Nondistortionary elicitation of beliefs in dynamic problems

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Introduction

- Many experiments involve incentivized truthful reporting of beliefs
  - examples: self-confidence, information acquisition, dynamic games,
  - incentives improve accuracy (Schlag et al. (2015) and many others),
  - many methods, including, binarized Becker-DeGroot-Marschak (BDM) scoring rule.
- But combined with actions, incentivization may distort behavior. Examples:
  - multiple-choice test and elicitation of the belief of being correct,
  - questions about future behavior.
- Distortions are problematic
  - interpretation of experimental results,
  - honest instructions (incentivization does not need to be explicit (Danz et al. (2022))),
  - ethical and legal issues.

Introduction

- Which questions are incentivizable questions, i.e. can be elicited without distorting behavior.
- Pęski and Stewart (2025) characterize all incentivizable questions in static problems:
  - Question must be aligned, i.e., an affine transformation, of payoffs.
- **Dynamic problems** have new issues:
  - richer learning environment (Chambers and Lambert (2021)),
  - multiple points of asking questions, and
  - multiple possible points of distortions.
- Different ways actions can affect arrival of new information:
  - action-independent,
  - actions affect the quality of information,
  - actions affect the distribution of states.

Introduction

Research objective

How to incentivize belief elicitation without distortion in dynamic experiments?
Which questions can be incentivized?

Answer

- Period 1 questions about payoffs and "affine" transformations.
  The meaning of "affine" depends on how actions affect the arrival of information.
- In period 2, only if actions do not affect information, payoffs are separable, and only questions about period-2 component of payoffs.
  Otherwise, no questions are incentivizable.

Outline

Model

Action-independent information

- Dynamic decision problem: (A_1, A_2, u)
  - A_t - finite action set in period t
  - $u : A_1 \times A_2 \times \Theta \longrightarrow \mathbb{R}$ payoffs
- Expected interim payoffs given period 2 beliefs $p \in \Delta\Theta$:

$$u^*(a_1, p) = \max_{a_2} \mathbb{E}_p u(a_1, a_2, \theta)$$

- $\mu \in \Delta(\Delta\Theta)$ beliefs in period 1 (beliefs about beliefs) in *action-independent information* case

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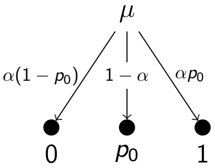
Example: Student

- Student writes a test and receives grade P.
- He is asked *before* and *after* the test about the prob. of crossing grade B threshold ($P > 75$).
  - The test is a signal about the difficulty.
- He is also asked about expected learning:

  What would be your expected grade if you had an option to choose after the test (but before seeing the test score) between
  - *(certain) grade B or*
  - *(uncertain) test score P?*
- Chambers and Lambert (2021) show how to incentivize answers to questions about second-order beliefs using supporting actions.



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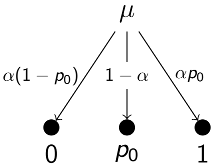

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Example: Student

- Student writes a test with $N = 100$ True-False questions.
- Pęski and Stewart (2025) show that questions are *incentivizable* "if and only if" they are are affine transformations (i.e., they are aligned) with payoffs $u(a_2, \theta)$.
- If the payoffs $u = P$, then question about probability of crossing threshold $P > 75$ will distort the incentives at the test taking.

- $\Theta = \{0, 1\}^N$
- $p \in \Delta\Theta$, $\mu \in \Delta\Delta\Theta$
- $A_2 = \{0, 1\}^N$
- class grade (test score)

$$P(a_2, \theta) = \sum_i 1\{a_{2,i} = \theta_i\}$$

No action choice in the first period, $A_1 = \{*\}$.

Model

Example: Student

- Student decides whether to write an optional test.

  If he stays out, he keeps his current average $h = 80$.

  If he writes the test, his weighted average is equal to:

$$u(a_1, a_2, \theta) = \frac{3}{4}h + \frac{1}{4}[a_1 P(a_2, \theta) + (1 - a_1)h]$$
$$= h + \frac{1}{4}a_1[P(a_2, \theta) - h].$$

- Question "How likely $P \geq 75$"? will distort incentives regardless when it is asked.
- Question "What is your expected average?"
  - will distort incentives if asked in period 2,
  - will not distort if asked in period 1.

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- *Actions affect the distribution of states* case if $\mu : A_1 \longrightarrow \Delta(\Delta\Theta)$ is a belief *profile* (no constraint).

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Action affect the quality of information

- Intermediate case:
- *Actions may affect the quality of information* case if $\mu : A_1 \longrightarrow \Delta(\Delta\Theta)$ is a *belief profile* st.

$$\mathbb{E}_{\mu(a_1)}p = \mathbb{E}_{\mu(b_1)}p \text{ for } a_1, b_1 \in A_1.$$

- Function $I : \Delta\Theta \longrightarrow \mathbb{R}$ is *affine* if for any p, q,

$$I(\alpha p + (1-\alpha)q) = \alpha I(p) + (1-\alpha)I(q).$$

- Affine functions can be written as $I(p) = I_0 + \sum_\theta p(\theta)I_\theta$ for some $I_0, I_\theta \in \mathbb{R}$.
- Actions may affect the quality of information if, for each affine I, each a_1, b_1

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- Function $l : \Delta\Theta \longrightarrow \mathbb{R}$ is *affine* if for any p, q,

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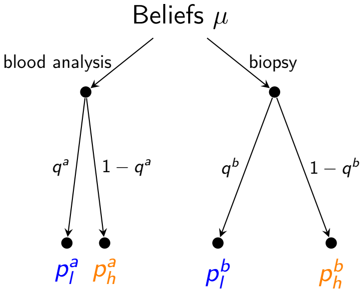
Example: Oncologist

- Oncologist chooses a method to diagnose tumor count:
  - blood analysis or biopsy,
  - biopsy improves information,
  - it is an invasive procedure and it costs $c > 0$
- After observing the outcome, she chooses treatment:
  - active surveillance or chemotheraphy
- Question "How likely are you to prescribe chemotherapy?" is never incentivizable.



