WAR-OF-ATTRITION WITH TWO-SIDED INCOMPLETE INFORMATION

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This is Online Appendix to "Reputational Bargaining with Incomplete Information about Preferences". We study a two-sided uncertainty version of the war-of-attrition model from "Reputational Bargaining with Incomplete Information about Preferences". There are two main results. First, we construct a two-type example to show that the war-of-attrition may have multiple equilibria. Second, we show that when there is continuum of types, and players demand linear menus, there is essentially one equilibrium of the war-of-attrition.

The Appendix is complete: it contains the description of the model, all required notations, and the results.

1. Model

Two players, Alice and Bob, $i = A, B$, bargain over a heterogeneous pie with $N = 2$ parts: *c*hocolate and *v*anilla. An allocation is defined as $(x_A, x_B) \in X := \left\{(a, b) \in [0,1]^N : a^n + b^n = 1 \text{ for each } n\right\}$. Each player has a linear preference over allocations $u_i \in \mathcal{U} := \left\{u \in \mathbb{R}_+^N : \sum u^n = 1\right\}$. (The normalization is w.l.o.g.) The payoffs from allocation x is equal to $u_A(x) = \sum_n u_A^n x_A^n$ for Alice type u_A and $u_B(x) = \sum_n u_B^n x_B^n = 1 - \sum_n u_B^n x_A^n$ for Bob's type u_B.

To simplify the exposition, we adopt the notation that for each player i, a tuple with a subscript$(a, b)_i$ denotes an allocation x such that $x_i = (a, b)$. Thus, $(a, b)_i = (1 - a, 1 - b)_{-i}$ denote the same allocation.

The bargaining takes form of a war of attrition. In alternating periods (starting with player $k = A, B$ in period 1), player i either continues or concedes. If he or she continues, the game moves to the next period and the other player. If she or he concedes, she must choose an allocation x from a (closed) menu of allocations $m_{-i} \subseteq X$. We refer to m_{-i} as the bargaining position of player $-i$.

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Player i start the war-of-attrition knowing their own preferences, and with beliefs $\pi_{-i} \in \Delta\mathcal{U}_{-i}$ about the preferences of her opponent, where $\mathcal{U}_{-i} = supp\pi_i \subseteq \mathcal{U}$ is the support of the beliefs. Additionally, and independently from the type distribution, each player is either strategic with probability $1-\lambda$ or stubborn with a strictly positive probability $\lambda \in (0,1)$. The stubborn player never concedes. The role of the stubborn types is to pin down the equilibrium; it is well known that, without them, the war-of-attrition games have a continuum of equilibria. The players maximize the expected utility and they discount future with a common factor $\mathrm{e}^{-\Delta}$, where Δ represents the length between two subsequent decision points.

Let T_i be the set of periods in which player i makes decision in the war-of-attrition. A strategy of the (strategic type of) player i is a pair $\sigma_i = \left(\sigma_i^T, \sigma_i^M\right)$ of measurable stopping time $\sigma_i^T : \mathcal{U} \to \Delta T_i$ and a choice $\sigma_i^M : \mathcal{U} \to \Delta m_{-i}$. A belief of player $-i$ is a pair of mappings $\lambda_i : T_i \to [0,1]$ and $\mu_i : T_i \to \Delta A_i$, with the interpretation that $\lambda_i(t)$ is the probability at the beginning of the period that player i is stubborn, and $\mu_i(.|t)$ is the probability distribution over the (strategic) types of player i who yield in period $t \in T_i$. Let $U_i^\sigma(u_i)$ denote the expected payoff of player i type $u_i \in \mathcal{U}_i$.

2. Two-type example

We describe an example with two types, and with two different equilibria.

Fix two constants a, b such that

$$\frac{a}{a+1} < b < a < \frac{1}{2}. \tag{2.1}$$

For each player i, let

$$m_{-i} = \left\{(a,0)_i, (0,b)_i\right\}$$

be the menu of choices when player i concedes. Each player has two types $u^c = (1,0)$ and $u^v = (0,1)$ and both types have a positive probability. The assumptions (2.1) imply that each player's type prefers to win regardless of the choice of the other player. See Figure 2.1a. The allocation $x_i = (a,b)_i$ is defined as the unique allocation such that the two types of player i are indifferent between their optimal concession allocation from menu m_{-i} and x_i. We refer to x_i as the *indifference point*.

