{2}\rho_{\kappa}(t - \lambda \Gamma_{\kappa} \tau)} \left(e^{\lambda \mathcal{K} \tau} h \right)(t).$$

Then, $\|y^0\|_{\mathcal{L}^{\infty}} \leq \|h\|_{\mathcal{L}^{\infty}}$. We are going to show that $\|y^{\tau}\|_{\mathcal{L}^{\infty}} \leq \|h\|_{\mathcal{L}^{\infty}}$ for each $\tau \geq 0$. Indeed, suppose that the claim holds for some $\tau \geq 0$. Then, for each $t > 0$,

$$\begin{aligned}
\frac{d \left\| \left(e^{\lambda \mathcal{K} \tau} h \right)(t) \right\|}{d\tau} &\leq \lambda \|A\| \left\| \left(e^{\lambda \mathcal{K} \tau} h \right)(t) \right\| + \lambda \int_{-\infty}^{\infty} P_{\kappa} e^{-\rho_{\kappa}|u|} \left\| \left(e^{\lambda \mathcal{K} \tau} h \right)(t-u) \right\| du \\
&\leq \lambda \|A\| \|y^{\tau}\| e^{-\frac{1}{2}\rho_{\kappa}(t - \lambda \Gamma_{\kappa} \tau)} + \lambda \int_{-\infty}^{\infty} P_{\kappa} e^{-\rho_{\kappa}|u|} \|y^{\tau}\| e^{-\frac{1}{2}\rho_{\kappa}((t-u) - \lambda \Gamma_{\kappa} \tau)} du \\
&\leq \lambda \left(\|A\| + \int_{-\infty}^{\infty} P_{\kappa} e^{-\frac{1}{2}\rho_{\kappa}|u|} du \right) \|y^{\tau}\| e^{-\frac{1}{2}\rho_{\kappa}(t - \lambda \Gamma_{\kappa} \tau)} \\
&\leq \lambda \left(\|A\| + \frac{4}{\rho_{\kappa}} P_{\kappa} \right) \|y^{\tau}\| e^{-\frac{1}{2}\rho_{\kappa}(t - \lambda \Gamma_{\kappa} \tau)}
\end{aligned}$$

and

$$\begin{aligned}
\frac{d \|y^{\tau}(t)\|}{d\tau} &= \frac{d \left\| \left(e^{\lambda \mathcal{K} \tau} h \right)(t) \right\|}{d\tau} e^{\frac{1}{2}\rho_{\kappa}(t - \lambda \Gamma_{\kappa} \tau)} - \frac{1}{2} \lambda \rho_{\kappa} \Gamma_{\kappa} \|y^{\tau}(t)\| \\
&\leq \lambda \left(\|A\| + \frac{4}{\rho_{\kappa}} P_{\kappa} - \frac{1}{2} \rho_{\kappa} \Gamma_{\kappa} \right) \|y^{\tau}\| \leq 0,
\end{aligned}$$

where the last inequality comes from the definition of Γ_{κ}. □

F.2.4. *Proof of Lemma 13.* We can conclude the proof of the Lemma. For any function $h \in \mathcal{L}^{\infty}(\mathcal{A})$ and each τ, t, define

$$h_{c,t}(s) = \mathbf{1}_{s\in[t-2\lambda\Gamma_{\kappa}\tau, t-2\lambda\Gamma_{\kappa}\tau]} h(s),$$
$$h_{f,t}(s) = \mathbf{1}_{s\notin[t-2\lambda\Gamma_{\kappa}\tau, t-2\lambda\Gamma_{\kappa}\tau]} h(s).$$

Then, $h = h_{c,t} + h_{f,t}$ and

$$\left(e^{\lambda\mathcal{K}\tau}h\right)(t) = e^{\lambda\mathcal{K}\tau}h_{c,t}(t) + e^{\lambda\mathcal{K}\tau}h_{f,t}(t) \tag{F.4}$$

We bound the two terms separately.

- Due to Lemma 14 and the fact that $\|h_{c,t}\|_{\mathcal{L}^2} \leq 4\lambda\Gamma_{\kappa}\tau \|h\|_{\mathcal{L}^{\infty}}$, we get

$$\left\|e^{\lambda\mathcal{K}\tau}h_{c,t}(t)\right\| \leq (4\lambda\Gamma_{\kappa}\tau + 1) P' e^{-\lambda\rho'\tau} \|h\|_{\mathcal{L}^{\infty}}.$$

- Due to Lemma 15, and the fact that $t - (t - 2\lambda\Gamma_{\kappa}t) = 2\lambda\Gamma_{\kappa}t$, for each t,

$$\left\|\left(e^{\lambda\mathcal{K}\tau}h_{f,t}\right)(t)\right\| \leq 2e^{-\frac{1}{2}\rho_{\kappa}(2\lambda\Gamma_{\kappa}\tau - \lambda\Gamma_{\kappa}\tau)} \|h\|_{\mathcal{L}^{\infty}},$$

Then, for each t,

$$\begin{aligned}\left\|e^{\lambda\mathcal{K}\tau}h(t)\right\| &\leq \left((4\lambda\Gamma_{\kappa}\tau + 1) P' e^{-\lambda\rho'\tau} + 2e^{-\frac{1}{2}\rho_{\kappa}(2\lambda\Gamma_{\kappa}\tau - \lambda\Gamma_{\kappa}\tau)}\right) \|h\|_{\mathcal{L}^{\infty}} \\ &\leq Pe^{-\lambda\rho\tau} \|h\|_{\mathcal{L}^{\infty}},\end{aligned} \tag{F.5}$$

where

$$P = 2 + \max_{\tau} (4\Gamma_{\kappa}\tau + 1) P' e^{-\frac{1}{2}\rho'\tau} < \infty,$$
$$\rho = \min\left(\frac{1}{2}\rho_{\kappa}\Gamma_{\kappa}, \frac{1}{2}\rho'\right) > 0.$$

The second part of the Lemma follows from Lemma 15.

F.3. **Sufficient conditions for stability of the best response dynamics.** Suppose that family $\{K(\omega) : \omega \in R\}$ is uniformly stable. We are going to show that the stationary equilibrium is λ-stable for sufficiently small λ. Let w^{τ} and v^{τ} be the dynamics initiated by the initial perturbation w^0 and v^0.

It is convenient to define an auxiliary dynamics on the paths of unbounded strategies $\overleftrightarrow{\mathcal{A}}$. For each $s \in R$ and $t \geq 0$, let

$$\varpi_s^t = \begin{cases} w_{s-t}^{t,E} - \alpha^*, & \text{if } s \geq t \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, for $t \leq s$, h_s^t is equal to the average action that is supposed to be played in period s as it is planned in period t. For $t > s$, h_s^t is equal to 0. Because the revision opportunities arrive independently across all players, the average action played in the population in period $s \geq t \geq 0$ is equal to

$$\varpi_s^t = e^{-\lambda t} h_s^0 + \lambda \int_0^t \left(b_{s-u}\left(w^u, v^u\right) - \alpha^*\right) e^{-\lambda(t-u)} du. \tag{F.6}$$

Using the second-order approximation from Lemma 12, we can rewrite (F.6) as

$$\varpi^t = \left(\mathcal{K} + I_{\mathcal{A}}\right) \left[\lambda \int_0^t \varpi^u e^{-\lambda(t-u)} du\right] + e^t + f^t,$$

where for all $s \in R$ and $t \geq 0$,

$$\begin{aligned} e_s^t =& \mathbf{1}_{s \geq t} e^{-\lambda t} \varpi_s^0 + \lambda \mathbf{1}_{s \geq t} \int_0^t c_{s-u}\left[v^u - v^*\right] e^{-\lambda(t-u)} du \\ &+ \mathbf{1}_{s \geq t} \left(\lambda \int_0^t \left(b_{s-u}\left(w^u, v^u\right) - \left(\mathcal{K} + I_{\mathcal{A}}\right)\left[\varpi^u\right] - c_{s-u}\left[v^u - v^*\right]\right)_s e^{-\lambda(t-u)}\right) du, \end{aligned}$$

and

$$f_s^t = \mathbf{1}_{s<t} \lambda \int_0^t \left(\left(\mathcal{K} + I_{\mathcal{A}}\right)\left[\lambda \int_0^t \varpi^u e^{-\lambda(t-u)} du\right]\right)_s e^{-\lambda(t-u)} du.$$