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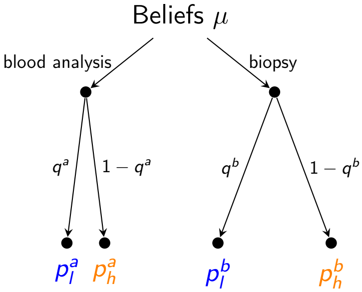

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Optimal behavior

- Optimal behavior in problem (A_1, A_2, u)

$$\mathbb{B}_2^u(a_1, p) = \arg\max_{a_2} \mathbb{E}_p u(a_1, a_2, \theta),$$
$$\mathbb{B}_1^u(\mu) = \arg\max_{a_1} \mathbb{E}_\mu(a_1) u^*(a_1, p).$$

  - Oncologist example: $\mathbb{B}_1^u(a_1, p)$ does not depend on a_1 (we will say that payoffs are separable)

- Actions a_1, b_1 are *adjacent* if a_1 and b_1 are best responses at the same belief μ, and there is no other optimal action: $\mathbb{B}_1^u(\mu) = \{a_1, b_1\}$.
- "Partition" $\mathcal{P}^u(a_1)$ of $\Delta\Theta$ into best response belief sets:

$$\mathcal{P}^u(a_1, a_2) = \{p : a_2 \in \mathbb{B}_2^u(a_1)\},$$
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Nondistortionary expansions

- Expansion $(A_1 \times S_1, A_2 \times S_2, v)$ of (A_1, A_2, u).
- Expansion is *nondistortionary* if optimal choices in the original problem remain optimal in the expansion (where the DM also makes choices from Bs):

no distortions in period 1: $\mathbb{B}_1^v(\mu) \subseteq \mathbb{B}_1^u(\mu) \times S_1$,

no distortions in period 2: $\mathbb{B}_2^v(a_1, b_1, p) \subseteq \mathbb{B}_2^u(a_1, p) \times S_2$.

- In principle, v does not need to be related to u. In practice, it will be.

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Period 1 elicitation

Question

- $X : A_1 \times \Delta\Theta \longrightarrow \mathbb{R}$ action-dependent period 1 question
- DM is asked to report $r = \mathbb{E}_{\mu(a_1)} X(a_1, \cdot)$

Example (Student)

1. **Expected test score:** let $p_i = P(\theta_i = 1)$,

$$X(a_1, p) = \max_{a_2} \sum_i E_p 1\{a_{2,i} = \theta_i\} = \sum_i \max(p_i, 1 - p_i)$$

2. **Expected average:**

$$X(a_1, p) = h + \frac{1}{4} a_1 \left[\sum_i \max(p_i, 1 - p_i) - h \right]$$

3. **Probability of average over 75**

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Period 1 elicitation

Question

- $X : A_1 \times \Delta\Theta \longrightarrow \mathbb{R}$ action-dependent period 1 question.
- Question about period 2 beliefs.
- W.l.o.g. to assume that X does not depend on a_2 or θ:
  - Instead of asking "What is the probability of action a_2?",
  - we can ask "What is the probability that your beliefs are such that a_2 is optimal?"

Period 1 elicitation

Incentivizability

- Dynamic decision problem (A_1, A_2, u) plus question $X(a_1, p)$.
- X is *incentivizable in period 1* if there exists nondistortionary expansion $(A_1 \times \mathbb{R} \times S_1, A_2 \times S_2, v)$ such that

$$\mathbb{B}^v(\mu) \subseteq \{(a_1, r, s_1) : a_1 \in \mathbb{B}^u(\mu), r = \mathbb{E}_{\mu(a_1)} X(a_1, p)\},$$

  - $r \in \mathbb{R}$ - period 1 report of the expectation of a variable $X(a, p)$
  - S_t - supporting actions in period t

Lemma

If X is incentivizable, then also

$$Y(a,p) = \gamma(a)X(a,p) + \kappa(a)$$

for all functions $\gamma, \kappa : A \longrightarrow \mathbb{R}$ st. $\gamma(a) \neq 0$

- We say that questions X and Y are *equivalent*.

Period 1 elicitation: Action-independent information

Incentivizability

- Function $f : \Delta\Theta \longrightarrow \mathbb{R}$ is *finitely continuous* if there are finite sets of affine functions L_+, L_- st.

$$f(p) = \max_{l \in L_+} l(p) - \max_{l \in L_-} l(p).$$

- Any continuous function can be approximated by finitely continuous functions.

Period 1 elicitation: Action-independent information

Incentivizability

Theorem

Suppose that actions do not affect information.
The following questions are incentivizable for any finitely continuous function $f : \Delta\Theta \longrightarrow \mathbb{R}$*:*

- $X(a, p) = f(p)$,
- $X(a, p) = u^*(a, p) + f(p)$.

For some[1] *classes of problems, all*[2] *incentivizable questions have the above form.*

- Proof: BDM argument + supporting actions as in Chambers and Lambert (2021). ▸ Proof
- [1] Conditions are characterized for static problems in Pęski and Stewart (2025).
- [2] Replace "finitely continuous" by "continuous".