Proposition 1. *There exists* $\pi^* \in (0,1)$ *such that for each* i*, if* $\pi\left(u_{-i}^v\right) \geq \pi^*$*, then there is a sequence of equilibria of the above game as* $\lambda \to 0, \Delta \to 1$ *such that player* i *concedes with a probability arbitrarily close to 1 in his first period of action.*

The reason for the multiplicity of equilibria is the lack of natural sorting to determine which types concede first. In the proof, we construct an equilibrium, in which the last types to concede are u_{-i}^v and u_i^c; if the roles i and $-i$ are exchanged, a different pair of types ends the game.

2.1. **Proof.** We briefly describe the construction. To fix attention, assume that $i = A$. The equilibrium has three phases:

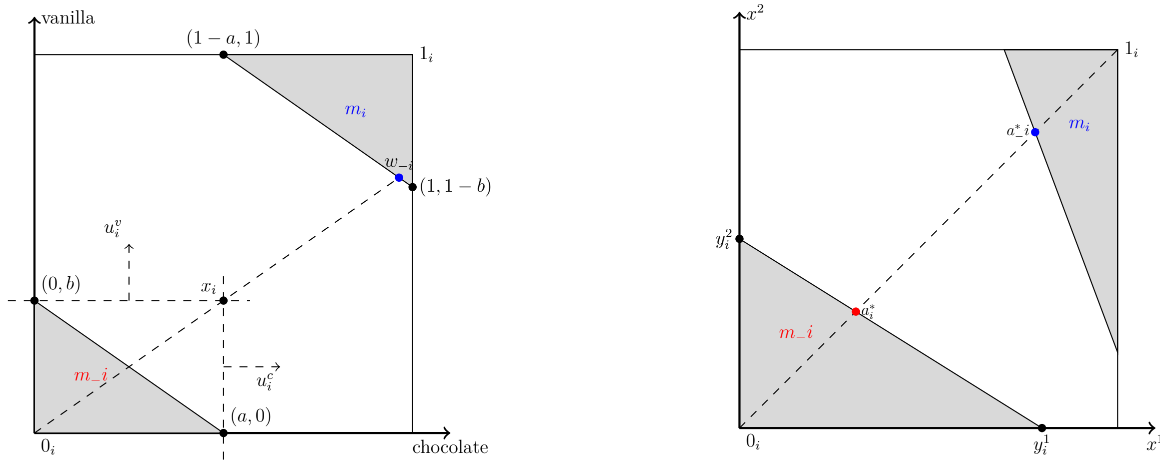
(1) *Atom concession.* In its first period of action t_A^0, each type u of Alice concedes with a positive probability. If Alice moves second, then Bob does not concede in his first period. For each subsequent period after the initial concession, the expected continuation payoff of each type of each player is equal to her immediate concession payoff.
(2) *War of attrition with both sides active.* In the intermediate phase, $t_i^0 < t < t_i^1$, each type $u = u_i^c, u_i^v$ of each player i concedes with a positive probability. The rates are chosen so that each type is indifferent between waiting and conceding.
(3) *War of attrition with one side active.* In the last phase of the game, $t_A^1 < t < T_A^*$, the two remaining types u_B^v and u_A^c concede at constant rates that make the opponent type indifferent between conceding and waiting. The concession rate of Bob is higher. The phase ends when the strategic types fully reveal themselves.

Next, we flesh out the details, starting from the end: Let $F_j^k(t)$ denote the probability that type u_j^k survives till period t.

(1) *War of attrition with one sides active.* In the last phase of the game, $t_i^1 < t < T_i^*$, the two remaining types u_B^v and u_A^c concede at constant rates that make the opponent type indifferent between conceding and waiting. One can calculate using Lemma 6 that the concession rates are equal to ,

$$p_A^2 = \left(\mathrm{e}^{\Delta} - \mathrm{e}^{-\Delta}\right)\frac{1}{\frac{1}{b} - \mathrm{e}^{-\Delta}} \text{ and } p_B^2 = \left(\mathrm{e}^{\Delta} - \mathrm{e}^{-\Delta}\right)\frac{1}{\frac{1}{a} - \mathrm{e}^{-\Delta}}.$$

where the approximation is when $\Delta \to 0$. Here, $\frac{1}{b}$ is the strength of type u_B^v facing u_A^c (winning payoff is 1 and the concession payoff is b); analogously, $\frac{1}{a}$ is



(A) Example with two types for each player.