We are going to show that the dynamics of the above equation are determined by the first term, and all the remaining term e_t becomes small for large t. More precisely, choose constants so that all the inequalities below are satisfied for all $\lambda \leq \lambda^*$:

$$\rho_{\varpi} < \frac{1}{2}, \frac{1}{4}\rho_{\kappa}$$

$$\lambda^* < \frac{1}{2}\left(\Gamma_{\kappa}\right)^{-1}, \frac{1}{\rho_{\varpi}}\gamma_1$$

$$P_{\varpi} < \left(P\left(1+\rho-\rho_{\varpi}\right)^{-1} + P_e\right) + \frac{\lambda}{\lambda+\rho_{\kappa}}, \tag{F.7}$$

$$P_e < 1 + \frac{\lambda}{\rho_c + \lambda\left(1-p_{\varpi}\right)} P_c^t P_v + \epsilon \frac{1}{1-2\rho_{\varpi}} K\left(P_w + P_v\right)^2, \tag{F.8}$$

$$P_w < 1 + \left(\|A+I\| + \|\kappa\|\right) P_{\varpi} + \frac{\lambda}{1+\lambda} P_c P_v + \epsilon K\left(P_w + P_v\right)^2, \tag{F.9}$$

$$P_v < P_1 + Q_1 P_w. \tag{F.10}$$

(Note that it is possible to satisfy all the above inequalities when λ^* and $\epsilon \leq \epsilon_1$ are sufficiently small.) Then, we show that if $\lambda < \lambda^*$ and $\|w^0 - \sigma^*\| + \|v^0 - v^*\| \leq \epsilon$, then for each $t > \hat{t}$, each $\omega \in \Omega$,

(1) $\sup_{s \geq t}\|\varpi_s^t\| \leq P_{\varpi} e^{-\rho_{\varpi}\lambda t}\epsilon$,
(2) $\|e^t\| \leq P_e e^{-\rho_{\varpi}\lambda t}\epsilon$,
(3) $\|w^t - \sigma^*\| \leq P_w e^{-\rho_{\varpi}\lambda t}\epsilon$,
(4) $\|v^t - v^*\| \leq P_v e^{-\rho_{\varpi}\lambda t}\epsilon$.

Indeed, let T^* be the set of periods in which at least one of the bounds above is not satisfied and let $t^* = \inf T^*$. If $t^* = \infty$, then $w^\tau \to \sigma^*$ and $v^\tau \to v^*$ and the result holds. On the contrary, suppose that $t^* < \infty$. Below, we show that given the appropriate choice of the constants, if all inequalities are satisfied for all $t \leq t^*$, then the remaining inequality is satisfied strictly at $t = t^*$. Because all the objects above are continuous, this contradicts the fact that $t^* < \infty$.

Suppose that all inequalities hold for all $t \leq t^*$.

(1) We show that inequality 2 is satisfied strictly at $t = t^*$. We can check directly that the unique solution to the integral equation (F.2) given e^t and f^t is given by

$$\varpi^t = \lambda \int_0^t e^{\mathcal{K}(t-s)-\lambda(t-s)}\left[e^s + f^s\right] ds + e_t + f_t.$$

Due to the inductive assumption and the first part Lemma 13,

$$\begin{aligned}
&\left\|\lambda\int_0^t e^{\mathcal{K}(t-s)-\lambda(t-s)}\left[e^s\right]ds+e_t\right\| \\
&\leq\left\|\lambda P\int_0^t e^{-\lambda\rho(t-s)}e^{-\lambda(t-s)}e^{-\lambda\rho_\varpi s}ds\right\|+P_e e^{-\rho_\varpi\lambda t}\epsilon \\
&\leq\left(P\left(1+\rho-\rho_\varpi\right)^{-1}+P_e\right)e^{-\lambda\rho_\varpi t}\epsilon.
\end{aligned}$$

Also, by the second part of Lemma 13,

$$\begin{aligned}
&\sup_{s\geq t}\left\|\lambda\int_0^t e^{-\lambda(t-u)}\left(e^{\mathcal{K}(t-u)}\left[f^u\right]\right)_s ds+f_s^t\right\| \\
&\leq\lambda\int_0^t e^{-\lambda(t-u)}\sup_{s\geq t}\left\|\left(e^{\mathcal{K}(t-u)}\left[f^u\right]\right)\right\|ds \\
&\leq\lambda\int_0^t e^{-\lambda(t-u)}\sup_{s\geq t}\left(e^{\lambda\frac{1}{2}\rho_\kappa\Gamma_\kappa t-\frac{1}{2}\rho_\kappa(t-u)}\right)\epsilon ds.
\end{aligned}$$

Because $\lambda\Gamma_\kappa<\frac{1}{2}$, the latter is not larger than

$$\leq\lambda\int_0^t e^{-\lambda(t-u)}e^{-\frac{1}{4}\rho_\kappa(t-u)}ds\leq\frac{\lambda}{\lambda+\rho_\kappa}e^{-\frac{1}{4}\rho_\kappa t}\epsilon.$$

The result follows from the choice of constants, and, specifically, inequality (F.7).

(2) We show that inequality 2 is satisfied strictly at $t = t^*$. Because $\rho_\varpi < 1$, we have $\left\| e^{-\lambda t}\varpi^0 \right\| \leq e^{-\lambda \rho_\varpi t}\epsilon$. Next, due to the inductive assumption,

$$\left\| \lambda \mathbf{1}_{s\geq t} \int_0^t c_{s-u}\left[v^u - v^*\right] e^{-\lambda(t-u)} du \right\|$$
$$\leq \sup_{s\geq t} \lambda \int_0^t P_c^t e^{-\rho_c(s-u)} P_v e^{-\rho_\varpi \lambda u} \epsilon e^{-\lambda(t-u)} du$$
$$\leq P_c^t P_v e^{-(\lambda+\rho_c)t} \left(\lambda \int_0^t e^{(\rho_c + \lambda(1-p_\varpi))u} du \right) \epsilon$$
$$\leq \frac{\lambda}{\rho_c + \lambda\left(1 - p_\varpi\right)} P_c^t P_v e^{-\lambda p_\varpi t}\epsilon.$$

Finally, due to Lemma 12,

$$\left\| b_{s-u}\left(w^u, v^u\right) - \left(\mathcal{K} + I_{\mathcal{A}}\right)\left[\varpi^u\right] - c_{s-u}\left[v^u - v^*\right] \right\| \tag{F.11}$$
$$\leq K\left(\left\|w^u - \sigma^*\right\| + \left\|v^u - v^*\right\|\right)^2$$
$$\leq K\left(P_w + P_v\right)^2 e^{-2\lambda p_\varpi u}\epsilon^2,$$

where the last inequality comes from the inductive assumption. Because $\rho_\varpi < \frac{1}{2}$,

$$\left\| \lambda \mathbf{1}_{s\geq t} \int_0^t \left(b_{s-u}\left(w^u, v^u\right) - \left(\mathcal{K} + I_{\mathcal{A}}\right)\left[\varpi^u\right] - c_{s-u}\left[v^u - v^*\right]\right)_s e^{-\lambda(t-u)} du \right\|$$
$$\leq e^{-\lambda t}\lambda \int_0^t K\left(P_w + P_v\right)^2 e^{-2\lambda p_\varpi u + \lambda u}\epsilon^2 du$$
$$\leq \epsilon \frac{1}{1 - 2\rho_\varpi} K\left(P_w + P_v\right)^2 e^{-\lambda \rho_\varpi t}\epsilon.$$

The result follows from the choice of constants, and, specifically, inequality (F.8).