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- [1] Conditions are characterized for static problems in Pęski and Stewart (2025).
- [2] Replace "finitely continuous" by "continuous".

Period 1 elicitation: Action-independent information

Incentivizability

Example (Student)

1. ✓ **Expected test score:**

$$X(a, p) = \sum_i \max(p_i, 1 - p_i).$$

2. ✓ **Expected final grade:**

$$X(a, p) = (1 - a) \sum_i \max(p_i, 1 - p_i) + ah$$

3. ✗ **Prob of crossing** 75**:**

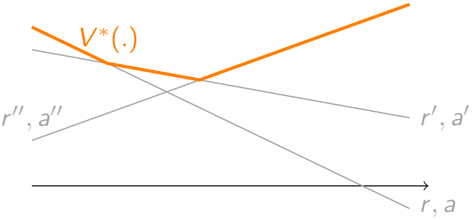
$$X(a, p) = 1\{(1 - a) \sum_i \max(p_i, 1 - p_i) + ah \geq 75\}$$

Period 1 elicitation: Action-independent information

Value of information

- Value function is convex:

$$V^*(\mu) = \max_{a_1, r} \mathbb{E}_\mu v^*(a_1, r, .)$$



- it is strictly convex at μ whenever there are multiple optimal actions and
- affine over intervals of beliefs μ' with a single optimal action.

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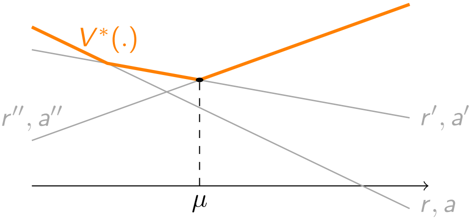

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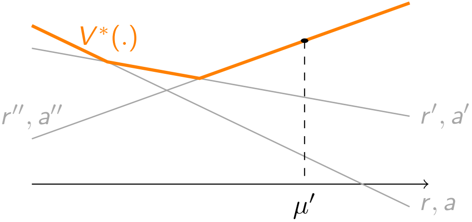

- it is strictly convex at μ whenever there are multiple optimal actions and
- affine over intervals of beliefs μ' with a single optimal action.

Period 1 elicitation: Action-independent information

Necessary conditions

- Take two adjacent actions a_1, b_1.
- Suppose that $\mathbb{B}_1^u(\mu_0) = \mathbb{B}_1^u(\mu_1) = \{a_1, b_1\}$ for a pair of two beliefs such that

$$\mathbb{E}_{\mu_0} X(a_1, \cdot) = \mathbb{E}_{\mu_1} X(a_1, \cdot).$$

- Hence, there is a unique report r following action a_1 and $V^*(.)$ is affine along the interval between μ_0 and μ_1.
- It means that there is a unique report r' following b_1, which implies

$$\mathbb{E}_{\mu_0} X(b_1, \cdot) = \mathbb{E}_{\mu_1} X(b_1, \cdot).$$

- If not, we have a contradiction with incentivizability.

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Period 1 elicitation: Action-independent information

Necessary conditions: Example

- Student chooses whether to write additional test $a_1 \in \{0,1\}$ to maximize the course average:

$$\frac{3}{4}h + \frac{1}{4}[a_1 \mathbb{E}_\mu P(p) + (1-a_1)h] = 80 + \frac{1}{4}a_1 \left(\mathbb{E}_\mu P(p) - 80\right)$$

where $P(p) = \max_{a_2} \mathbb{E}_p \sum_i 1\{a_i = \theta_i\}$ is the expected test score.

- Student is indifferent only if $\mathbb{E}_\mu P(p) = 80$.
- Question "*What's the probability that the final grade is* ≥ 75*?*"

$$X(a_1, p) = \begin{cases} 1 & a_1 = 0 \\ 1\{P(p) \geq 60\} & a_1 = 1. \end{cases}$$

- We show that X is not incentivizable.

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Period 1 elicitation: Action-independent information

Necessary conditions: Example

- Take beliefs
  - μ_0: 40 questions $p_i = 1$ and 60 questions $p_i = \frac{2}{3}$.
  - μ_1: 60 questions $p_i = 1$ and 40 questions $p_i = \frac{1}{2}$.

  Student is indifferent at each belief μ_0 and μ_1.
- But

$$\mathbb{E}_{\mu_0} X(0, .\cdot) = 1 = \mathbb{E}_{\mu_1} X(0, .\cdot),$$
$$\mathbb{E}_{\mu_0} X(1, .\cdot) = \mu_0(P(p) \geq 60) < 1 = \mu_1(P(p) \geq 60) = \mathbb{E}_{\mu_1} X(1, .\cdot).$$

- If V incentivizes X, then V^* must be, at the same time, affine and strictly convex along the interval between μ_0 and μ_1.
- Contradiction.

Period 1 elicitation: Action-independent information

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- Contradiction.

Period 1 sufficient conditions

Actions affect the quality of information

Theorem

Suppose that actions affect the quality of information.
The following questions are incentivizable for any affine function l:

- $X(a, p) = l(p)$,
- $X(a, p) = u^*(a, p) + l(p)$.