(B) Theorem 1.

the strength of type u_A^c facing u_B^v. The concession rate of Bob is higher. Importantly, the two concession rates are too slow for the other two types (u_A^v and u_B^c)

$$p_A^2 < \left(\mathrm{e}^{\Delta} - \mathrm{e}^{-\Delta}\right) \frac{1}{\frac{1-a}{a} - \mathrm{e}^{-\Delta}} \text{ and } p_B^2 < \left(\mathrm{e}^{\Delta} - \mathrm{e}^{-\Delta}\right) \frac{1}{\frac{1-b}{b} - \mathrm{e}^{-\Delta}};$$

each of them would prefer to concede immediately. (To see it, notice that type u_B^c winning payoff against u_A^c is equal to $1-a$. Hence, the strength of u_B^c is equal to $\frac{1-a}{a}$.) This ensures that none of those two types has a profitable deviation to reach the third phase. We have

$$F_B\left(t_B^1+1\right) = F_B^v\left(t_B^1+1\right) = \left(1-p_B^2\right)^{-\frac{1}{2}\left(T_B^*-t_B^1\right)}\lambda - \lambda,$$

$$F_A\left(t_A^1+1\right) = F_A^c\left(t_A^1+1\right) = \left(1-p_A^2\right)^{-\frac{1}{2}\left(T_A^*-t_A^1\right)}\lambda - \lambda,$$

When $\Delta \to 0$, this is approximately equal to

$$F_j\left(t_j^1+1\right) + \lambda \approx \mathrm{e}^{-\gamma_j^2\left(T^*-t_j^1\right)\Delta}\lambda v.$$

where $\gamma_A^2 = \frac{b}{1-b} < \frac{a}{a-1} = \gamma_B^2$.
The phase ends when the strategic types fully reveal themselves.

(2) *War of attrition with both sides active.* In the intermediate phase, $t_i^0 < t < t_i^1$, each type $u = u_i^c, u_i^v$ of each player i concedes with a positive probability. The

rates are chosen so that each type is indifferent between waiting and conceding. In order to satisfy the indifference condition, the average winning allocation of player j conditionally on $-j$ concession must lie on the ray that connects the 0 payoff and the indifference point:

$$w_j = \gamma x_j = \alpha \left(1, 1-b\right)_j + \left(1-\alpha\right)\left(1-a, 1\right)_j$$

for some $\gamma > 1$ and $\alpha \in (0,1)$ (see Figure 2.1a). It follows that

$$\alpha = \frac{1}{2} + \frac{1}{2}\left(\frac{1}{b} - \frac{1}{a}\right).$$

By Lemma 6, the concession rate of each player is equal to

$$p^1 = \left(\mathrm{e}^{\Delta} - \mathrm{e}^{-\Delta}\right)\frac{1}{\frac{1}{2a} + \frac{1}{2b} - \frac{1}{2} - \mathrm{e}^{-\Delta}}.$$

(Note that $\frac{1}{2a} + \frac{1}{2b} - \frac{1}{2} = \frac{w_j^{(c)}}{a} = \frac{w_j^{(v)}}{b}$ is equal to the strength of type u_j^c and/or u_j^v who wins with allocation w_j.) In order to ensure that the average concession allocation is equal to w_j, the types must concede with probability $p_j^1\left(u^k\right) = \alpha^k p^1$, where

$$\alpha^k = \begin{cases} \alpha, & \text{if } k = v, \\ 1-\alpha, & \text{if } k = c. \end{cases}$$

Hence,

$$\begin{aligned} F_j\left(t_j^0 + 1\right) + \lambda &= \left(1 - p^1\right)^{-\frac{1}{2}\left(t_j^1 - t_j^0\right)} \lambda \left(F_j\left(t_j^0 + 1\right) + \lambda\right) \\ &\approx \mathrm{e}^{-\frac{1}{2}\gamma^1\left(t_j^1 - t_j^0\right)\Delta - \frac{1}{2}\gamma_j^2\left(T^* - t_j^1\right)\Delta}\lambda. \end{aligned} \tag{2.2}$$