(3) We show that inequality 3 is satisfied strictly at $t = t^*$. By (F.11) and the inductive assumption, for each $u \geq 0$,

$$\begin{aligned}\|b(w^u, v^u)\| =& \sup_{s\geq u} \|b_{s-u}(w^u, v^u)\| \\ &\leq (\|A+I\| + \|\kappa\|) P_\varpi e^{-\rho_\varpi \lambda t}\epsilon + P_c P_v e^{-(s-u)} e^{-\lambda p_\varpi t}\epsilon \\ &\quad + \epsilon K (P_w + P_v)^2 e^{-\lambda \rho_\varpi t}\epsilon.\end{aligned}$$

Thus, because $p_\varpi < 1$,

$$\begin{aligned}\left\|w^t - \sigma^*\right\| =& e^{-\lambda t}\left\|w^0 - \sigma^*\right\| + \int_0^t \|b(w^u, v^u)\| e^{-\lambda(t-u)} du \\ &\leq \left(1 + (\|A+I\| + \|\kappa\|) P_\varpi + +\epsilon K (P_w + P_v)^2\right) e^{-\lambda p_\varpi t}\epsilon \\ &\quad + \lambda P_c P_v \int_0^t e^{-(t-u)} e^{-\lambda(t-u)} du \\ &\leq \left(1 + (\|A+I\| + \|\kappa\|) P_\varpi + \frac{\lambda}{1+\lambda} P_c P_v + \epsilon K (P_w + P_v)^2\right) e^{-\lambda p_\varpi t}\epsilon.\end{aligned}$$

The result follows from the choice of constants, and, specifically, inequality (F.9).

(4) We show that inequality 4 is satisfied strictly at $t = t^*$. By Lemma 9 and the inductive assumption,

$$\begin{aligned}\left\|v^t - v^*\right\| \leq& P_1 e^{-\gamma_1 t}\left\|v^0 - v^*\right\| + Q_1 \max_{s\leq t} e^{-\gamma_1(t-s)} \|w^s - v^*\|. \\ \leq& P_1 e^{-\gamma_1 t}\epsilon + Q_1 P_w \max_{s\leq t} e^{-\gamma_1(t-s)} e^{-\lambda\rho_\varpi s}\epsilon.\end{aligned}$$

Because $\lambda\rho_\varpi < \gamma_1$, the latter is not larger than $(P_1 + Q_1) e^{-\lambda\rho_\varpi t}\epsilon$. The result follows from the choice of constants, and, specifically, inequality (F.10).

F.4. **Instability .**

Appendix G. Proof of Theorem 3: Learning Dynamics

G.1. **Approximate predictions.** We are going to define an approximation to the prediction strategies defined in (???). Let $w \in \overline{\mathcal{A}}$ be a (continuous) path of actions.

For each τ and each $\omega \in \Omega$, define $a^t_{\sin}(\omega; w)$ and $a^t_{\cos}(\omega, w)$ so to minimize

$$\int_0^t \left(w_s - \sum_{\omega \in \Omega} \left(a^t_{\sin}(\omega; w) \sin(2\pi\omega s) + a^t_{\cos}(\omega; w) \cos(2\pi\omega s) \right) \right)^2.$$

Additionally, for each $s > t > 0$,define a complex generalized actions

$$\begin{aligned} a^t(\omega; w) &= \frac{2}{t} \int_0^t \left(w_0^{s,E} - \alpha^* \right) e^{-i2\pi\omega s} ds, \\ w^{\text{app},t}_{s-t} &= \text{Re}\left(a^t(\omega; w) \right) \cos(2\pi\omega s) - \text{Im}\left(a^t(\omega; w) \right) \sin(2\pi\omega s) \\ &= \text{Re}\left(a^t(\omega; w) e^{2\pi\omega s} \right). \end{aligned}$$

,

Lemma 16. *For each $\delta > 0$, there exists $t_\delta < \infty$ such that for each $\tau \geq t_\delta$,*

$$\max_{\omega \in \Omega} \left\| a^\tau(\omega; w) - \alpha^* \mathbf{1}_{\omega=0} - a^\tau_{\cos}(\omega; w) + i a^\tau_{\sin}(\omega; w) \right\| \leq \delta \max_{\omega} \left\| a^\tau(\omega; w) \right\|.$$

Proof. Let $F = \{\mathbf{1}\} \cup \{\sin(2\pi\omega.), \cos(2\pi\omega.) : \omega \in \Omega \setminus \{0\}\}$ be a finite collection of functions We show that □

Let A and $\kappa(.)$ be as in Lemma 12. For each complex generalized action a, compute

$$\begin{aligned} & A w^{\text{app},\tau}_{t-\tau} + \int_{-\infty}^{\infty} \kappa(t-s) w^{\text{app},\tau}_{s-\tau} ds \\ =& A\left(\text{Re}(a) \cos(2\pi\omega t) - \text{Im}(a) \sin(2\pi\omega t) \right) \\ & + \int_{-\infty}^{\infty} \kappa(t-s) \left(\text{Re}(a) \cos(2\pi\omega s) - \text{Im}(a) \sin(2\pi\omega s) \right) ds \\ =& A\left(\text{Re}\left(a e^{i2\pi\omega t} \right) \right) + \text{Re} \int_{-\infty}^{\infty} \kappa(t-s) a e^{i2\pi\omega s} ds \\ =& \text{Re}\left(\left(K(\omega) + I_{\mathcal{A}} \right) a e^{i2\pi\omega t} \right). \end{aligned}$$

Finally, notice that

$$\int_{\hat{t}}^{t} \left(\text{Re}\left(a^E_s(\omega') e^{2\pi i s \omega'} \right) \right) e^{-2\pi i \omega s} ds$$

$$\frac{2}{t}\int\limits_0^t \mathrm{Re}\left(\left(K\left(\omega\right)+I_{\mathcal{A}}\right)ae^{i2\pi\omega s}\right)e^{-i2\pi\omega s}ds$$
$$=\frac{2}{t}\int\limits_0^t \mathrm{Re}\left(\left(K\left(\omega\right)+I_{\mathcal{A}}\right)ae^{i2\pi\omega s}\right)e^{-i2\pi\omega s}ds$$

G.2. **Sufficient conditions for stability of the learning dynamics.**

Structure of the proof. Suppose that family $\{K(\omega):\omega\in\Omega\}$ is uniformly stable. We are going to show that the stationary equilibrium is (Ω,λ)-stable for sufficiently small λ. Let w^τ and v^τ be the dynamics initiated by the initial perturbation w^0 and v^0. A preliminary result says that the best response dynamics do not grow infinitely quickly. The proof can be found at the end of this section.

Lemma 17. *For each $\phi>0$ and $\epsilon>0$, there exists $\epsilon'_{\epsilon,\phi}>0$ such that if $\|w^0-\sigma^*\|+\|v^0-v^*\|\leq\epsilon'_{\epsilon,\phi}$, then $\|w^t-\sigma^*\|+\|v^t-v^*\|\leq\epsilon'_{\epsilon,\phi}$ for each $t\leq\hat{t}=\lambda^{-1}\phi$. (The $\epsilon'_{\epsilon,\phi}$ does not depend on λ.)*

Below, we are going to show that there exists $\rho,\epsilon>0$ and $P_w,\phi<\infty$ such that if $\left\|w^{\hat{t}}-\sigma^*\right\|+\left\|v^{\hat{t}}-v^*\right\|\leq\epsilon$, then

$$\left\|w^t-\sigma^*\right\|+\left\|v^t-v^*\right\|\leq P_w t^{-\rho}$$

for each $t\geq\hat{t}=\lambda^{-1}\phi$. Together with the above Lemma, this shall conclude the proof of the stability of the dynamics.

For each $t>0$, and each $\omega\in\Omega$, define

$$a_t^E\left(\omega\right)=\frac{\left(1+\mathbf{1}_{\omega\neq0}\right)}{t}\int\limits_0^t\left(w_0^{s,E}-\alpha^*\right)e^{-i2\pi\omega s}ds,\ \text{and}\tag{G.1}$$

$$e_t\left(\omega\right)=a_t^E\left(\omega\right)-\left(K\left(\omega\right)+I_{\mathcal{A}}\right)\left[\frac{1}{t}\int\limits_{\hat{t}}^t a_s^E\left(\omega\right)ds\right].\tag{G.2}$$

We are going to show that the error term $e_t(\omega)$ is small relative to $a_t^E(\omega)$. Moreover, in a sequence of approximations, we are going to show that there exist constants

$\rho, \epsilon > 0 > 0$, and $\phi, P_a, P_e, P_w, P_{wp}, P_v < \infty$ such that if $\lambda < \lambda^*$ and $\left\|w^{\hat{t}} - \sigma^*\right\| + \left\|v^{\hat{t}} - v^*\right\| \leq \epsilon$, then for each $t > \hat{t}$, each $\omega \in \Omega$,

(1) $\left\|a_t^E(\omega)\right\| \leq P_a \left(\frac{\hat{t}}{t}\right)^\rho \epsilon$,
(2) $\left\|e_t(\omega)\right\| \leq P_e \left(\frac{\hat{t}}{t}\right)^\rho \epsilon$,
(3) $\left\|w^t - \sigma^*\right\| \leq P_w \left(\frac{\hat{t}}{t}\right)^\rho \epsilon$,
(4) $\left\|w^{P,t} - \sigma^*\right\| \leq P_{wP} \left(\frac{\hat{t}}{t}\right)^\rho \epsilon$,
(5) $\left\|v^t - v^*\right\| \leq P_v \left(\frac{\hat{t}}{t}\right)^\rho \epsilon$.