▸ Proof

Period 1 necessary conditions
Example 2

- Oncologist chooses diagnostics first, and then treatment to maximize QALY

$$Q_0 + (Dp - c_{\text{chemo}})1\{a_2 = \text{chemo}\} - c1\{a_1 = \text{biopsy}\}$$

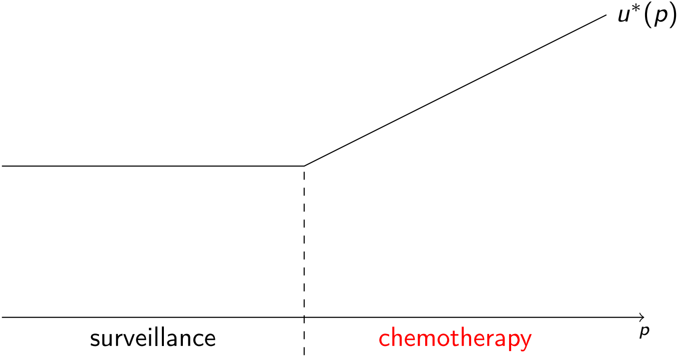
  - Q_0 - quality years under survaillance
  - D improvement due to chemo for high tumor count
  - c_{chemo} cost of chemo in QALY
  - c cost of biopsy in QALY.
- *How likely are you going to prescribe chemotherapy?*

$$X(a_1, p) = 1\{Dp > c_{\text{chemo}}\}$$

- We show that X is not incentivizable.

Period 1 necessary conditions

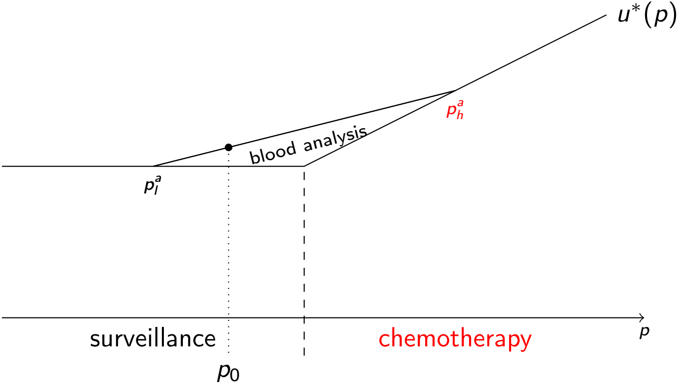
Example 2: Oncologist



- Second-period expected optimal payoffs (do not depend on first period diagnostic decision)

Period 1 necessary conditions

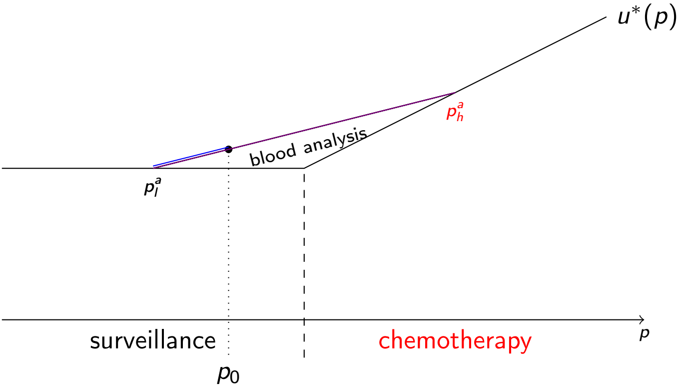
Example 2: Oncologist



- Blood analysis

Period 1 necessary conditions

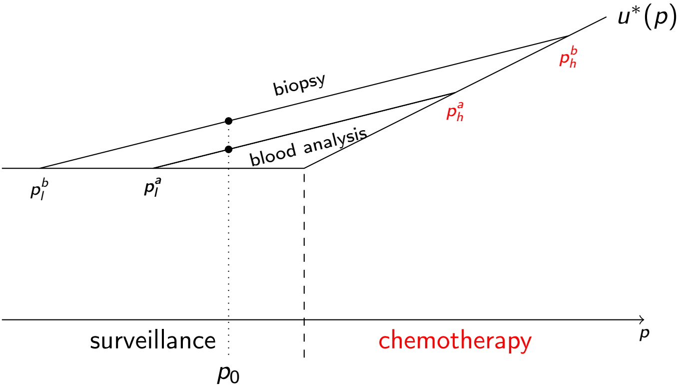
Example 2: Oncologist



- Probability of chemotherapy after blood analysis $\mathbb{E}_{\mu(a)}X(a,p) = \frac{p_0 - p_l^a}{p_h^a - p_l^a}$

Period 1 necessary conditions

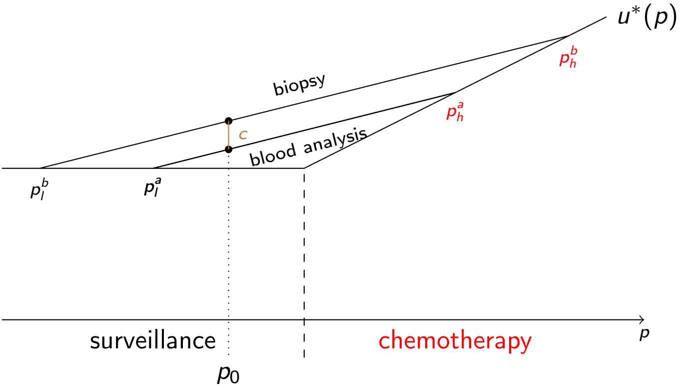
Example 2: Oncologist



- Biopsy

Period 1 necessary conditions

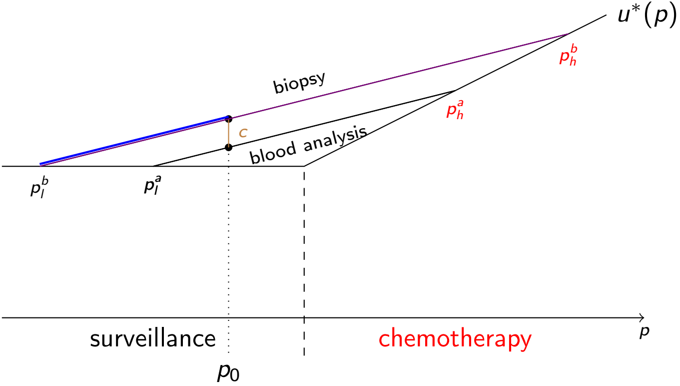
Example 2: Oncologist



- If the difference between expected payoffs is equal to the cost of biopsy c, the oncologist is indifferent between two diagnostics.