Moreover, because $F_j^k(t) = F_j^k(t-2) - \alpha^k p^1 F_j(t)$, it is easy to check that for each $t > t_j^0$,

$$\frac{F_j^k(t-2)}{F_j(t-2)} - \alpha^k\left(1 - p^1\right) = \frac{1}{1-p^1}\left(\frac{F_j^k(t)}{F_j(t)} - \alpha^k\left(1 - p^1\right)\right).$$

Hence,

$$\frac{F_j^k\left(t_A^0+1\right)}{F_j\left(t_A^0+1\right)} = \alpha^k\left(1-p^1\right)\left(1-\left(1-p^1\right)^{-\frac{1}{2}\left(t_j^1-t_j^0\right)-O(1)}\right) + \left(1-p^1\right)^{-\frac{1}{2}\left(t_j^1-t_j^0\right)-O(1)}\frac{F_j^k\left(t_j^1\right)}{F_j\left(t_j^1\right)},$$

where $O(1)\leq 1$. If we take $\gamma^1=\frac{2}{\frac{1}{a}+\frac{1}{b}-3}$, the latter is approximately equal to

$$\frac{F_j^k\left(t_A^0+1\right)}{F_j\left(t_A^0+1\right)} \approx \alpha^k\left(1-\mathrm{e}^{-\gamma^1\left(t_j^1-t_j^0\right)\Delta}\right)+\mathrm{e}^{-\gamma^1\left(t_j^1-t_j^0\right)\Delta}\mathbf{1}_{(k,j)\in\{(c,A),(v,B)\}}. \tag{2.3}$$

The end date of the phase, t_i^1, is chosen as the last period when types u_A^v for Alice and u_B^c for Bob concede with a positive probability. (To make sure that it is possible, we need to assume that the initial probability of type u_B^v is sufficiently high.)

(3) *Atom concession.* In its first period of action t_A^0, each type u of Alice concedes with a positive probability $1-F_A^k\left(t_A^0+1\right)$. If Alice moves second, then Bob does not concede in his first period. For each subsequent period after the initial concession, the expected continuation payoff of each type of each player is equal to her immediate concession payoff.

Let $X^1=\left(t_j^1-t_j^0\right)\Delta$ and $X^2=\left(T^*-t_j^1\right)\Delta$ for some j (neither of the two quantities depends on j but for more that $O(\Delta)$). It follows from (2.3) implies that

$$\pi\left(u_B^c\right)\approx\alpha^c\left(1-\mathrm{e}^{-X^2}\right).$$

Hence, (2.3) allows to determine X^2 if $\pi\left(u_B^c\right)\leq 1-\pi^*$ and $\pi^*>\alpha$. Further, (2.2) implies that

$$X^1\approx -2\frac{1}{\gamma^1}\log\lambda-\frac{1}{\gamma^1}\gamma_B^2X^2.$$

This makes sure that the probabilities add up for Bob. For Bob, let $\rho=\gamma_A^2/\gamma_B^2<1$. Then, (2.2) implies that

$$\begin{aligned}F_A\left(t_A^0+1\right)+\lambda &\approx \mathrm{e}^{-\frac{1}{2}\gamma^1X^1}\mathrm{e}^{-\frac{1}{2}\gamma_A^2X^2}\lambda\\ &\approx \mathrm{e}^{-\frac{1}{2}\gamma^1X^1}\left(\mathrm{e}^{-\frac{1}{2}\gamma_B^2X^2}\lambda\right)^{\rho}\lambda^{1-\rho}\\ &\approx \mathrm{e}^{-\frac{1}{2}\gamma^1X^1(1-\rho)}\lambda^{1-\rho},\end{aligned} \tag{2.4}$$

where in the last equality we used the fact that $\mathrm{e}^{-\frac{1}{2}\gamma^1 X^1}\mathrm{e}^{-\frac{1}{2}\gamma_B^2 X^2}\lambda = 1$. Hence, for appropriately small λ, $F_A\left(t_A^0+1\right) < \min_k \pi_A\left(u_A^k\right)$. This verifies that the probabilities add up for Alice as well.

3. Continuum types

Next, we assume that players bargaining positions take form of linear menus: $m_i = \{x : \psi_{-i}(x) \leq v_{-i}\}$ for some preference $\psi_{-i} \in \mathcal{U}_{-i}$ and $v_{-i} > 0$ and each i.[1] Additionally, we make two assumptions. The first assumption ensure that the beliefs about i's types are sufficiently regular in the neighborhood of vectors β_i.