Constants. We begin with defining the constants. Let $\gamma_1, \epsilon_1 > 0$ and $P_1, Q_1 < \infty$ be the constants from the first part of Lemma 9. By Lemma 3, there exist constants $P < \infty$ and $\rho > 0$such that for each ω and each $a \in \mathcal{A}^{\mathcal{C}}$,

$$\left\|e^{K(\omega)t}a\right\| \leq Pe^{-2\rho t}\left\|a\right\|. \tag{G.3}$$

We assume w.l.o.g. that $\rho < \frac{1}{2}$. Let exponentially bounded functions κ and c be as in Lemma 12. Let $K < \infty$ be the constant that determines the quality of the second-order approximation in Lemma 12. It follows from Lemma 12 that there exists a constant $K_1 < \infty$ such that for each w and v,

$$\left\|b(w, v) - \sigma^*\right\| \leq K_1 \left(\left\|w - \sigma^*\right\| + \left\|v - v^*\right\|\right).$$

Let $P_c < \infty$ and $\rho_c > 0$ be the constants from the definition of the exponentially bounded function c. Let $\|\kappa\| = \sup_t \|\kappa(t)\|$.

For each t^*, let

$$r_{t^*} = \max_{t \geq t^*} \max_{\omega, \omega' \in \Omega \setminus \{0\}} \begin{pmatrix} \frac{1}{t}\left|\int_0^t \cos^2(2\pi\omega s)\, ds - \frac{1}{2}\right|, \frac{1}{t}\left|\int_0^t \sin^2(2\pi\omega s)\, ds - \frac{1}{2}\right| \\ \left|\frac{1}{t}\int_0^t \cos(2\pi\omega s)\, ds\right|, \left|\frac{1}{t}\int_0^t \sin(2\pi\omega s)\, ds\right|, \\ \left|\frac{1}{t}\int_0^t \cos(2\pi\omega' s)\sin(2\pi\omega s)\, ds\right|. \end{pmatrix}$$

Find $\lambda^*, \epsilon > 0$, $\phi < \infty$, and $P_1, ..., P_5 < \infty$ such that for each $\lambda < \lambda^*$,

$$\phi > 1, \tag{G.4}$$

$$\rho < \gamma_1, \tag{G.5}$$

$$P_a > P_e \sum_{\omega'} \left(P \left\| K(\omega') + I \right\| \frac{1}{\rho} + 1 \right),$$

$$P_e > 1 + \frac{1}{\phi} + \frac{\lambda}{1+\lambda} P_c P_v \frac{2}{1-\rho} + \epsilon \left(K (P_{wP} + P_v)^2 \right) \frac{2}{1-\rho} \tag{G.6}$$

$$\sum_{\omega' \in \Omega} \left\| K(\omega') + I_{\mathcal{A}} \right\| \left(\left(\lambda^{-1} \phi \right)^{-\rho} \left(3 + P_a + P_w \frac{1}{1-\rho} \right) + \epsilon P_{wP} \frac{2}{1-\rho} \right)$$

$$+ 2 \sum_{\omega' \in \Omega, \omega' \neq \omega} \frac{1}{\lambda^{-1}\phi} \left\| K(\omega') + I_{\mathcal{A}} \right\| \frac{8}{\omega' - \omega} \left(P_w \frac{1}{\rho} + 1 \right).$$

$$P_w > 1 + \|\kappa\| \, |\Omega| \, P_a + \frac{\lambda}{\lambda + \gamma_c} P_c P_v + \epsilon K (P_{wP} + P_v)^2,$$

$$P_{wP} = 2 P_w,$$

$$P_v = P_1 + Q_1 P_w,$$

$$r_{\lambda^{-1}\phi} \leq \frac{1}{10 |\Omega|} \epsilon. \tag{G.7}$$

(To see that all these inequalities can be satisfied, choose first constants P_a, P_w, P_{wP}, P_v assuming that $P_e = 2$ and then choose ϕ large enough, and λ and ϵ small enough so that $P_e \leq 2$.) Let $\hat{t} = \lambda^{-1}\phi$. Then for each $s \geq \hat{t}$,

$$e^{\lambda \left(s - \hat{t} \right)} \geq 1 + \lambda \left(s - \hat{t} \right) \geq 1 + \lambda \frac{s - \hat{t}}{\hat{t}} \left(\frac{1}{\lambda} \phi \right) \geq 1 + \frac{s - \hat{t}}{\hat{t}} = \frac{s}{\hat{t}}, \tag{G.8}$$

where the last inequality holds because of (G.4).

It is easy to check that inequalities (1)-(5) hold for $t = \hat{t}$. The proof that the inequalities are satisfied for all $t \geq \hat{t}$ follows by a continuous version of induction on t. Below, we are going to assume that all but one inequality holds for all $t \leq t'$, and conclude that the remaining inequality is satisfied strictly at $t = \hat{t}$.

Inequality 1. Notice that (G.2) implies that

$$a_t^E(\omega) = \int_{\hat{t}}^{t} s^{-1} \exp\left(K(\omega)(\log t - \log s)\right)(K(\omega) + I_{\mathcal{A}})\, e_s(\omega)\, ds + e_t(\omega).$$

(This follows from the uniqueness of the solution to the integral equation (G.2) given a path of error terms $(e_t(\omega))_t$.) Then, by (G.3) and the inductive assumption,

$$\begin{aligned}
\left\|a_t^E(\omega)\right\| \leq & P \int_{\hat{t}}^{t} s^{-1} P\left(\frac{s}{t}\right)^{2\rho} \|K(\omega) + I_{\mathcal{A}}\| P_e \left(\frac{\hat{t}}{s}\right)^{\rho} \epsilon ds + P_e \left(\frac{\hat{t}}{t}\right)^{\rho} \epsilon \\
= & P_e \left(P \|K(\omega) + I_{\mathcal{A}}\| \left(t^{-2\rho}\hat{t}^{\rho} \int_{\hat{t}}^{t} (s)^{-1+\rho} ds\right) + \left(\frac{\hat{t}}{t}\right)^{\rho}\right) \epsilon \\
\leq & P_e \left(P \|K(\omega) + I_{\mathcal{A}}\| \frac{1}{\rho} + 1\right) \left(\frac{\hat{t}}{t}\right)^{\rho} \epsilon < P_a \left(\frac{\hat{t}}{t}\right)^{\rho} \epsilon.
\end{aligned}$$

Inequality 2. Because the revision opportunities rive independently across all players, notice that the average profile of actions played in the population in period $s \geq \hat{t}$ is equal to

$$w_0^{s,E} - \alpha^* = e^{-\lambda(s-\hat{t})}\left(w_{s-\hat{t}}^{\hat{t},E} - \alpha^*\right) + \lambda \int_{\hat{t}}^{s} \left(b_{s-u}\left(w^{P,u}, v^u\right) - \alpha^*\right) e^{-\lambda(s-u)} du. \quad \text{(G.9)}$$

