Properly universal type space

Jeffrey C. Ely, Marcin Peski

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1 Notation and definitions

The notation and definitions mostly follow [HBIR]. We remind some of them to make the note self-contained. For the sake of clarity, we restrict the discussion to the case of two players only. Everything extends to more players.

1.1 Type spaces

Let Ω be a compact Polish space. A type space T over Ω is a tuple $(T_i, \mu_i)_{i=1,2}$, where T_i are measurable spaces and $\mu_i : T_i \to \Delta(\Omega \times T_{-i})$ is a measurable mapping. If T_i can be given a topology under which T_i are Polish spaces and μ_i are Borel measurable, then we refer to T as a Polish type space.

A belief-preserving type mapping between type spaces T, T' over Ω is a profile $\phi = (\phi_i)_{i=1,2}$ of measurable mappings $\phi_i : T_i \to T'_i$ such that for any player i, any type $t_i \in T_i$, any measurable subset $S_0 \subseteq \Omega, S' \subseteq T'_{-i}$,

$$\mu'\left(S_0 \times S' | \phi_i(t_i)\right) = \mu\left(S_0 \times \phi_{-i}^{-1}(S') | t_i\right).$$

We say that measure preserving type mapping ϕ is *injective* if ϕ_i is an injection for each i.

1.2 Universal type space

Let $V(\Omega)$ be the smallest Polish space with the property that

$$V(\Omega) \simeq \Delta(\Omega \times V(\Omega)).$$

The existence (and the uniqueness up to isomorphism) is shown in [MZ UTS]. Moreover, space $V(\Omega)$ is compact. Let $\mu^{\Omega} : U(\Omega) \to \Delta(\Omega \times U(\Omega))$ be the isomorphsim. Let $U_i(\Omega), i = 1, 2$ be two disjoint copies of set $V(\Omega)$ and let $\mu_i^{\Omega} : U_i(\Omega) \to \Delta(\Omega \times U_{-i}(\Omega))$ be isomorphisms derived from μ^{Ω}. Then, $U(\Omega) = \left(U_i(\Omega), \mu_i^{\Omega}\right)_{i=1,2}$ is a Polish type space with the property that

$$U_i(\Omega) \simeq \Delta(\Omega \times U_{-i}(\Omega)).$$

[MZ UTS] shows that $U(\Omega)$ has the following universal property for the category of type spaces equipped with belief-preserving type mappings:

Theorem 1. *For each Polish type space $T = (T_i, \mu_i)_{i=1,2}$ over Ω, there exist a belief-preserving type mapping $u^T : T \to U(\Omega)$. The mapping u^T is the unique mapping with such a property.*

1.3 Universal Polish space

Finally, we recall a basic fact about Polish spaces. Let H be a Hilbert cube, i.e. topological product of intervals $\left[0, \frac{1}{n}\right], n = 1, 2, ...$ with l^2-metric. Then, H is a Polish compact space. Moreover, every Polish space S is homeomorphic to a G^δ subset of H. Let us fix one of such possible homeomorphisms $h^S : S \to H$.

2 Properly universal type space

Say that a type space P over Ω is a *properly universal type space (PUTS),* if it satisfies the following universal property: For each Polish type space $T = (T_i, \mu_i)_{i=1,2}$ over Ω, there exist an injective belief-preserving type mapping $u^T : T \to P$. The purpose of this note is to establish the existence of the PUTS.

Theorem 2. *There exists a propertly universal Polish type space over Ω.*

As compared to Theorem 1, the key contribution is that the type mappings into PUTS are injective. On the other hand, we cannot guarantee the uniqueness of the type mapping (and there are easy examples to show that it is impossible . Moreover, the universal property is restricted to the category of Polish type spaces. We do not know whether it is possible to construct a universal type space for the category of all (measurable) type spaces (we suspect not).

