COMPARISON OF INFORMATION STRUCTURES IN ZERO-SUM GAMES

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Abstract. This note provides simple necessary and sufficient conditions for the comparison of information structures in zero-sum games. This solves an open problem of (Gossner and Mertens 2001). The conditions are phrased in terms of Blackwell garbling of information of each of the players.

1. Introduction

(Blackwell 1953) provides an explicit connection between the "informativeness" of a signal and the value of information in one-person decision problems. Signal T about the state of the world $\omega \in \Omega$ is *more informative* than signal T' if the latter is a *garbling* of the former. Blackwell shows that acting upon signal T leads to higher payoff than acting upon T' in *any* decision problem if and only if signal T is more informative than T'.

It has been noticed that Blackwell's theorem does not generalize to two- or more person decision problems (The first example of this form has been provided in (Hirshleifer 1971).) With more than one player, an information structure is defined as a Harsanyi's type space, where each type of each player has beliefs about the state of the world and the types of the other players. An improvement in the information of a player may decrease the equilibrium payoff of the player in some games.

In order to get any positive results about the value of information structure in games, one needs to consider a smaller class of games. This is the approach taken in (Gossner and Mertens 2001) (see also (Lehrer, Rosenberg, and Shmaya 2006b) and (Lehrer, Rosenberg, and Shmaya 2006a)). Suppose that there are two players, called *maximizer* and *minimizer*. Players play zero-sum game with incomplete information. Prior to choosing their actions, they receive information about their type drawn from a common prior distribution $\mu \in \Delta(\Omega \times T_{\min} \times T_{\max})$, where T_i is a finite set of types of player $i = \min, \max$. The value of the a zero-sum game on information structure μ is equal to the equilibrium

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payoff of the maximizer in a Bayesian Nash equilibrium. By the minmax theorem, this payoff is unique. Say that information structure μ is *(weakly) better for maximizer* than μ', if there is no zero-sum game for which the value on information structure μ is smaller than the value on μ'. The "better for maximizer" relation introduces partial order on the space of information structures $\Delta(\Omega \times T_{\min} \times T_{\max})$. As a partial result, (Gossner and Mertens 2001) provide the necessary and sufficient conditions for two information structures to be equivalent in the above order.

The purpose of this note is to provide a complete characterization of the order in terms of Blackwell's garbling. Information structure μ is better for maximizer than information structure ψ if and only if there exists an information structure ξ such that (a) information of the minimizer in ξ is obtained by garbling information of the minimizer in ψ and information of the maximizer is the same in ξ and ψ; (b) information of the maximizer in ξ is obtained from by garbling the information of the maximizer in μ and information of the minimizer is the same in ξ and ψ. In a sense, (a) the information of the minimizer in ξ is worse in Blackwell's sense than in ψ, whereas information of the maximizer is the same, and (b) the information of the minimizer in ξ is worse in Blackwell's sense than in μ, whereas the information of the minimizer is kept the same

This result has the following interpretation. It is intuitive that improving the information of the maximizer increases his payoff in any zero-sum game. Similarly, worsening the information of the minimizer should increase the payoff to the maximizer in any zero-sum game. The Theorem says that these two operations exhaust all the possible changes in the information structure that lead to higher payoff of the maximizer in any zero-sum game.

2. Model

There are two players, minimizer and maximizer. I use $i = \max, \min$ to denote a generic player and $-i$ to denote the other player.