Assumption 1. *(Regularity) For each player i, $\mathcal{U}_i = supp\pi_i$ has a nonempty interior in $\mathcal{U}$, $\psi_i \in int\mathcal{U}_i$, and π_i has a strictly positive Lipschitz continuous density with respect to the Lebesgue measure on $\mathcal{U}$.*

Recall that the payoff from winning the war of attrition depends on the choice made by the other player when conceding. The next assumption says that, no matter what is the choice, all types of player i would rather win than lose.

Assumption 2. *(Large Gap). For each $u_i \in \mathcal{U}_i$, for each $x_i \in m_i$ and each $y_i \in m_{-i}$, $\inf_{x\in m_i} u_i(x) > \sup_{y\in m_{-i}} u_i(y)$.*

Let

$$\alpha_i^* = \sup\left\{\alpha : \alpha \mathbf{1}_i + (1-\alpha)\mathbf{0}_i \in m_{-i}\right\},$$
$$a_i^* = \alpha_i^* \mathbf{1}_i + (1-\alpha_i^*)\mathbf{0}_i.$$

Here, $a_i^*$is the unique allocation that lies in the intersection of the diagonal and the boundary of menu m_{-i}. Let

$$S_i^* = \frac{1-\alpha_{-i}^*}{\alpha_i^*} = \frac{u_i\left(a_{-i}^*\right)}{u_i\left(a_i^*\right)},$$

where the last equality holds for arbitrary preference type $u_i \in \mathcal{U}$. Thus, S_i^* is the strength of player i defined as the winning/concession ratio under the restriction that,

[1] It is convenient notationally describe the menu as the set of all allocations that the opponent type ψ_{-i} payoff that is no more v_{-i}. (Of course, such a menu is equal to the set of allocations that give the same type of player i a payoff of at least $1 - v_{-i}$.) Because in this ssection we assume $N = 2$, any linear menu is equivalent to a menu that consists of two most extreme allocations in the menu.

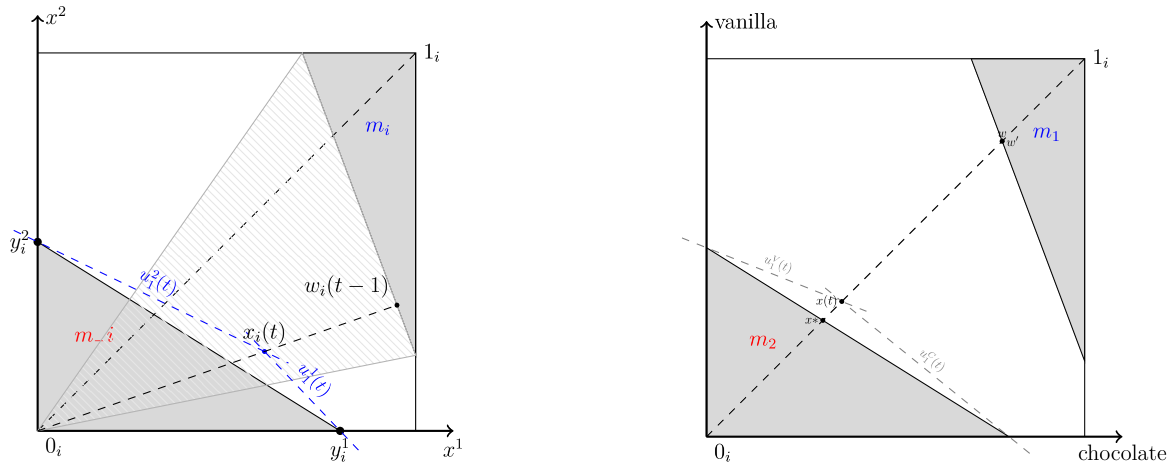


FIGURE 3.1. Illustration of the proof .

when conceding, the player must choose an allocation that belongs to the diagonal. Because of linearity of preferences, so defined strength does not depend on the player's type. The main result of this section shows that the strength characterizes the behavior in the war of attrition.

Theorem 1. *Suppose that Assumptions 1 and 2 hold. Suppose that* $S_i^* > S_{-i}^*$. *For each* $\delta > 0$, *there exist* $\lambda^*, \Delta^* > 0$ *such that if* $\lambda \leq \lambda^*$ *and* $\Delta \leq \Delta^*$, *then there is* $T < \infty$ *such that* $\mathrm{e}^{-r\Delta T} > 1 - \delta$ *and, in any equilibrium, player* $-i$ *concedes with probability at least* $1 - \delta$ *before the end of period* T.