Using the above, we can write

$$\begin{aligned}
e_t(\omega) = & e_t^1(\omega) + \frac{2}{t} \int_{\hat{t}}^{t} \left(\lambda \int_{\hat{t}}^{s} e_{s,u}^2(\omega)\, e^{-\lambda(s-u)} du\right) e^{-2\pi i \omega s} ds \\
& + \frac{2}{t} \int_{\hat{t}}^{t} \left(\lambda \int_{\hat{t}}^{s} \left(\operatorname{Re}\left(\sum_{\omega' \in \Omega} (K(\omega') + I_{\mathcal{A}}) \left[e_u^3(\omega')\right] e^{2\pi i s \omega'}\right)\right) e^{-\lambda(s-u)} du\right) e^{-2\pi i \omega s} ds \\
& + \frac{2}{t} \int_{\hat{t}}^{t} \left(\operatorname{Re}\left(\sum_{\omega' \in \Omega} (K(\omega') + I_{\mathcal{A}}) \left[e_s^4(\omega')\right] e^{2\pi i s \omega'}\right)\right) e^{-2\pi i \omega s} ds \\
& + \sum_{\omega' \in \Omega, \omega' \neq \{\omega\}} e_{t,\omega'}^5(\omega) + e_t^6(\omega)
\end{aligned}$$

Let

$$e_t^1(\omega) = \frac{\hat{t}}{t} a_{\hat{t}}^E(\omega) + \frac{1}{t}\int_{\hat{t}}^{t} \left(e^{-\lambda\left(s-\hat{t}\right)}\left(w_{s-\hat{t}}^{\hat{t},E} - \alpha^*\right)\right) e^{-2\pi i\omega s} ds,$$

$$e_{s,u}^2(\omega) = b_{s-u}\left(w^{P,u}, v^u\right) - \alpha^* - \sum_{\omega'\in\Omega} \mathrm{Re}\left(\left(K(\omega') + I_{\mathcal{A}}\right)\left[a_{\cos}^{\tau,E}(\omega') - i a_{\sin}^{u,E}(\omega')\right] e^{2\pi i s\omega'}\right),$$

$$e_u^3(\omega') = \begin{cases} a_{\cos}^{u,E}(\omega') - i a_{\sin}^{u,E}(\omega') - a_u^E(\omega'), & \text{if } \omega' \neq 0 \\ a_{\cos}^{u,E}(\omega') - \mathrm{Re}\left(a_u^E(\omega')\right), & \text{if } \omega' = 0. \end{cases},$$

$$e_s^4(\omega') = \lambda \int_{\hat{t}}^{s} a_u^E(\omega') e^{-\lambda(s-u)} - a_s^E(\omega'),$$

$$e_{t,\omega'}^5(\omega) = \frac{2}{t}\int_{\hat{t}}^{t} \mathrm{Re}\left(\left(K(\omega') + I_{\mathcal{A}}\right)\left[a_s^E(\omega')\right] e^{2\pi i s\omega'}\right) e^{-2\pi i\omega s} ds,$$

$$e_t^6(\omega) = \frac{2}{t}\int_{\hat{t}}^{t} \mathrm{Re}\left(\left(K(\omega) + I_{\mathcal{A}}\right)\left[a_s^E(\omega)\right] e^{2\pi i s\omega}\right) e^{-2\pi i\omega s} ds - \left(K(\omega) + I_{\mathcal{A}}\right)\left[\frac{1}{t}\int_{\hat{t}}^{t} a_s^E(\omega) ds\right]$$

Next, we are going to provide bounds on terms $e_{\cdot}(\omega)$. The bounds are collected at the end of this subsection.

(1) Because $\|w^s - \sigma^*\| \leq \epsilon$ for each $s \leq \hat{t}$ and because of the definition of a_s^E, we have

$$\left\|e_t^1(\omega)\right\| = \left\|\frac{\hat{t}}{t} a_{\hat{t}}^E(\omega) + \frac{1}{t}\int_{\hat{t}}^{t} e^{-\lambda s}\left(w_{s-\hat{t}}^{\hat{t},E} - \alpha^*\right) e^{-2\pi i\omega s} ds\right\|$$
$$\leq \frac{\hat{t}}{t}\epsilon + \frac{1}{\lambda}\frac{1}{t}\epsilon \leq \left(1 + \frac{1}{\phi}\right)\left(\frac{\hat{t}}{t}\right)^{\rho}\epsilon.$$

(2) Recall that

$$w_x^{P,u,E} - \alpha^* = \sum_{\omega'\in\Omega} a_{\sin}^{u,E}(\omega') \sin\left(2\pi\omega'(u+x)\right) + \sum_{\omega'\in\Omega} a_{\cos}^{\tau}(\omega) \cos\left(2\pi\omega'(u+x)\right),$$
$$= \sum_{\omega'\in\Omega} \mathrm{Re}\left(\left(a_{\cos}^{\tau}(\omega') - i a_{\sin}^{u,E}(\omega')\right) e^{2\pi\omega'(u+x)}\right)$$

which implies,

$$\begin{aligned}
&\left(\kappa\star\left(w^{P,u,E}-\alpha^{*}\right)\right)(s-u)\\
&=\int_{-\infty}^{\infty}\kappa\left(s-u-x\right)\left[w_{x}^{P,u,E}-\alpha^{*}\right]ds\\
&=\sum_{\omega'\in\Omega}\operatorname{Re}\left(\left(\int_{-\infty}^{\infty}\kappa\left(s-u-x\right)\left[\left(a_{\cos}^{u,E}\left(\omega'\right)-ia_{\sin}^{u,E}\left(\omega'\right)\right)\right]e^{2\pi\omega'(u+x-s)}dx\right)e^{2\pi\omega's}\right)\\
&=\sum_{\omega'\in\Omega}\operatorname{Re}\left(\left(\left(\mathcal{F}\kappa\right)(s-u)\left[a_{\cos}^{u,E}\left(\omega'\right)-ia_{\sin}^{u,E}\left(\omega'\right)\right]\right)e^{2\pi i\omega's}\right).
\end{aligned}$$

(We use the fact that the operator function $\kappa(.)$ is real-valued.) Because of Lemma 12 and the inductive assumption,

$$\begin{aligned}
\left\|e_{s,u}^{2}\left(\omega\right)\right\|&=\left\|b_{s-u}\left(w^{P,u},v^{u}\right)-\alpha^{*}-\operatorname{Re}\left(\sum_{\omega'\in\Omega}\left(K\left(\omega'\right)+I_{\mathcal{A}}\right)\left[a_{\cos}^{u,E}\left(\omega'\right)-ia_{\sin}^{u,E}\left(\omega'\right)\right]e^{2\pi is\omega'}\right)\right\|\\
&\leq\left\|c_{s-u}\left[v^{u}-v^{*}\right]\right\|+K\left(\left\|w^{P,u}-\sigma^{*}\right\|+\left\|v^{u}-v^{*}\right\|\right)^{2}\\
&\leq P_{c}P_{v}e^{-(s-u)}\left(\frac{\hat{t}}{u}\right)^{\rho}\epsilon+K\left(P_{wP}+P_{v}\right)^{2}\left(\frac{\hat{t}}{u}\right)^{2\rho}\epsilon^{2}\\
&\leq\left(P_{c}P_{v}e^{-(s-u)}+\epsilon K\left(P_{wP}+P_{v}\right)^{2}\right)\left(\frac{\hat{t}}{u}\right)^{\rho}\epsilon.
\end{aligned}$$

(3) Using the facts about linear regression and (G.7), one shows that for each $\omega'\in\Omega\setminus\{0\}$, each u,

$$\left\|a_{\cos}^{u,E}\left(\omega'\right)-\operatorname{Re}\left(a_{u}^{E}\left(\omega'\right)\right)\right\|,\ \left\|a_{\sin}^{u,E}\left(\omega'\right)-\operatorname{Im}\left(a_{u}^{E}\left(\omega'\right)\right)\right\|\leq\epsilon\max_{\omega'\in\Omega}\left\|a_{u}^{E}\left(\omega'\right)\right\|.$$

Thus, by the inductive assumption,

$$\left\|e_{t}^{3}\left(\omega\right)\right\|\leq2\epsilon\left\|w^{P,t,E}-\sigma^{*}\right\|\leq$$

An analogous bound holds when $\omega'=0$.