Let Ω be a finite space of payoff-relevant uncertainty. A *(common prior) type space* over Ω is a pair of finite sets of types T_i for both players i and a distribution $\mu \in \Delta(T_{\min} \times T_{\max} \times \Omega)$. I keep sets of types T_i fixed. I refer to μ as an *information structure.*

A *zero-sum game* is a tuple $G = (A_{\min}, A_{\max}, u)$ of finite action sets A_i for both players i and a payoff function of the maximizer $u : A_{\min} \times A_{\max} \times \Omega \rightarrow R$. An incomplete information game (G, μ) is a pair of zero-sum game G and an information structure μ. Let $\sigma_i : T_i \rightarrow \Delta A_i$ be a strategy of player i in the incomplete information game and let

Σ_i be a set of strategies of player i. For any strategy profile $(\sigma_{\min}, \sigma_{\max})$, let

$$u(\sigma_{\min}, \sigma_{\max}; \mu) := \sum_{t_{\min}\in T_{\min}, t_{\max}\in T_{\max}} u(\sigma_{\min}(t_{\min}), \sigma_{\max}(t_{\max}), \omega)\, \mu(t_{\min}, t_{\max}, \omega).$$

By the minmax theorem, for each incomplete information game there exists a value $V(G, \mu)$, such that for any Bayesian Nash equilibrium profile $(\sigma^*_{\min}, \sigma^*_{\max})$,

$$\begin{aligned} V(G, \mu) &= u(\sigma^*_{\min}, \sigma^*_{\max}; \mu) \\ &= \max_{\sigma_{\max}} \min_{\sigma_{\min}} u(\sigma_{\min}, \sigma_{\max}; \mu) \\ &= \min_{\sigma_{\min}} \max_{\sigma_{\max}} u(\sigma_{\min}, \sigma_{\max}; \mu). \end{aligned} \tag{2.1}$$

Definition 1. *Say that information structure μ is* (weakly) better for maximizer *than information structure ψ, (write $\psi \lesssim \mu$) if and only if for each game G,*

$$V(G, \mu) \geq V(G, \psi).$$

For any player i, a *kernel* of player i is a mapping $Q_i : T_i \to \Delta T_i$. Let $\mathcal{K}_i$ be the space of all kernels of player i. Let μ be an information structure and Q_i be a kernel of player i. Define information structure $Q_i\mu \in \Delta(T_{\min} \times T_{\max} \times \Omega)$ as

$$Q_i\mu(t_i, t_{-i}, \omega) := \sum_{t'_i} \mu(t'_i, t_{-i}, \omega)\, Q(t'_i)(t_i).$$

The subsequent Lemma presents some useful properties of kernels: For any information structure μ, let $\mathcal{K}_i\mu := \{Q_i\mu : Q_i \in \mathcal{K}_i\}$.

Lemma 1. *$\mathcal{K}_i\mu$ is convex and compact.*

Proof. This is an immediate consequence of the fact that $\mathcal{K}_i\mu$ is the image of compact and convex $\mathcal{K}_i$ under the linear map $Q \to Q\mu$. $\square$

Lemma 2. *For any information structure μ, any kernels $Q_i \in \mathcal{K}_i$,*

$$Q_{\max}\mu \lesssim \mu \text{ and } \mu \lesssim Q_{\min}\mu.$$

Proof. I consider only the second inequality, as the first one is analogous. Take any game $G = (A_{\min}, A_{\max}, u)$. Let $(\sigma^*_{\min}, \sigma^*_{\max})$ be a strategy profile in game $(G, Q_{\min}\mu)$, such that

$$\begin{aligned} V(G, Q_{\min}\mu) &= \min_{\sigma_{\min}} \max_{\sigma_{\max}} u(\sigma_{\min}, \sigma_{\max}; Q_{\min}\mu) \\ &= u(\sigma^*_{\min}, \sigma^*_{\max}; Q_{\min}\mu). \end{aligned}$$

Consider a strategy of the minimizer $\sigma^{**}_{\min}$ in game (G, μ) defined as

$$\sigma^{**}_{\min}(t_{\min}) = \sum_{t\in T_{\min}} Q_{\min}(t_{\min})(t)\, \sigma^*_{\min}(t) \text{ for each } t_{\min} \in T_{\min}.$$

Then, for any strategy of the maximizer $\sigma_{\max}$,

$$u\left(\sigma^*_{\min}, \sigma_{\max}; Q_{\min}\mu\right) = u\left(\sigma^{**}_{\min}, \sigma_{\max}; \mu\right).$$

