Value-based distance

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Outline

Introduction

- Small differences in information may lead to significant differences in the behavior (Rubinstein, 1989).
- "Topologies on types" literature (Dekel *et al.* (2006), Chen *et al.* (2016)).
- What about payoffs?

Introduction

- Goal: Define a payoff-based notion of distance between information structures.
- Problem A:
  - earlier literature focused on rationalizability as solution concept,
  - characterization through Mertens-Zamir hierarchies
  - no existence issues,
  - but to talk about payoffs we need an "ex ante", equilibrium-like solution concept.

Introduction

- Goal: Define a payoff-based notion of distance between information structures

  $$d(u,v) = \sup_{g \in G} |\pi(u,g) - \pi(v,g)|.$$

  where
  - u and v are information structures (i.e., type spaces) and
  - $\pi(g,u)$ are "equilibrium payoffs" in Bayesian game with payoffs g on u,
  - G are all "bounded" game payoffs.
- A tight bound on the value of information

Introduction

- Goal: Define a payoff-based notion of distance between information structures

$$d(u, v) = \sup_{g \in G} |\pi(u, g) - \pi(v, g)|.$$

- Problem B: multiplicity
  - Gossner (1996) and Kajii and Morris (1998) compute distance between sets,
- Problem C: because of freedom to choose games, the notion of payoff-based distance is trivial,
  - approximate equilibrium (Kajii and Morris (1998)),
- Problem D: existence issues (Simon (2003)).

Introduction

- Idea: restrict the games to zero-sum

$$d(u,v) = \sup_{g \text{ is zero-sum}} \left|\mathrm{val}\,(u,g) - \mathrm{val}(v,g)\right|.$$

  - $\mathrm{val}(u,g)$ is the value of zero-sum game (g,u),
  - value-based distance

Introduction

- Idea: restrict the games to zero-sum

$$d(u, v) = \sup_{g \text{ is zero-sum}} \left| \mathrm{val}\,(u, g) - \mathrm{val}(v, g) \right|.$$

  - no multiplicity, no existence issues, non-trivial,
  - a tight bound on the willingness to pay for information.
- Zero-sum games have a natural comparative statics wrt information (Peski (2008)).
- Re-examine the "topologies on types" literature
  - most constructions and counter-examples are about coordination games.
- Still, an important class of games.

Introduction

Results:

1. Characterization of the distance
2. Value of (different pieces of) information: substitutes, complements, joint information
3. Value-based topology on countable types is equal to the weak (i.e., product) topology.
   But ...
4. Value-based distance is not pre-compact.
   4.1 last unsolved problem of Mertens (86)
5. Payoff-based distance.

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Outline

Value-based distance

- u, v are countable information structures over finite K,
  - common prior, countable types $t_i \in \mathbb{N}$,
  - basic distance (L^1-norm):

$$\|u - v\| = \sum_{k,t,s} |u(k,t,s) - v(k,t,s)|.$$

- G_0 is a class of zero-sum payoff functions $g : A_1 \times A_2 \times K \to \mathbb{R}$ (payoffs of player 1) st.
  - A_i are finite or countable, and
  - $\sup_{a_1 a_2} |g(a_1, a_2)| \leq 1$,
  - player 1 is the maximizer, and

$$\begin{aligned} \text{val}(u, g) &= \max_{\sigma_1} \min_{\sigma_2} E_{\sigma_1,\sigma_2} g(a_1, a_2, k) \\ &= \min_{\sigma_2} \max_{\sigma_1} E_{\sigma_1,\sigma_2} g(a_1, a_2, k). \end{aligned}$$

Value-based distance

Characterization

Definition

Value-based distance (VBD)

$$d(u, v) = \sup_{g \in G_0} \left|\mathrm{val}\,(u, g) - \mathrm{val}(v, g)\right|.$$

- Tight bound on cost/benefits of moving from one to another information structures,
- $d(u, v) \leq \|u - v\| \leq 2.$
  - the first inequality is a property of zero-sum games.