Theorem 1 shows that when the type distribution is continuous, there is an unique equilibrium. The equilibrium concession behavior is the same *as if* the players choices were restricted to the diagonal. In the equilibrium, almost all of the types choose one of the extreme allocations in the menus; however, we show that the ratios with which the extreme allocations are chosen balance so that their average lies on the diagonal. The Large Gap assumption ensures that the concession rates in the early game are bounded; because the late game is arbitrarily long, it means that the late game effects dominate over anything that happens in the early game.

3.1. **Proof intuition.** We describe the intuition behind the proof in few steps. As in the rest of the paper, the argument relies on the analysis of the late game. The goal is to show that after sufficiently many periods, the players behave *as if* they conceded with outcomes a_i^* for each i. Then, their concession behavior is determined by strengths S_i^*. Because $S_j^* > S_{-j}^*$, player j concedes significantly faster than her opponent. The rest of the argument proceeds in the same way as in the case of Lemma 1 of the main paper.

Sorting. The main difficulty with two-sided incomplete information is the lack of natural sorting. When the menus are linear, a partial sorting can be restored. Let $y_i^1, y_i^2 \in m_{-i}$ be two extreme points of menu m_{-i}. (See the left panel on Figure 3.1.) Let $\mathcal{U}_i^k$ be the subset of types of player i who strictly prefer allocation y_i^k to allocation y_i^{-k}, i.e., the types who care about issue k relatively more than the type ψ_i, and,, as follows, than all types in $\mathcal{U}_i^{-k}$. We say that such types are on side k. Take any two types $u, u' \in \mathcal{U}_i^k$ and suppose that $u^k > u'^k > \psi_i^k$. Using a similar argument as in the previous sections, we can show that for any allocation $y \notin m_{-i}$, we have

$$\frac{u\left(y\right)}{u\left(y^k\right)} < \frac{u'\left(y\right)}{u'\left(y^k\right)}.$$

In other words, type u cares relatively less about winning and obtaining y rather than losing than type u'. This implies that type u is going to concede before type u' in the war of attrition. From now on, we rank player i types according to their distance to the last type ψ_i.

Let $u_i^k\left(t\right)$ denote the largest type on side k who survives till period t. (See the left panel of Figure 3.1.) We say that player i *is active on side* k in period t if $u_i^k\left(t\right) \neq u_i^k\left(t+2\right)$, i.e, if outcome y_i^k is chosen with strictly positive probability in period t. Because of the general properties of the war-of-attrition games, each player must be active on at least one side in each period before the final concession of the strategic player.

Indifference condition. If the player is active on side k in two consecutive periods $t-2$ and t, then types $u_i^k\left(t\right)$ must be indifferent between conceding in those two periods. There is a simple geometric characterization of this indifference. For each $t \in T_i$, let $p_{-i}\left(t-1\right)$ be the concession rate, i.e. the probability of $-i$ conceding conditionally on reaching period $t-1$ and let

$$w_i\left(t-1\right) = \sum_k \text{Prob}\left(-i \text{ chooses } y_{-i}^k | -i \text{ concedes at } t\right) y_{-i}^k$$

be the average allocation left to her by player $-i$ conditionally on him conceding. Then, type $u_i^k(t)$ is indifferent if

$$\begin{aligned}&u_i^k(t)\left(y_i^k\right),\\=&p_{-i}(t-1)\,\mathrm{e}^{-\Delta}u_i^k(t)\left(w_i(t-1)\right)+\left(1-p_{-i}(t-1)\right)\left(1-\mathrm{e}^{-2\Delta}\right)\left(u_i^k(t)\left(y_i^k\right)\right)\end{aligned}$$

or, if allocation

$$q_i(t-1)=\frac{p_{-i}(t-1)\,\mathrm{e}^{-\Delta}}{\mathrm{e}^{-2\Delta}+p_{-i}(t-1)\left(1-\mathrm{e}^{-2\Delta}\right)}w_i(t-1)$$

belongs to the indifference curve of type $u_i^k(t)$ that passes through her optimal choice in the menu. We refer to $w_i(t-1)$ as the *win outcome* and to $q_i(t-1)$ as the *anticipated payoff* in period $t-1$. The latter belongs to the ray between the win outcome and the allocation $\mathbf{0}$.