Hence,

$$\begin{aligned} V\left(G,\mu\right) &= \min_{\sigma_{\min}} \max_{\sigma_{\max}} u\left(\sigma_{\min}, \sigma_{\max}; \mu\right) \\ &\leq \max_{\sigma_{\max}} u\left(\sigma^{**}_{\min}, \sigma_{\max}; \mu\right) \\ &= \max_{\sigma_{\max}} u\left(\sigma^*_{\min}, \sigma_{\max}; Q_{\min}\mu\right) = V\left(G, Q_{\min}\mu\right). \end{aligned}$$

□

The Lemma provides simple sufficient conditions for one information structure to be better than the other. The main result of this note shows that the sufficient conditions are, up to compositions, also necessary:

Theorem 1. *For any two information structures* μ, ψ*,* μ *is better for the maximizer than* ψ *if and only if there exist kernels* $Q_i \in \mathcal{K}_i$*,* $i = \min, \max$*, such that*

$$Q_{\min}\psi = Q_{\max}\mu. \tag{2.2}$$

Proof. The "if" part is a consequence of Lemma 2. Suppose that there are no kernels $Q_{\min}$ and $Q_{\max}$, such that (2.2) holds. Then, $\mathcal{K}_{\min}\psi \cap \mathcal{K}_{\max}\mu = \varnothing$. By Lemma 2, both sets $\mathcal{K}_{\min}\psi$ and $\mathcal{K}_{\max}\mu$ are compact and convex. By the separating hyperplane theorem, there is a function $u : T_{\min} \times T_{\max} \times \Omega \to R$, such that

$$\min_{\psi' \in \mathcal{K}_{\min}\psi} \psi'\left[u\right] > \max_{\mu' \in \mathcal{K}_{\max}\mu} \mu'\left[u\right],$$

where, for any $\varpi \in \Delta\left(T_{\min} \times T_{\max} \times \Omega\right)$, I denote

$$\varpi\left[u\right] := \sum_{t_{\min}, t_{\max}, \omega} \varpi\left(t_{\min}, t_{\max}, \omega\right) u\left(t_{\min}, t_{\max}, \omega\right).$$

Consider a game $G^* = \left(T_{\min}, T_{\max}, u\right)$. Note that kernels $Q \in \mathcal{K}_{\min}, \mathcal{K}_{\max}$ correspond to players' strategies in game G^*. Let σ^{id}_i denote the strategy of player i in which each type

t_i plays $t_i : \sigma_i(t_i) = t_i$. Notice that

$$\begin{aligned}
&\min_{\psi' \in \mathcal{K}_{\min}\psi} \psi'[u] \\
&= \min_{Q_{\min} \in \mathcal{K}_{\min}} \sum_{t_{\min}, t_{\max}, \omega} \sum_{t'_{\min}} \psi\left(t'_{\min}, t_{\max}, \omega\right) Q_{\min}\left(t'_{\min}\right)\left(t_{\min}\right) u\left(t_{\min}, t_{\max}, \omega\right) \\
&= \min_{Q_{\min} \in \mathcal{K}_{\min}} \sum_{t_{\min}, t_{\max}, \omega} \psi\left(t_{\min}, t_{\max}, \omega\right) u\left(Q_{\min}\left(t_{\min}\right), \sigma^{id}_{\max}\left(t_{\max}\right), \omega\right) \\
&= \min_{Q_{\min} \in \mathcal{K}_{\min}} u\left(Q_{\min}, \sigma^{id}_{\max}, ; \psi\right) \leq V\left(G, \psi\right).
\end{aligned}$$

Similarly,

$$\max_{\mu' \in \mathcal{K}_{\max}\mu} \mu'[u] \geq V\left(G, \mu\right).$$

Hence, it cannot be that $\psi \lesssim \mu$. $\square$

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