Value-based distance

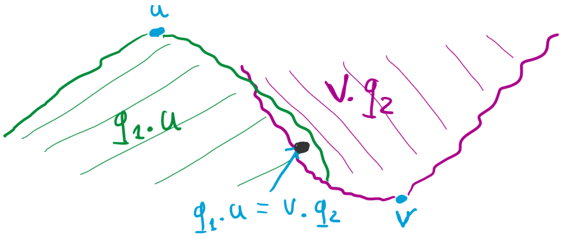
Characterization

- A garbling is a mapping $q : \mathbb{N} \to \Delta\,\mathbb{N}$.
- $q.u$ and $u.q$ denote garbled information structure obtained from u.
  - $q.u$ means worse information for player 1,
  - $u.q$ means worse information for player 2.
- Peski, 08:

$$\forall_{g \in G_0} \mathrm{val}\,(u, g) \geq \mathrm{val}\,(v, g) \iff \exists_{q_1, q_2} q_1.u = v.q_2 .$$

Value-based distance
Characterization

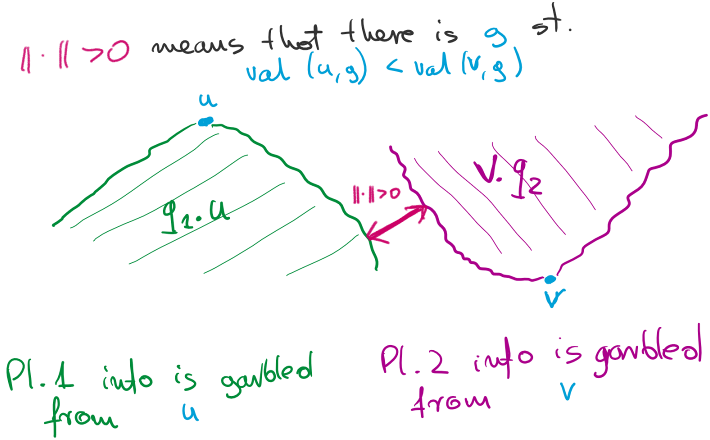
u is better (for pl. 1) than v iff $\exists q_1, q_2$ st.



Pl. 1 info is garbled from u Pl. 2 info is garbled from v

Value-based distance

Characterization



Value-based distance

Characterization

Theorem

$$\sup_{g \in G} \left(\mathrm{val}(v,g) - \mathrm{val}(u,g)\right) = \min_{q_1,q_2} \|q_1.u - v.q_2\|.$$

It follows that

$$d\left(u,v\right) = \max\left(\min_{q_1,q_2} \|q_1.u - v.q_2\|, \min_{q_1,q_2} \|u.q_1 - q_2.v\|\right).$$

Value-based distance

Characterization

Theorem

$$\sup_{g \in G} \left(\mathrm{val}(v, g) - \mathrm{val}(u, g)\right) = \min_{q_1, q_2} \|q_1.u - v.q_2\|.$$

- interpretation,
- reduces the complexity of the problem (max-max-mins to min-min),
- sufficiently easy to use in calculations and applications.

Value-based distance

Characterization: Proof

- Part 1:
- Value is monotonic wrt. information:

$$\mathrm{val}(v,g) - \mathrm{val}(u,g) \leq \mathrm{val}(v.q_2,g) - \mathrm{val}(q_1.u,g) \leq \|v.q_2 - q_1.u\|.$$

- Take inf over garblings.

Value-based distance

Characterization: Proof

- identify each garbling with a mixed strategy,
  - *Id* is a special case
- the expected payoff: $\langle g, q_1.u.q_2\rangle$, where $\langle g, u\rangle = \sum_{k,c,d} g(k,c,d)u(k,c,d)$,
- Using the Minmax Theorem,

$$
\begin{aligned}
\mathrm{val}(v,g) - \mathrm{val}(u,g) &\geq \inf_{q_2}\langle g, v.q_2\rangle - \sup_{q_1}\langle g, q_1.u\rangle \\
&= \sup_{g}\inf_{q_1,q_2}\langle g, v.q_2 - q_1.u\rangle \\
&= \inf_{q_1,q_2}\sup_{g}\langle g, v.q_2 - q_1.u\rangle \\
&= \inf_{q_1,q_2}\left\| q_1.u - v.q_2 \right\|.
\end{aligned}
$$

Outline

Applications

- Impact of marginal over K.
- Single-agent problems
- Value of additional information: Substitutes
- Value of additional information: Complements
- Value of jointly-held information

Applications

Impact of marginal over K.