Period 1 necessary conditions

Example 2: Oncologist



- Probability of chemotherapy after biopsy $\mathbb{E}_{\mu(b)} X(b,p) = \frac{p_0 - p_l^b}{p_h^b - p_l^b}$

Period 1 necessary conditions

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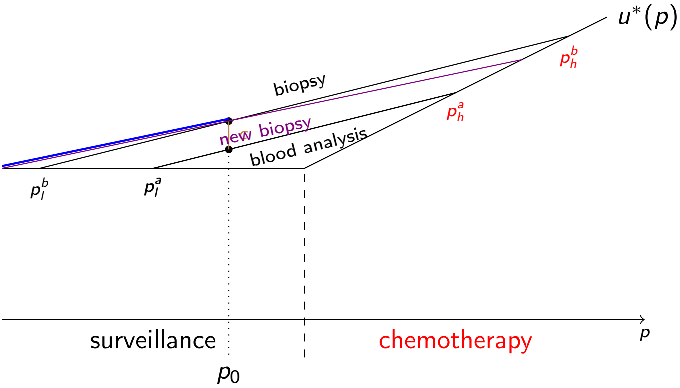

- There are beliefs μ' that keep the indifference and

$$\mathbb{E}_{\mu(a)}X(a,p) = \mathbb{E}_{\mu'(a)}X(a,p), \text{ but } \mathbb{E}_{\mu(b)}X(b,p) \neq \mathbb{E}_{\mu'(b)}X(b,p)$$

Period 1 necessary conditions

Example 2

- The above argument relies on the information being useful in the second-period decision making, i.e., partition $\mathcal{P}^u(a_1)$ is nontrivial.

Theorem

Suppose that actions may affect the quality of information and $\mathcal{P}^u(a_1)$ is nontrivial for all a_1.
X is incentivizable in the first period if and only if X is equivalent to

$$u^*(a,p) + l(p) \text{ for some affine } l \in L(\Theta).$$

Period 1 necessary conditions

Actions affect the state distribution

Theorem

Suppose that actions may affect the state distribution.
X is incentivizable in the first period if and only if X is equivalent to

$$u^*(a, p).$$

- If X is not equivalent to payoffs, then one can change the distributions $\mu(a_1)$ and $\mu(b_1)$ at the point of indifference so to
  - the indifference stays unchanged,
  - the expected value of the question after a_1 is unchanged, and
  - the expected value of the question after b_1 changes.

Outline

Period 2 elicitation

Question

- $X : A_1 \times A_2 \times \Theta \longrightarrow \mathbb{R}$ action-dependent period 2 question
- DM is asked to report $r = \mathbb{E}_p X(a_1, a_2, \cdot)$ in period 2

Example (Student)

1. expected test score: $X(a_1, a_2, \theta) = \sum_i 1\{a_{2,i} = \theta_i\}$
2. expected average: $X(a_1, a_2, \theta) = h + \frac{1}{4} a_1 [\sum_i 1\{a_{2,i} = \theta_i\} - h]$

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Period 2 elicitation

Incentivizability

- Dynamic decision problem (A_1, A_2, u) plus question $X(a_1, a_2, \theta)$.
- X is *incentivizable in period 2* if there exists nondistortionary expansion $(A_1 \times S_1, A_2 \times \mathbb{R} \times S_2, v)$ such that

$$\mathbb{B}^v(a_1, p) \subseteq \{(a_2, r, s_2) : a_2 \in \mathbb{B}^u(a_1, p), r = \mathbb{E}_p X(a_1, a_2, \theta)\},$$

    - S_t - supporting actions in period t
    - $r \in \mathbb{R}$ - period 2 report of the expectation of a variable $X(a_1, a_2, \theta)$
- Because the expansion is nondistortionary, we also have

$$\mathbb{B}^v(\mu) \subseteq \mathbb{B}^u(\mu) \times S_1.$$

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Period 2 sufficient conditions

Action-independent information

- We show that incentivizability second-period questions may affect the first-period incentives.
- Payoffs are *separable*, when we can represent them as

$$u(a_1, a_2, \theta) = u_1(a_1, \theta) + u_2(a_2, \theta).$$

- Payoffs are *essentially separable*, when $P(a_1, a_2) = P(b_1, a_2)$ for any a_2 and any pair of first-period actions a_1, b_1
  - first-period actions do not change second-period incentives.
  - example: Oncologist.

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Period 2 sufficient conditions

Action-independent information

Theorem

Suppose that information does not depend on actions.
Suppose that payoffs are separable.
For each $d \in \mathbb{R}^{\Theta}$, question

$$X(a_1, a_2, \theta) = u_2(a_2, \theta) + d(\theta)$$

is incentivizable.

- Proof: BDM in the second period plus the original first-period payoff.
- BDM is nondistortionary in the second period.
- Because the expected second period BDM payoff does not depend on a_1, it does not cause distortions in the first period.
- The Theorem has a generalization for essentially separable payoffs.

Period 2 sufficient conditions

Action-independent information

- Separability is an important assumption for incentivizability.
- Without separable payoffs, a question about payoffs

$$X(a_1, a_2, \theta) = X(a_1, a_2, \theta)$$

might not be incentivizable.