2.1 Construction

Let $H_i, i = 1, 2$ be two distinct copies of Hilbert cube and let $U(\Omega \times H_1 \times H_2)$ be the universal type space over $\Omega \times H_1 \times H_2$. For each $h_i \in H_i$, let

$$\begin{aligned}
U_i^a(h_i) &= \{u_i \in U_i(\Omega \times H_1 \times H_2) : u_i(\Omega \times \{h_i\} \times H_{-i} \times U_{-i}(\Omega \times H_1 \times H_2)) = 1\}, \\
U_i^a &= \bigcup_{h_i} U_i^a(h_i), \\
P_i^0 &= \{(h_i, u_i) \in H_i \times U_i^a : u_i = U_i^a(h_i)\}, \\
U_i^b &= \left\{u_i : u_i\left(\Omega \times H_i \times P_{-i}^0\right) = 1\right\}.
\end{aligned}$$

Intuitively, $u_i \in U_i^a$ if (u.t.s.) typeu_i assigns probability 1 to one of the coordinate elements H_i of its space of the basic uncertainty. Set U_i^b contains types u_i that assign probability 1 to the coordinate associated with the opponent is known by her.

It is easy to see that sets $U_i^0(h_i)$ for all h_i, U_i^0, and P_0^0 are closed (hence, compact) subset of Polish compact space $U_i(\Omega \times H_1 \times H_2)$. Let $h_i^0 : U_i^0 \to H_i$ be the mapping such that $u_i \in U_i^0\left(h_i^0(u_i)\right)$ for each $u_i \in U_i^0$. It is easy to see that mapping h_i^0 is continuous.

By induction on $k \geq 0$, define $U_i^0 = U_i^a \cap U_i^b$ and

$$U_i^{k+1} = U_i^a \cap U_i^b \cap \left\{u_i : u_i\left(\Omega \times H_i \times H_{-i} \times U_{-i}^k\right) = 1\right\}.$$

As intersection of closed sets, sets U_i^k are closed (hence, compact) subsets of $U_i(\Omega \times H_1 \times H_2)$. Let

$$U_i^* = \bigcap_k U_i^k.$$

We have the following result:

Lemma 3. $U_i^* \simeq U_i^a \cap U_i^b \cap \Delta\left(\Omega \times H_i \times H_{-i} \times U_{-i}^*\right)$.

Proof. By definition,

$$U_i^{k+1} \simeq U_i^a \cap U_i^b \cap \Delta\left(\Omega \times H_i \times H_{-i} \times U_{-i}^k\right).$$

Because U_i^* can be understood as an inverse limit of spaces U_i^k, the claim can be proven using the same ideas as the proof of Theorem 1. □

Intuitively, U_i^* is a set of player i types for which it is common knowledge that "each player i assigns probability 1 to one of the coordinate elements H_i of its space of the basic uncertainty," and $U_i^*(h_i) = U_i^0(h_i) \cap U_i^*$ is a subset of types that assign probability 1 to the coordinate being equal to h_i.

Notice that $U^* = \left(U_i^*, \mu_i^{\Omega \times H_1 \times H_2}|_{\Omega \times U_{-i}^*}\right)$ is a well-defined type space over $\Omega \times H_1 \times H_2$ with a belief mapping obtained from the u.t.s. $U(\Omega \times H_1 \times H_2)$.

Define

$$P_i(\Omega) = \{(h_i, u_i) \in H_i \times U_i^* : h_i^*(u_i) = h_i\} = \bigcup_{h_i} \{h_i\} \times U_i^*(h_i).$$

It follows from Lemma 3 that for each $u_i \in U_i^*$,

$$\mu_i^{\Omega \times H_1 \times H_2}(u_i) \in \Delta\left(\Omega \times \{h_i^*(u_i)\} \times P_{-i}(\Omega)\right).$$

Then, set $P_i(\Omega)$ is closed, hence compact Polish. Define measurable mapping $\psi_i^\Omega : P_i(\Omega) \to \Delta(\Omega \times P_{-i}(\Omega))$ so that for each $(h_i, u_i) \in P_i(\Omega)$, each measurable subset $A \subseteq \Omega \times H_{-i} \times U_{-i}^*$, we have

$$\psi_i^\Omega(A|h_i, u_i) = \mu_i^{\Omega \times H_1 \times H_2}(A \times \{h_i\}|u_i).$$

Then, $P(\Omega) = \left(P_i(\Omega), \psi_i^\Omega\right)$ is a well-defined Polish type space over Ω.