Proposition

$\forall u, v$, *if* $\operatorname{marg}_K u = p, \operatorname{marg}_K v = q$, *we have*

$$\sum_k |p_k - q_k| \leq d(u,v) \leq 2\left(1 - \max_{p',q' \in \Delta K} \sum_k \min\left(p_k q'_k, p'_k q_k\right)\right). \tag{1}$$

- If $p = q$, the upper bound is equal to $2(1 - \max_k p_k)$,
- a bound on the strategic value of information.

Applications

Single-agent problems

- Single-agent problems $g \in G_0$ (minimizer's action is irrelevant):

$$d_1(u, v) = \sup_{g \in G_1} \left|\text{val}\left(u, g\right) - \text{val}(v, g)\right| \leq d\left(u, v\right).$$

- d_1 is (relatively) easy to characterize (especially when $K = \{0, 1\}$).

Applications

Single-agent problems

Proposition

If

$$\mu \in \Delta\left(K \times T \times T' \times S\right),$$
$$u = marg_{K \times T \times S}\mu,$$
$$v = marg_{K \times T' \times S}\mu,$$

and $(T$ and $S)$ and $(T'$ and $S)$ are conditionally independent given K, then

$$d\left(u, v\right) = d_1\left(u, v\right).$$

- We say that information under μ is conditionally independent (ICI).
- ICI is needed because otherwise $T \to T'$ may not change information about K, but improve about S.

Applications

Single-agent problems

Proposition

If

$$u \sim \Delta\left(K \times T \times \not{T'} \times S\right),$$
$$v \sim \Delta\left(K \times \not{T} \times T' \times S\right),$$

and $T \times T'$ and S are conditionally independent given K, then

$$d\left(u, v\right) = d_1\left(u, v\right).$$

- We say that information under μ is conditionally independent (μ has ICI).
- ICI is needed because otherwise $T \to T'$ may reduce information about K, but improve about S.

Applications

Value of additional information

- Special case:

$$u \sim \Delta(K \times T \times T' \times S),$$
$$v \sim \Delta\left(K \times T \times \cancel{T'} \times S\right)$$

If u has ICI,

$$d(u, v) = d_1(u, v).$$

- Value of additional (conditionally independent) information can be bounded by its value in single-agent problems.

Applications

Value of additional information: Substitutes

Proposition

Suppose that

$$u \sim \Delta\left(K \times (T \times T_1 \times T_2) \times S\right) \text{ and } v \sim \Delta\left(K \times (T \times T_1 \times \cancel{T_2}) \times S\right),$$
$$u' \sim \Delta\left(K \times (T \times \cancel{T_1} \times T_2) \times S\right) \text{ and } v' \sim \Delta\left(K \times (T \times \cancel{T_1} \times \cancel{T_2}) \times S\right)$$

and that, under u, T_1 is conditionally independent from $T \times T_2 \times S$ given K. Then,

$$d(u, v) \leq d(u', v').$$

- $d(u, v)$ is the value of T_2 in the presence of T_1,
- $d(u, v)$ is the value of T_2 in the absence of T_1,
- Two additional pieces of player's information are substitutes.

Applications

Value of additional information: Complements

Proposition

Suppose that

$$u \sim \Delta\left(K \times (T \times T_1) \times (S \times S_1)\right) \text{ and } v \sim \Delta\left(K \times (T \times \cancel{T_1}) \times (S \times S_1)\right),$$
$$u' \sim \Delta\left(K \times (T \times T_1) \times (S \times \cancel{S_1})\right) \text{ and } v' \sim \Delta\left(K \times (T \times \cancel{T_1}) \times (S \times \cancel{S_1})\right)$$

and that, under u, $T \times T_1$ is conditionally independent from S given K. Then,

$$d(u, v) \geq d(u', v').$$

- $d(u, v)$ is the value of T_1 in the presence of S_1,
- $d(u, v)$ is the value of T_1 in the absence of S_1,
- Two additional pieces of opposing players' information are complements.