Period 2 necessary conditions

Value of information (2nd fact)

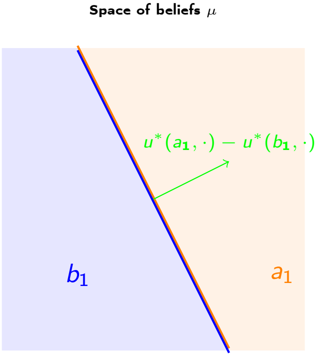
- If X is *incentivizable*, the first-period indifference condition must be preserved:

$$\mathbb{B}^u(\mu) = \{a_1, b_1\}$$

implies that for some s_1, t_1

$$(a_1, s_1), (b_1, t_2) \in \mathbb{B}^v(\mu).$$

- Hence, vectors $u^*(a_1, \cdot) - u^*(b_1, \cdot)$ and $v^*(a_1, s_1, \cdot) - v^*(b_1, t_1, \cdot)$ are collinear.



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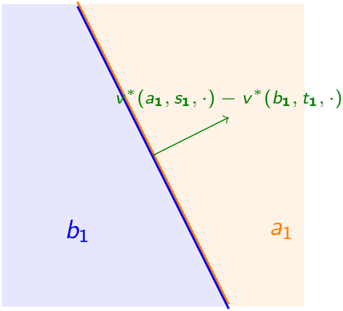
Space of beliefs μ



Period 2 necessary conditions

Example Student

- $a_1 = 0, 1$ decision whether to write the test.
- Assume $N = 1$, i.e., $a_2 \in \{0, 1\}$.
- Payoffs:

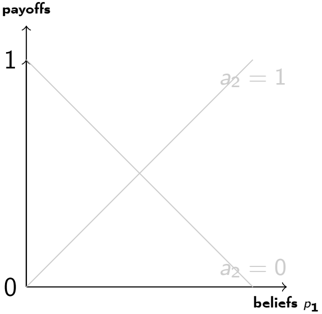
$$u(a_1, a_2, \theta) = h + \frac{1}{4} a_1 (1\{a_2 = \theta\} - h).$$

- Question: "What are the payoffs?"

$$X(a_1, a_2, \theta) = u(a_1, a_2, \theta).$$

Period 2 necessary conditions

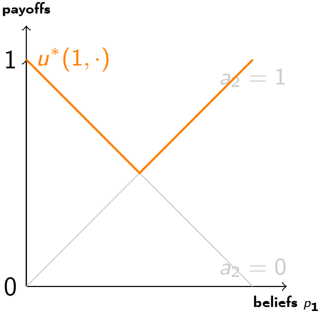
Example Student



- Payoffs from actions a_2 after test $a_1 = 1$.

Period 2 necessary conditions

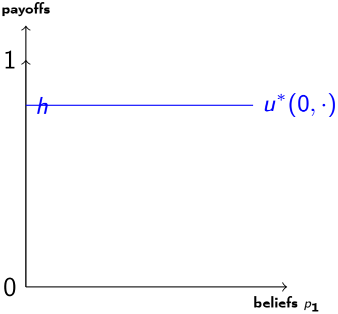
Example Student



- Expected payoffs after the test $a_1 = 1$.

Period 2 necessary conditions

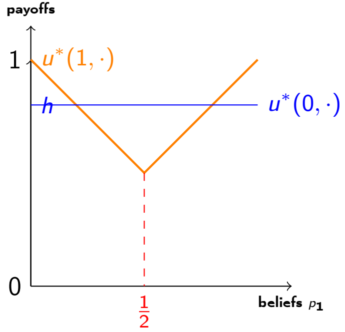
Example Student



- Expected payoffs after no test $a_1 = 0$.

Period 2 necessary conditions

Example Student



- The payoff difference

$$u^*(1,p) - u^*(0,p) \text{ is strictly convex only at } p = \frac{1}{2}.$$

Period 2 necessary conditions

Example Student

- Because $\max_{a_2} \mathbb{E}_p X(1, a_2, \theta) = u^*(1, p)$ is changing for *each* p,

$$v^*(1, s_1, p) \text{ is strictly convex at each } p.$$

- Because $\max_{a_2} \mathbb{E}_p X(0, a_2, \theta) = u^*(0, p)$ is constant at *each* p,

$$v^*(0, t_1, p) \text{ is affine.}$$

- Hence,

$$v^*(1, s_1, p) - v^*(0, t_1, p) \text{ is strictly convex at each } p \in [0, 1].$$

- Contradiction to colinearity with $u^*(1, p) - u^*(0, p)$. which is strictly convex only at $p = \frac{1}{2}$.

Period 2 necessary conditions

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$$v^*(0, t_1, p) \text{ is affine.}$$

- Hence,

$$v^*(1, s_1, p) - v^*(0, t_1, p) \text{ is strictly convex at each } p \in [0, 1].$$

- Contradiction to colinearity with $u^*(1, p) - u^*(0, p)$. which is strictly convex only at $p = \frac{1}{2}$.

Period 2 necessary conditions

Example Student

- Because $\max_{a_2} \mathbb{E}_p X(1, a_2, \theta) = u^*(1, p)$ is changing for *each* p,

$$v^*(1, s_1, p) \text{ is strictly convex at each } p.$$

- Because $\max_{a_2} \mathbb{E}_p X(0, a_2, \theta) = u^*(0, p)$ is constant at *each* p,

$$v^*(0, t_1, p) \text{ is affine.}$$

- Hence,

$$v^*(1, s_1, p) - v^*(0, t_1, p) \text{ is strictly convex at each } p \in [0, 1].$$

- Contradiction to colinearity with $u^*(1, p) - u^*(0, p)$. which is strictly convex only at $p = \frac{1}{2}$.