2.2 Proof of Theorem 2

First, we construct a Polish type space $T^H = \left(T_i, \mu_i^H\right)_{i=1,2}$ over $\Omega \times H_1 \times H_2$. Let $h_i = h^{T_i}$ be the homeomoprhic (in particular, both way Borel measurable) injection from T_i to H_i. Let $PT_i = \{(h_i(t_i), t_i) : t_i \in T_i\}$ and define a measurable mapping $\eta_i : T_i \to PT_i$ so that $\eta_i(t_i) = (h_i(t_i), t_i)$. Define a Borel-measurable mapping $\mu_i^H : T_i \to \Delta\left(\Omega \times H_i \times T_{-i}^P\right)$ so that for all measurable sets $A \subseteq \Omega, B \subseteq T_{-i}$

$$\mu_i^H\left(A \times \{h_i(t_i)\} \times \eta_{-i}(B)\,|t_i\right) = \mu_i\left(A \times B|t_i\right).$$

By Theorem 1, there exists a belief-preserving type mapping $u^H : T^H \to U(\Omega \times H_i \times H_{-i})$. Due to definition of the belief function μ^H, and because u^H preserves beliefs, for each player i, $u_i^H(T_i) \subseteq U_i^0$. Moreover, $h_{-i}^0\left(u_{-i}^H(t_{-i})\right) = h_{-i}(t_{-i})$, which implies that $u_i^H(T_i) \subseteq U_i^b$. An inductive argument shows that $u_i^H(T_i) \subseteq U_i^k$ for each k, hence, $u_i^H\left(T^H\right) \subseteq U_i^*$. It follows that u^H is a belief-preserving type mapping from T^H to U^*. Moreover, because h_i is incjective for each i, u^H is injective.

Because u_i^H preserves beliefs, we have for any measurable subsets $S_0 \subseteq \Omega, S' \subseteq P_{-i}(\Omega)$,

$$\mu_i^{\Omega \times H_1 \times H_2}\left(S_0 \times \{h_i(t_i)\} \times S'|u_i^H(t_i)\right) = \mu_i^H\left(S_0 \times \{h_i(t_i)\} \times \eta_{-i}\left(\left(u_{-i}^H\right)^{-1}(S')\right)|t_i\right)$$

Define a measurable mapping $p_i : T_i \to P_i(\Omega)$ as

$$p_i(t_i) = \left(h_i(t_i), u_i^H(t_i)\right).$$

We are going to show that $p^T = (p_1, p_2)$ show that is an injective and belief-prerserving type mapping. Notice that p_i^T is an injection for each i because h_i is an injection. Fix player i and type t_i and take any measurable subsets $S_0 \subseteq \Omega, S' \subseteq P_{-i}(\Omega)$. Let $S_U = \{u_i \in U_i^* : (h_i, u_i) \in S'\}$.

$$\begin{aligned}
\psi_i^\Omega\left(S_0 \times S'|h_i(t_i), u_i^*(t_i)\right) &= \mu_i^{\Omega \times H_1 \times H_2}\left(S_0 \times \{h_i(t_i)\} \times S'|u_i^H(t_i)\right) \\
&= \mu_i^{\Omega \times H_1 \times H_2}\left(S_0 \times \{h_i(t_i)\} \times S'|u_i^H(t_i)\right) \\
&= \mu_i^H\left(S_0 \times \{h_i(t_i)\} \times \eta_{-i}\left(\left(u_{-i}^H\right)^{-1}(S')\right)|t_i\right) \\
&= \mu\left(S_0 \times p_{-i}^{-1}(S')\,|t_i\right).
\end{aligned}$$

The first equality follows from the definition of beliefs ψ_i. The second equality follows from the above observation. The last line follows from the definition of beliefs μ_i^H and the fact that $\left(u_{-i}^H\right)^{-1}(S') = p_{-i}^{-1}(S')$ due to the injectivity of u_i^H. This shows that p^T preserves beliefs and concludes the proof of the theorem.

References

[HBIR] Jeffrey C. Ely and Marcin Peski, Hierarchies of belief and interim rationalizability, joint with , Theoretical Economics, 2006, Vol. 1(1)

[MZ UTS] Mertens, J. F., and S. Zamir (1985): "Formulation of Bayesian Analysis for Games with Incomplete Information," International Journal of Game Theory, 14, 1–29.