Applications

Value of jointly-held Information

- Example: states and types are equally likely

$t \backslash s$	$*$
$*$	$k = 0^{1/2}1^{1/2}$

$t \backslash s$	$+$	-
$+$	$k = 1$	$k = 0$
-	$k = 0$	$k = 1$

- On the right, information about the state k is held jointly.

Applications

Value of jointly-held Information

- Example: states and types are equally likely

$v: t \backslash s$	$*$
$*$	$k = 0^{1/2}1^{1/2}$

$u: t \backslash s$	$+$	-
$+$	$k = 1$	$k = 0$
-	$k = 0$	$k = 1$

- On the right, information about the state k is held jointly:
  - t is independent from k,
  - s is independent from k,
  - (t, s) is NOT independent from k.
- We show that $d(u, v) = 0$.

Applications

Value of jointly-held Information

- Consider a distribution $\mu \in \Delta(X \times Y \times Z)$
- Random variables x and y are ε-conditionally independent given z if

$$\sum_{z} \mu(z) \sum_{x,y} |\mu(x, y|z) - \mu(x|z)\mu(y|z)| \leq \varepsilon.$$

  - "ex ante" notion

Proposition

Suppose that

$$u \sim \Delta\left(K \times (T \times T_1) \times (S \times S_1)\right)$$
$$v \sim \Delta\left(K \times (T \times \cancel{T_1}) \times (S \times \cancel{S_1})\right),$$

and

- T_1 *is* ε*-conditionally independent from* (K, S) *given* T*, and*
- S_1 *is* ε*-conditionally independent from* (K, T) *given* S*.*

Then,

$$d(u, v) \leq \varepsilon.$$

Applications

Value of jointly-held Information

- Example: State $k = 0, 1$ equally likely
  - u= common knowledge of the state
  - v=Rubinstein's email game
    - pl. 1 observes the state,
    - If state 1, pl. 1 sends an email that goes back and forth, with probability of being lost $\alpha > 0$,
  - starting from v, learning the true state for pl. 2 is $C\alpha$-conditionally independent from the state and pl. 1's info for some constant C.
  - $d(u, v) \leq C\alpha$.

Outline

Value-based topology

- Mertens-Zamir constructed the *universal type space* U
  - distributions over consistent common prior hirerachies,
  - value of the zero-sum game depends only on the representation of an information structure in U,
  - natural plays to study the distance,
- U is compact under the weak (i.e. the product) topology of the convergence of belief hierarchies,
  - countable info structures $U_0 \subseteq U$ are dense in U.
- "Topologies on types" literature: the weak topology is too weak to capture continuous strategic behavior.

Value-based topology

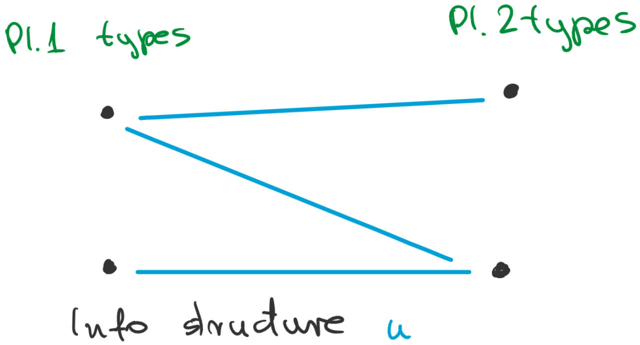
Theorem

For any (countable) u and any sequence $u_n \to u$ in the weak topology,

$$d(u_n, u) \to 0.$$

- convergence of higher order belief ensures convergence of values across all zero-sum games,
- "countable" is important.

Value-based topology
Proof



Value-based topology

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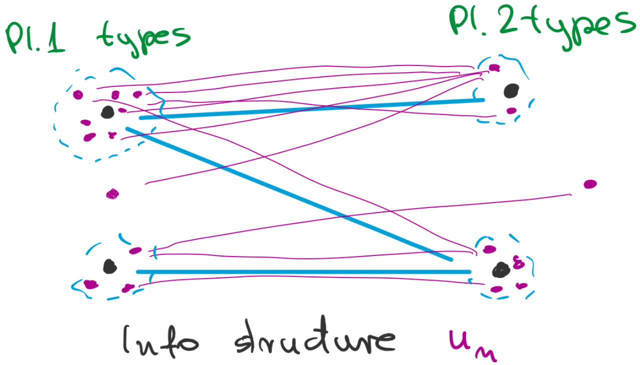

Outline

(Non)-compactness of VBD

Last open problem of Mertens (86)

- VBD convergence is equivalent to convergence in product topology around countable type spaces.
- Countable type spaces are dense in product topology.
- Does it mean that VBD topology is equivalent to product topology everywhere?
- In particular, product topology is compact.
- Does it mean that VBD topology is also compact?