Period 2 necessary conditions

Action-independent information

- For each first-period action a_1, let

$$D(a_1) = \{u(a_1, a_2, \cdot) - u(a_1, b_2, \cdot) : a_2, b_2 \in A_2 \text{ st. } a_2 \neq b_2\} \subseteq \mathbb{R}^{\Theta}.$$

- Payoffs are *generic* if sets $D(a_1)$ and $D(b_1)$ are disjoint for any pair of essential best responses a_1, b_1.
- intuitively: partitions $\mathcal{P}(a_1)$ and $\mathcal{P}(b_1)$ are not aligned.

Theorem

Suppose that information does not depend on actions.
For generic payoffs, if X is incentivizable, then it is equivalent to action-independent question.

- Proof: generalization of the idea from the example.

Period 2 necessary conditions

Action-independent information

Theorem

Suppose that information does not depend on actions and X is incentivizable in the second period.
If the decision problem has essentially separable payoffs, then, X is equivalent to a question profile that depends only on the second-period actions.

- Positive result for separable payoff case.

Period 2 necessary conditions

Actions affect the quality of information

Theorem

Suppose that actions may affect the quality of information or state distribution. There are no nontrivial (non-constant) incentivizable questions in the second period.

- Any second-period incentivization changes the first-period value of information.
- If actions affect the quality of information, this changes the first-period incentives.

Outline

Conclusions

- Nondistortionary elicitation of dynamic beliefs.
- Two types of information:
  - action-independent,
  - actions affect the quality of information,
  - actions affect the distribution of states.
- Two types of questions:
  - period 1,
  - period 2.
- Many types of distortions.

Conclusions

Incentivizable questions:

	Action-independent information	Actions may affect the quality of information	Actions may affect the states
$t = 1$	$u^*(a_1, p) + f(p)$ PS25 conditions	$u^*(a_1, p) + l(p)$ nontrivial continuation	$u^*(a_1, p)$ always
$t = 2$	$u_2(a_2, \theta) + d(\theta)$ only if payoffs are separable	✗	✗

Outline

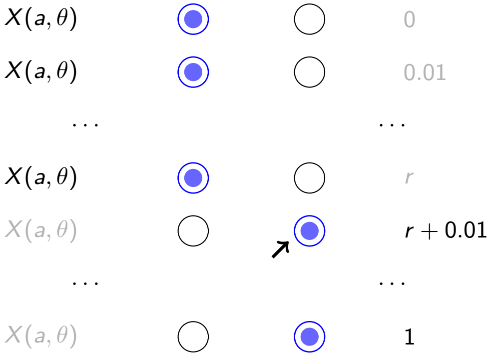
Period 1 sufficient conditions

Action-independent information

- Assume first that p is observable. (It is not!)
- W.l.o.g. normalize $0 < X < 1$.
- Becker-DeGroot-Marschak (Becker et al. (1964)):
  - subject reports $r \in [0, 1]$,
  - random number x is drawn uniformly from $[0, 1]$,
  - if $x \leq r$, the subject receives $X(a, p)$,
  - otherwise, if $r \leq x$, the subject receives x.

Period 1 sufficient conditions

Action-independent information



Period 1 sufficient conditions

Action-independent information

- Becker-DeGroot-Marschak: Let

$$\bar{V}(r, a, p) = \int_0^r X(a; p)\, dx + \int_r^1 x dx = X(a; p)\, r - \frac{r^2}{2} + \frac{1}{2}$$

- Then, the expected payoff

$$\mathbb{E}_\mu V(r, a, \cdot) = (\mathbb{E}_\mu X) r - \frac{r^2}{2} + \frac{1}{2} \text{ is maximized by } r = \mathbb{E}_\mu X(a, .), \text{ and}$$

- the expected r-optimal payoff

$$\max_r \mathbb{E}_\mu V(r, a, \cdot) = \frac{1}{2}(\mathbb{E}_\mu X)^2 + \frac{1}{2} = \frac{1}{2}(\mathbb{E}_\mu u^*(a, .) + \mathbb{E}_\mu f(\cdot))^2 + \frac{1}{2}$$

is maximized by $a \in \arg\max \mathbb{E}_p u(a, .)$.

- Problem: p is not observable!

Period 1 sufficient conditions

Action-independent information

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  is maximized by $a \in \arg\max \mathbb{E}_p u(a,.)$.
- Problem: p is not observable!

Period 1 sufficient conditions

Action-independent information

- Suppose first that $X(a_1, p) = u^*(a_1, p) = \max_{a_2} \mathbb{E}_p u(a_1, a_2, \theta)$.
- Instead of $X(a_1, p)$, the DM chooses a_2, and receives $u(a_1, a_2, \theta)$. Let

$$V(a_1, r, a_2, \theta) = u(a_1, a_2, \theta)r - \frac{r^2}{2} + \frac{1}{2}$$

- Then

$$\mathbb{E}_p V(a_1, r, a_2, \theta) = r\mathbb{E}_p u(a_1, a_2, \theta) - \frac{r^2}{2} + \frac{1}{2} \text{ is maximized by } a_2 \in \mathbb{B}^u(a_1, p)$$

$$\mathbb{E}_\mu \max_{a_2} \mathbb{E}_p V(a_1, r, a_2, \theta) = r\mathbb{E}_\mu u^*(a_1, p) - \frac{r^2}{2} + \frac{1}{2} \text{ is maximized by } r = \mathbb{E}_\mu u^*(a_1, .)$$

$$\max_r \mathbb{E}_\mu \max_{a_2} \mathbb{E}_p V(a_1, r, a_2, \theta) = \frac{1}{2}(\mathbb{E}_\mu u^*(a_1, \theta))^2 + \frac{1}{2} \text{ is maximized by } a_1 \in \mathbb{B}^u(\mu).$$