(4) For each $\omega' \in \Omega$,

$$\lambda \int_{\hat{t}}^{s} a_u^E(\omega') e^{-\lambda(s-u)} du - a_s^E(\omega')$$
$$= \lambda \int_{-\infty}^{s} \left(a_u^E(\omega') - a_s^E(\omega')\right) e^{-\lambda(s-u)} du - \lambda \int_{-\infty}^{\hat{t}} a_u(\omega') e^{-\lambda(s-u)} du,$$

where we take $a_u^E(\omega') = 0$ for $u \leq 0$. By the definition of $a_\cdot^E(\omega')$, and by the inductive assumption, for each $\hat{t} \leq u \leq s \leq t$,

$$\left\|a_u^E(\omega') - a_s^E(\omega')\right\|$$
$$\leq \frac{s-u}{s}\left\|a_u^E(\omega')\right\| + \frac{1}{s}\int_u^s \left\|w_0^{x,E}\right\| dx$$
$$\leq \frac{s-u}{s} P_a \left(\frac{\hat{t}}{u}\right)^{\rho} \epsilon + \frac{1}{s}\int_u^s P_w \left(\frac{\hat{t}}{x}\right)^{\rho} \epsilon dx$$
$$= \frac{1}{s}\left((s-u) P_a + \frac{1}{1-\rho} P_w \left(u^{\rho} s^{1-\rho} - u\right)\right)\left(\frac{\hat{t}}{u}\right)^{\rho} \epsilon$$
$$\leq \frac{s-u}{s}\left(P_a + P_w \frac{1}{1-\rho}\right)\left(\frac{\hat{t}}{u}\right)^{\rho} \epsilon.$$

A similar calculations for $u \leq \hat{t} \leq s$ yield

$$\left\|a_u^E(\omega') - a_s^E(\omega')\right\| \leq \frac{s-u}{s}\left(2 + P_a + P_w \frac{1}{1-\rho}\right)\epsilon.$$

(We use the fact that $\left\|a_x^E(\omega')\right\| \leq \epsilon$ for each $x \leq \hat{t}$.) Because$\lambda \int_{-\infty}^{s} (s-u) e^{-\lambda(s-u)} = \frac{1}{\lambda}$, we get

$$\left\|\lambda \int_{-\infty}^{s} \left(a_u^E(\omega') - a_s^E(\omega')\right) e^{-\lambda(s-u)} du\right\| \leq \frac{1}{\phi}\left(2 + P_a + P_w \frac{1}{1-\rho}\right)\frac{\hat{t}}{s}\epsilon.$$

Next,

$$\left\|\lambda \int_{-\infty}^{\hat{t}} a_u^E(\omega') e^{-\lambda(s-u)} du\right\| \leq e^{-\lambda\left(s-\hat{t}\right)}\epsilon \leq \frac{\hat{t}}{s}\epsilon,$$

where the last inequality comes from (G.8). It follows that

$$\left\|e_s^4(\omega)\right\| \leq \left(3+P_a+P_w\frac{1}{1-\rho}\right)\frac{\hat{t}}{s}\epsilon.$$

(5) Fix $\omega' \neq \omega$. Using the fact that for any complex number $\mathrm{Re}(z)=\frac{1}{2}(z+\overline{z})$, we obtain

$$\begin{aligned}
&\frac{2}{t}\int_{\hat{t}}^{t}\mathrm{Re}\left((K(\omega')+I_{\mathcal{A}})\left[a_s^E(\omega')\right]e^{2\pi is\omega'}\right)e^{-2\pi i\omega s}ds\\
&=(K(\omega')+I_{\mathcal{A}})\left[\frac{1}{t}\int_{\hat{t}}^{t}a_s^E(\omega')\,e^{2\pi is(\omega'-\omega)}ds\right]\\
&\quad+\overline{(K(\omega')+I_{\mathcal{A}})}\left[\frac{1}{t}\int_{\hat{t}}^{t}\overline{a_s^E(\omega')}e^{-2\pi is(\omega+\omega')}ds\right]
\end{aligned}$$

We are going to bound the first term - the second term is bounded analogously. Let $\Delta=\omega'-\omega\neq 0$. Then, by the definition of $a_s^E(\omega)$ and the change of the variables,

$$\begin{aligned}
&\frac{1}{t}\int_{\hat{t}}^{t}a_s^E(\omega')\,e^{2\pi is\Delta}ds\\
=&\frac{1}{t}\int_{\hat{t}}^{t}\left(\frac{2}{s}\int_{\hat{t}}^{s}\left(w_0^{u,E}-\alpha^*\right)e^{-2\pi i\omega' u}du\right)e^{2\pi i\Delta s}ds+\frac{1}{t}\left(\int_{\hat{t}}^{t}\frac{2}{s}e^{2\pi i\Delta s}ds\right)\left(\int_{0}^{\hat{t}}\left(w_0^{u,E}-\alpha^*\right)e^{-2\pi i\omega' u}du\right)\\
=&\frac{2}{t}\int_{\hat{t}}^{t}\left(w_0^{u,E}-\alpha^*\right)e^{-2\pi i\omega u}\left(\int_{u}^{t}\frac{1}{s}e^{2\pi i\Delta s}ds\right)du+\frac{2}{t}\left(\int_{\hat{t}}^{t}\frac{1}{s}e^{2\pi i\Delta s}ds\right)\left(\int_{0}^{\hat{t}}\left(w_0^{u,E}-\alpha^*\right)e^{-2\pi i\omega' u}du\right).
\end{aligned}$$

Notice that $\left|\int_u^t \frac{1}{s}e^{2\pi i\Delta s}ds\right| \leq \left(\frac{1}{u}+\frac{1}{t}\right) \leq \frac{2}{|\Delta|}\frac{1}{u}$ and that $|\Delta| \leq \omega+\omega'$. The inductive assumption implies that

$$\begin{aligned}\left\|e^5_{t,\omega'}(\omega)'\right\| \leq& 2\left\|K(\omega')+I_{\mathcal{A}}\right\|\frac{4}{\Delta}\frac{1}{t}\left\|\left(\int_{\hat{t}}^{t}\left(w_0^{u,E}-\alpha^*\right)e^{-2\pi i\omega' u}\frac{1}{u}du+\frac{1}{\hat{t}}\int_{0}^{\hat{t}}\left(w_0^{u,E}-\alpha^*\right)e^{-2\pi i\omega' u}du\right)\right\| \\ \leq& \left\|K(\omega')+I_{\mathcal{A}}\right\|\frac{8}{\Delta}\frac{1}{t}\left(\int_{\hat{t}}^{t}P_w u^{-1-\rho}du+1\right)\epsilon \\ \leq& \left\|K(\omega')+I_{\mathcal{A}}\right\|\frac{8}{\Delta}\frac{1}{t}\left(P_w\frac{1}{\rho}\hat{t}^{-\rho}+1\right)\epsilon \\ \leq& \lambda\frac{1}{\phi}\left\|K(\omega')+I_{\mathcal{A}}\right\|\frac{8}{\Delta}\left(P_w\frac{1}{\rho}+1\right)\left(\frac{\hat{t}}{t}\right)^{\rho}\epsilon.\end{aligned}$$

(6) if $\omega=0$, then $e^6_t(\omega)=0$. If $\omega\neq 0$, then, applying complex conjugates, we obtain

$$\begin{aligned}e^6_t(\omega) =& \frac{1}{t}\int_{\hat{t}}^{t}\left(\left(K(\omega)+I_{\mathcal{A}}\right)\left[a^E_s(\omega)\right]e^{2\pi i s\omega}\right)e^{-2\pi i\omega s}ds-\left(K(\omega)+I_{\mathcal{A}}\right)\left[\frac{1}{t}\int_{\hat{t}}^{t}a^E_s(\omega)\,ds\right] \\ &+\overline{\left(K(\omega')+I_{\mathcal{A}}\right)}\left[\frac{1}{t}\int_{\hat{t}}^{t}\left(\overline{a^E_s(\omega)}e^{-2\pi i s\omega}\right)e^{-2\pi i\omega s}ds\right] \\ =& \overline{\left(K(\omega')+I_{\mathcal{A}}\right)}\left[\frac{1}{t}\int_{\hat{t}}^{t}\left(\overline{a^E_s(\omega)}e^{-2\pi i s\omega}\right)e^{-2\pi i\omega s}ds\right].\end{aligned}$$

Using the calculations from above, we show that

$$\left\|e^6_t(\omega)\right\| \leq \lambda\frac{1}{\phi}\left\|K(\omega')+I_{\mathcal{A}}\right\|\frac{8}{2\omega}\left(P_w\frac{1}{\rho}+1\right)\left(\frac{\hat{t}}{t}\right)^{\rho}\epsilon.$$

Combing the bounds together shows that $\left\|e_t(\omega)\right\| \leq P_e\left(\frac{\hat{t}}{u}\right)^{\rho}\epsilon$.