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(Non)-compactness of VBD

Last open problem of Mertens (86)

- In 1986, Mertens asked whether family of functions $(u \to \mathrm{val}\,(u, g))_g$ is uniformly equicontinuous,
  - or, equivalently, where U is compact under VBD.
- Existence of finite classifications of type spaces.
- The question is important for zero-sum repeated games:
  - convergence of value $v_\delta \to v_1$ proven in some classes of games (large lit. started from Mertens 71),
  - a proof for general stochastic games is still missing,
  - uniform equicontinuity of value would deliver it immediately.

(Non)-compactness of VBD

Theorem

There exists $\varepsilon > 0$ and a sequence u_n such that for each $k \leq n$ u_n and u_k has the same kth hierarchy of belief and such that for each $k \neq n$

$$d(u_k, u_n) > \varepsilon.$$

- U is not compact under VBD.
- $\varepsilon > 0$ in the proof is v. small, but our proof is not careful.

(Non)-compactness of VBD

- Answer to Mertens is negative.
- Universe of type spaces is large, even if we restrict ourselves to zero-sum games only.
- Be wary of (over)interpreting “topological” results.

(Non)-compactness of VBD
Proof

- Markov chain $a_1, a_2, a_3, \ldots$ over $\{1, \ldots, N\}$,
  - a_1 depends on $k \in \{0, 1\}$.
- Pl. 1 observes $t^{(n)} = (a_1, \ldots, a_{2n+1})$,
  Pl. 2 observes $s^{(n)} = (a_2, \ldots, a_{2n+2})$.
- In game g^k, players are asked to report k first signals and we check whether their reports are consistent.

Outline

Payoff-based distance

- What can we say about non-zero sum games?
- Payoff-based distance

$$d_{NZS}(u,v) = \sup_{g \in G} d^H_{\max}\left(\mathrm{Eq}(u,g), \mathrm{Eq}(v,g)\right), \qquad (2)$$

  where
  - G are all non-zero-sum games,
  - $\mathrm{Eq}(u,g)$ is the set of ex-ante equilibrium payoffs and
  - d^H is the Hausdorf distance.
- Kajii and Morris (98) define two structures to be strategically ε-close if for any equilibrium on one structure, there is ε-interim equilibrium on the other with ε-close payoffs (roughly).
  Our definition is easier to interpret for large ε.
  But, there is a cost.

Payoff-based distance

- Information structures are simple if they have a unique (up to measure 0) common knowledge event.
- $\tilde{u}$ is the representation of u in Mertens-Zamir uts.

Theorem

Suppose that u, v are non-redundant information structures. If u and v are simple, then

$$d_{NZS}(u,v)(u,v) = \begin{cases} 0, & \text{if } \tilde{u} = \tilde{v}, \\ 2 & \text{otherwise.} \end{cases}$$

Payoff-based distance

- Information structures are simple if they have a unique (up to measure 0) common knowledge event

Theorem

More generally, suppose that $u = \sum p_\alpha u_\alpha$ *and* $v = \sum q_\alpha v_\alpha$ *are the decompositions into simple information structures. We can always choose the decompositions so that* $\tilde{u}_\alpha = \tilde{v}_\alpha$ *for each* α*. Then,*

$$d_{NZS}(u, v) = \sum_\alpha |p_\alpha - q_\alpha| .$$

Outline

Conclusions

- Value-based notion on distance on information structures.
  - a tight bound on the willingness to pay for information
  - tractable -> characterization, applications.
- Some predictions of the literature do not hold when restricted to zero-sum games,
  - no email-game type of examples,
- But, VBD is not compact.

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