Period 1 sufficient conditions

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Period 1 sufficient conditions

Action-independent information

- To elicit
$$X(a_1, p) = u^*(a_1, p) + f(p)$$
for general finitely continuous
$$f(p) = \max_{l \in L_+} \mathbb{E}_p l(\theta) - \max_{l \in L_-} \mathbb{E}_p l(\theta),$$
- we use BDM, like above,
- enhanced with period 2 supporting actions L_+, L_-, like in Chambers and Lambert (2021).

Period 1 sufficient conditions

Action-independent information

- The DM chooses a_2 and $l_+ \in L_+, l_- \in L_-$, and receives

$$V(a_1, r, a_2, l_+, l_-, \theta) = (u(a_1, a_2, \theta) + l_+(\theta) - l_-(\theta))r - \frac{r^2}{2} + \frac{1}{2} + 2l_-(\theta).$$

The last term ensures that period 2 choice of l_- maximizes $\mathbb{E}_p l_-(\theta)$.

- Then,

$$\mathbb{E}_p V(a_1, r, a_2, l_+, l_-, \theta) = r(u(a_1, , a_2, \theta) + l_+(p) - l_-(p)) - \frac{r^2}{2} + \frac{1}{2} + 2l_-(p)$$

is maximized by $a_2 \in \mathbb{B}^u(a_1, p)$ and correct l_+, l_-, and

$$\mathbb{E}_\mu \max_{a_2} \mathbb{E}_p V(a_1, r, a_2, l_+, l_-, \theta) = r\mathbb{E}_\mu(u^*(a_1, p) + f(p)) - \frac{r^2}{2} + \frac{1}{2} + 2\mathbb{E}_\mu \max_{l \in L_-} l(p)$$

is maximized by $r = \mathbb{E}_\mu(u^*(a_1, .) + f(p))$

$$\max_r \mathbb{E}_\mu \max_{a_2} \mathbb{E}_p V(a_1, r, a_2, l_+, l_-, \theta) = \frac{1}{2}(\mathbb{E}_\mu u^*(a_1, \theta) + f(p))^2 + \frac{1}{2} + 2\mathbb{E}_\mu \max_{l \in L_-} l(p)$$

is maximized by $a_1 \in \mathbb{B}^u(\mu)$.

◂ Back (sufficiency independent)

Period 1 sufficient conditions

Action-independent information

- The DM chooses a_2 and $l_+ \in L_+, l_- \in L_-$, and receives

$$V(a_1, r, a_2, l_+, l_-, \theta) = (u(a_1, a_2, \theta) + l_+(\theta) - l_-(\theta))r - \frac{r^2}{2} + \frac{1}{2} + 2l_-(\theta).$$

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$$\mathbb{E}_\mu \max_{a_2} \mathbb{E}_p V(a_1, r, a_2, l_+, l_-, \theta) = r\mathbb{E}_\mu(u^*(a_1, p) + f(p)) - \frac{r^2}{2} + \frac{1}{2} + 2\mathbb{E}_\mu \max_{l \in L_-} l(p)$$

is maximized by $r = \mathbb{E}_\mu(u^*(a_1, .) + f(p))$

$$\max_r \mathbb{E}_\mu \max_{a_2} \mathbb{E}_p V(a_1, r, a_2, l_+, l_-, \theta) = \frac{1}{2}(\mathbb{E}_\mu u^*(a_1, \theta) + f(p))^2 + \frac{1}{2} + 2\mathbb{E}_\mu \max_{l \in L_-} l(p)$$

is maximized by $a_1 \in \mathbb{B}^u(\mu)$.

◂ Back (sufficiency independent)

Period 1 sufficient conditions

Action-independent information

- The DM chooses a_2 and $l_+ \in L_+, l_- \in L_-$, and receives

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- Then,

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is maximized by $a_2 \in \mathbb{B}^u(a_1, p)$ and correct l_+, l_-, and

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is maximized by $r = \mathbb{E}_\mu(u^*(a_1, .) + f(p))$

$$\max_r \mathbb{E}_\mu \max_{a_2} \mathbb{E}_p V(a_1, r, a_2, l_+, l_-, \theta) = \frac{1}{2}(\mathbb{E}_\mu u^*(a_1, \theta) + f(p))^2 + \frac{1}{2} + 2\mathbb{E}_\mu \max_{l \in L_-} l(p)$$

is maximized by $a_1 \in \mathbb{B}^u(\mu)$.

Back (sufficiency independent)

Period 1 sufficient conditions

Action-independent information

- The DM chooses a_2 and receives

$$V(a_1, r, a_2, \theta) = (u(a_1, a_2, \theta) + I(\theta))r - \frac{r^2}{2} + \frac{1}{2}.$$

- Then,

$$\max_r \mathbb{E}_\mu \max_{a_2} \mathbb{E}_p V(a_1, r, a_2, \theta) = \frac{1}{2}(\mathbb{E}_{\mu(a_1)}(u^*(a_1, \theta) + I(p)))^2 + \frac{1}{2}$$

is maximized by $a_1 \in \mathbb{B}^u(\mu)$.

- Note that $E_{\mu(a_1)} I(p)$ does not depend on a_1.

Back (sufficiency quality)