Inequality 3. Notice that

$$\begin{aligned}\left\|w^{t}-\sigma^{*}\right\| =& \int_{C}\left\|w_{c}^{t}-\sigma^{*}\right\| d\mu^{\lambda}(c)\\ =& e^{-\lambda\left(t-\hat{t}\right)}\int_{C}\left\|w_{c,t-\hat{t}+.}^{\hat{t}}-\sigma^{*}\right\| d\mu^{\lambda}(c)\\ &+\int_{\hat{t}}^{t}\lambda e^{-\lambda(t-s)}\left\|b_{t-s+.}^{s}\left(w^{F,s},v^{s}\right)-\sigma^{*}\right\| dt.\end{aligned}$$

By Lemma 12,

$$\begin{aligned}&\left\|b_{t-s+.}^{s}\left(w^{F,s},v^{s}\right)-\sigma^{*}\right\|\\ \leq& P_{c}e^{-\gamma_{c}(t-s)}\left\|v^{s}-v^{*}\right\|+\|\kappa\|\left\|w^{F,s,E}-\sigma^{*}\right\|+K\left(\left\|w^{F,s}-\sigma^{*}\right\|+\left\|v^{s}-v^{*}\right\|\right)^{2}.\end{aligned}$$

Notice that $\left\|w^{F,s,E}-\sigma^{*}\right\|=\sum_{\omega'}\left\|a_{s}^{E}(\omega)\right\|$. By the inductive assumption,

$$\begin{aligned}&\left\|b_{t-s+.}^{s}\left(w^{F,s},v^{s}\right)-\sigma^{*}\right\|\\ \leq&\left(P_{c}P_{v}e^{-\gamma_{c}(t-s)}+\|\kappa\|\left|\Omega\right|P_{a}+\epsilon K\left(P_{wP}+P_{v}\right)^{2}\right)\left(\frac{\hat{t}}{s}\right)^{\rho}\epsilon.\end{aligned}$$

Hence, because of (G.8),

$$\begin{aligned}&\left\|w^{t}-\sigma^{*}\right\|\\ \leq& e^{-\lambda\left(t-\hat{t}\right)}\epsilon+P_{c}P_{v}\int_{\hat{t}}^{t}\lambda e^{-\lambda(t-s)}e^{-\gamma_{c}(t-s)}dt\\ &+\left(\|\kappa\|\left|\Omega\right|P_{a}+\epsilon K\left(P_{wP}+P_{v}\right)^{2}\right)\int_{\hat{t}}^{t}\lambda e^{-\lambda(t-s)}\left(\frac{\hat{t}}{s}\right)^{\rho}\epsilon dt\\ \leq&\left(1+\|\kappa\|\left|\Omega\right|P_{a}+\frac{\lambda}{\lambda+\gamma_{c}}P_{c}P_{v}+\epsilon K\left(P_{wP}+P_{v}\right)^{2}\right)\left(\frac{\hat{t}}{t}\right)^{\rho}\epsilon<P_{w}\left(\frac{\hat{t}}{t}\right)^{\rho}\epsilon.\end{aligned}$$

Inequality 4. Inequality 4 is satisfied strictly at $t=t^{*}$ due to the inductive assumption, the definition of w^{P}, the definition of $\alpha_{t}^{*}(\omega)$, and inequality (3).

Inequality 5. By Lemma 9 and the inductive assumption,

$$\begin{aligned}\left\|v^t - v^*\right\| \leq & P_1 \left\|v_0 - v^*\right\| + Q_1 \max_{s\leq t} e^{-\gamma_1(t-s)} \left\|w^s - \sigma^*\right\|, \\ \leq & P_1 e^{-\gamma_1 t}\epsilon + Q_1 P_w \max_{s\leq t} e^{-\gamma_1(t-s)} \left(\frac{\hat{t}}{s}\right)^\rho \epsilon.\end{aligned}$$

By inequality (G.8), and because $\rho < \gamma_1$, the latter is not larger than

$$\leq (P_1 + Q_1 P_w)\left(\frac{\hat{t}}{t}\right)^\rho \epsilon < P_v \left(\frac{\hat{t}}{t}\right)^\rho \epsilon.$$

Proof of Lemma 17. It is enough to show that $\|w^t - \sigma^*\| + \|v^t - \sigma^*\| \leq Qe^{\lambda qt}(\|w^0 - \sigma^*\| + \|v^0 - \sigma^*\|)$ for some q. The idea of the proof is a simpler version of the argument presented above.
Choose constants $P^0_w, P^0_{wP}, P^0_v, q < \infty$ so that

$$\begin{aligned}P^0_w <& \frac{1}{q-1} K\left(P^0_{wF} + P^0_v\right) + 1, \\ P^0_{wP} =& 2P^0_w, \\ P^0_v <& P_1 + Q_1 P^0_w.\end{aligned}$$

Let $\epsilon = \|w^0 - \sigma^*\| + \|v^0 - \sigma^*\|$. Let T^* be the non-empty set of periods for which at least one of the below inequalities fail:

(1) $\|w^t - \sigma^*\| \leq P^0_w e^{q\lambda t}\epsilon$,
(2) $\left\|w^{F,t} - \sigma^*\right\| \leq P^0_{wP} e^{q\lambda t}\epsilon$,
(3) $\|v^t - v^*\| \leq P^0_v e^{q\lambda t}\epsilon$.

Let $t^* = \inf T^*$. If $t^* = \infty$, then the result holds. On the contrary, suppose that $t^* < \infty$. Because of the continuity, all the inequalities are satisfied for all $t \leq t^*$. We show that each of the inequalities must, in fact, be satisfied strictly. This shall contradict the choice of t^*.

(1) We show that inequality 1 is satisfied strictly at $t = t^*$. By the inductive assumption,

$$\begin{aligned}\left\|w^t - \sigma^*\right\| \leq & e^{-\lambda t}\left\|w^0 - \sigma^*\right\| + \int_0^t \lambda e^{-\lambda(t-s)}\left\|b^s\left(w^{F,s}, v^s\right) - \sigma^*\right\| ds \\ \leq & e^{-\lambda t}\epsilon + K\int_0^t \lambda e^{-\lambda(t-s)}\left(\left\|w^{F,s} - \sigma^*\right\| + \|v^s - v^*\|\right) ds \\ \leq & e^{-\lambda t}\epsilon + K\left(P^0_{wP} + P^0_v\right)\int_0^t \lambda e^{-\lambda(t-s)} e^{q\lambda s}\epsilon ds \\ \leq & \left(\frac{1}{q-1}K\left(P^0_{wP} + P^0_v\right) + 1\right) e^{q\lambda t}\epsilon < P^0_w e^{q\lambda t}\epsilon.\end{aligned}$$

(2) Inequality 2 is satisfied strictly at $t = t^*$ due to the inductive assumption, the definition of w^F, and the choice of constant P^0_{wP}.

(3) We show that inequality 3 is satisfied strictly at $t = t^*$. By Lemma 9 and the inductive assumption,

$$\begin{aligned}\left\|v^t - v^*\right\| \leq & P_1\|v_0 - v^*\| + Q_1\max_{s\leq t} e^{-\gamma_1(t-s)}\|w^s - \sigma^*\|, \\ \leq & P_1 e^{-\gamma_1 t}\epsilon + Q_1 P^0_w \max_{s\leq t} e^{-\gamma_1(t-s)} e^{q\lambda t}\epsilon \\ \leq & \left(P_1 + Q_1 P^0_w\right) e^{q\lambda t}\epsilon < P^0_v e^{q\lambda t}\epsilon.\end{aligned}$$

G.3. **Necessary conditions for stability.** We use the following lemma:

Lemma 18. *Fix complex ϕ such that $Re(\phi) > 0$. Let y_t be a process such that $y_0 = 1$ and*

$$y_t = \frac{1}{t}\int_0^t \left(e^{-(t-s)} + \left(1 - e^{-(t-s)}\right)(1+\phi)\, y_s\right) ds,$$

The process is well-defined and $y_t \to \infty$ when $t \to \infty$.

Proof. Notice that

$$\frac{d}{dt}y_t = -\frac{1}{t}y_t + \frac{1}{t}1 - \frac{1}{t}\int_0^t \left(e^{-(t-s)} - (1+\phi)\, y_s e^{-(t-s)}\right) ds$$
$$= \left((1+\phi)\frac{1}{t}\int_0^t y_s ds - y_t\right) + \frac{1}{t}(1-y_t)\,. \tag{G.10}$$

Let $x_t = \frac{1}{t+1}(1+\phi)\int_0^t y_s ds + \frac{1}{t+1}$. Then, the pair $x_.$ and $y_.$ are the unique solutions to a system of differential equations

$$\frac{d}{dt}y_t = \frac{t+1}{t}(x_t - y_t) \text{ and } \frac{d}{dt}x_t = \frac{1}{t+1}((1+\phi)\, y_t - x_t)$$

with the initial condition $x_t = y_t = 1$. For each y, let

$$A(y) = \{y + \alpha y + \beta i y : \alpha \geq 0\}$$

be the half space of the complex plane bounded away from the circle of radius $|y|$ by the line tangent to the circle at y. We show that for each t, $x_t \in A(y_t)$. Indeed, if $x = y + \varpi i y \in \mathrm{bd}A(y)$, then

$$x' = y + \beta i y + \frac{1}{t+1}(\phi - \beta i)\, y\epsilon$$
$$= y + \frac{1}{t+1}\mathrm{Re}(\phi)\,\epsilon y + \left(\varpi + \frac{1}{t+1}(\mathrm{Im}(\phi) - \beta)\, y\epsilon\right) iy \in \mathrm{int}A(y)\,.$$

It follows that $y_t \to \infty$ when $t \to \infty$. □

Suppose that there exists $\omega_0 \in \Omega$ and an eigenvalue of linear operator $K(\omega)$ with a strictly positive real part. Let $a_0^* \in \mathcal{A}^{\mathcal{C}}$ be the corresponding (complex) eigenvector. Chhose small $\epsilon > 0$ and let $a(\omega) = \begin{cases} \epsilon a_0^*, & \text{if } \omega = \omega_0 \\ 0, & \text{otherwise} \end{cases}$ for each $\omega \in \Omega$. Define a strategy profile

$$w_{sc}^0 = \alpha^* + 2\mathrm{Re}\left(\sum_{\omega\in\Omega} a_0^*(\omega)\, e^{2\pi i\omega s}\right)$$
$$= \alpha^* + \epsilon a_0^* e^{2\pi i\omega s} + \epsilon\overline{a_0^*} e^{-2\pi i\omega s}.$$

(The second equality follows from the properties of complex numbers.) We are going to show that if λ is sufficiently small, then there exists $\eta > 0$ such that for each $\epsilon > 0$, there exists t so that the learning dynamics innitiated by w^0 and $v^0 = v^*$ diverges to the distance at least $\eta > 0$.

Define $a_t^E(\omega)$ for each $\omega \in \Omega$ as in (G.1). In order to isolate the effect of small λ, it is convenient to change variables $t^\lambda := \lambda t$ and write $a_{t^\lambda} = a^E_{\lambda t}$. Then, for each ω,

$$\begin{aligned}
a_{t^\lambda}(\omega) =& a^E_{\frac{1}{\lambda}t^\lambda}(\omega) = \frac{1}{\frac{1}{\lambda}t^\lambda}\int_0^{\frac{1}{\lambda}t^\lambda}\left(w_0^{s,E} - \alpha^*\right)e^{-2\pi i\omega s}ds \\
=& \frac{\lambda}{t^\lambda}\int_0^{\frac{1}{\lambda}t^\lambda} e^{-\lambda s}\left(w_s^{0,E} - \alpha^*\right)e^{-2\pi i\omega s}ds \\
&+ \frac{\lambda}{t^\lambda}\int_0^{\frac{1}{\lambda}t^\lambda}\lambda\int_0^s\left(b_{s-u}\left(w^{P,u}, v^u\right) - \alpha^*\right)e^{-\lambda(s-u)}e^{-2\pi i\omega s}duds \\
=& 2\frac{\lambda}{t^\lambda}\sum_{\omega\in\Omega}\int_0^{\frac{1}{\lambda}t^\lambda} e^{-\lambda s}\mathrm{Re}\left(a_0^*(\omega)e^{2\pi i\omega s}\right)e^{-2\pi i\omega s}ds \\
&+ \frac{\lambda}{t^\lambda}\int_0^{\frac{1}{\lambda}t^\lambda}\left(1 - e^{-\lambda(t-s)}\right)\left(K(\omega) + I_{\mathcal{A}}\right)\left[a_s^E(\omega)\right]ds + e_{t^\lambda}(\omega) \\
=& \frac{1}{t^\lambda}\sum_{\omega\in\Omega}\int_0^{t^\lambda} e^{-s^\lambda}a_0^*(\omega)\,ds^\lambda \\
&+ \frac{1}{t^\lambda}\int_0^{t^\lambda}\left(1 - e^{-\left(t^\lambda - s^\lambda\right)}\right)\left(K(\omega) + I_{\mathcal{A}}\right)\left[a_{s^\lambda}(\omega)\right]ds^\lambda + e_{t^\lambda}(\omega),
\end{aligned}$$

where $e_{t^\lambda}(\omega) = e^1_{t^\lambda}(\omega) + e^2_{t^\lambda}(\omega)$, and

$$\begin{aligned}
e^1_{t^\lambda}(\omega) =& \frac{\lambda}{t^\lambda}\int_0^{\frac{1}{\lambda}t^\lambda} e^{-\lambda s}\left(\overline{a_0^*}e^{-4\pi i\omega s} + 2\sum_{\omega'\neq\omega} a_0^* e^{-4\pi i(\omega'-\omega)s}\right)ds, \\
e^2_{t^\lambda}(\omega) =& \frac{\lambda}{t^\lambda}\int_0^{\frac{1}{\lambda}t^\lambda}\left(\lambda\int_0^s\left(b_{s-u}\left(w^{P,u}, v^u\right) - \alpha^*\right)e^{-\lambda(s-u)} - \left(K(\omega) + I_{\mathcal{A}}\right)\left[a_s^E(\omega)\right]e^{2\pi i\omega s}\right)e^{-2\pi i\omega s}duds.
\end{aligned}$$

Using the approximations from the sufficiency part of the proof of the Theorem, we can show that for arbitrarily high $t^{\lambda *} < \infty$, there exists constant $P < \infty$ such that for sufficiently small $\eta_0 > 0$ there exists $\lambda^* > 0$, such that for each $t^\lambda \in \left[0, t^{\lambda *}\right]$ and each $\epsilon \leq \eta_0$, each $\lambda \leq \lambda^*$,

$$\|a_{t^{\lambda *}}(\omega_0) - y_{t^\lambda} a_0^*\| \leq P(\lambda + \eta_0)\|a_0^*\|\epsilon, \text{ and}$$
$$\|a_{t^\lambda}(\omega_0)\| \leq P(\lambda + \eta_0)\|a_0^*\|\epsilon \text{ for each } \omega \neq \omega_0.$$

We omit the details.

Let

$$e_t'(\omega) = e_t(\omega) + \frac{1}{t^\lambda}\int_0^{t^\lambda}\left(e^{-\left(t^\lambda - s^\lambda\right)}\right)(K(\omega) + I_{\mathcal{A}})\left[a_{s^\lambda}(\omega)\right] ds^\lambda.$$

For any $\epsilon' > 0$, we can find sufficiently low λ and sufficiently high $t^{\lambda *}$ so that if $\|a_{t^\lambda}\|$ is increasing (or at least, there exists a constant $p > 0$ such that $\|a_{t^\lambda}\| \leq p\|a_{s^l}\|$ for each $t^\lambda < s^\lambda$, then, using the approximations from the sufficiency part of the proof, we can show that $\|e_t'\| \leq \epsilon' \max_{s \leq t}\|a_s\|$ for all for all t.

Let

$$x_t = \frac{1}{t}\int_0^t (K + I_{\mathcal{A}^\Omega})[a_s]\, ds.$$

Then, because process a_t is continuous, process x_t is differentiable, and

$$\frac{d}{dt}x_t = \frac{1}{t}(Kx_t - Ke_t).$$

Lemma 5 concludes the argument.

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