If the player is active on both sides, then the anticipated payoff must be equal to the *indifference point* $x_i(t)$, i.e., the unique allocation such that each type $u_i^k(t)$ is indifferent between $x_i(t)$ and her optimal concession allocation y_i^k. For future reference, note that this is only possible if the indifference point belongs to the convex hull spanned by the allocations y_{-i}^1, y_{-i}^2 and $\mathbf{0}$ (the dashed area of Figure 3.1).

Structure of the late game. We show in the proof that the players must be active on both sides in each period of the late game, i.e., when the remaining types are sufficiently close to the lowest type ψ_i. There are two steps to the argument. First, we show that the indifference point must remain in the convex hull of y_{-i}^1, y_{-i}^2 and $\mathbf{0}$ (the dashed area of Figure 3.1). Otherwise, say if at some t the indifference point leaves the convex hull one the side k, then, we show using the indifference condition that the player must be only active on side k for each $t' < t$. But that leads to the contradiction as there must be a substantial revelation of types on side $-k$ before the late game is reached. TBA

The diagonal. Finally, we can show that the late behavior must remain close to the diagonal. We can estimate the late game rate of movement of the indifference point by the distance between $x_i(t)$ and the win outcome $w_{-i}(t)$:

$$\Delta x_i(t)=x_i(t)-x_i(t+2)\approx c_i(t)\left[w_{-i}(t)-x_i(t+2)\right],\tag{3.1}$$

where the proportionality constant $c_i(t)$ depends on the concession rate, etc. The idea is simple: if player i chooses y_i^k with a relatively high probability in period t, then the gap between types $u_i^k(t+2)$ and $u_i^k(t)$ is relatively large. But it also means that the indifference point is moving towards side k. A careful calculation that relies on the Lipschitzness of the density in the neighborhood of ψ_i shows that the indifference point does not change (much) only if the win outcome is very close.

Suppose that in the late game, the indifference points $x_i(t)$ remain in the close neighborhood of some constant x_i^*. In such a case, (3.1) implies that $w_{-i}(t) \approx x_i^*$ for both players i. A the same time, the indifference condition implies that $x_i(t)$ is a convex combination of allocations $\mathbf{0}$ and $w_i(t-1) \approx x_{-i}^*$. Putting those two conditions together, we obtain that x_i^* must lie on a diagonal for each i (see the right panel of Figure 3.1).

If the indifference points do not converge, we provide an argument based on equation () that shows that in such a case, the indifference point must diverge away from the diagonal. Another argument, similar to the one used above in the discussion of the structure of the late game, shows that it leads to a contradiction while in the late game.

3.2. **Outline of the proof.** Here, we describe the main structure of the argument, with notation and key steps. The proofs of the key lemmas can be found in the rest of the section.

3.2.1. *Notation: Menus.* We begin with defining notation that is specific to linear menus. For each player i, define two extreme allocations in menu m_{-i}: for each $k = c, v$, let

$$y_i^k = \begin{cases} \left(\frac{v_i}{\psi_i}{}_{k\text{th coordinate}}, 0_{-k\text{th coordinate}} \right), & \text{if } \psi_i \geq v_i, \\ \left(1_{k\text{th coordinate}}, \frac{v_i - \psi_i}{\psi_i}{}_{-k\text{th coordinate}} \right) & \text{otherwise.} \end{cases}$$

Then, $y_i^k \in m_{-i}$ for each player i and each side k. Let

$$\text{bd} m_{-i} = \text{con}\left\{ y_i^1, y_i^2 \right\}$$

be the outer boundary of menu m_{-i}.

For each allocation, we define projections on the menu boundary. For each player i, each side k, and each allocation $x \neq \mathbf{0}$, let $P_i^k x \geq, R_{-i}^k x \in \mathbb{R}$ be uniquely defined by

$$\sum_k \left(P_i^k x \right) = 1 \text{ and } \sum_k \left(P_i^k x \right) y_i^k = \alpha x \text{ for some } \alpha > 0,$$
$$\sum_k \left(R_{-i}^k x \right) = 1 \text{ and } \sum_k \left(R_{-i}^k x \right) y_{-i}^k = \alpha x \text{ for some } \alpha > 0.$$

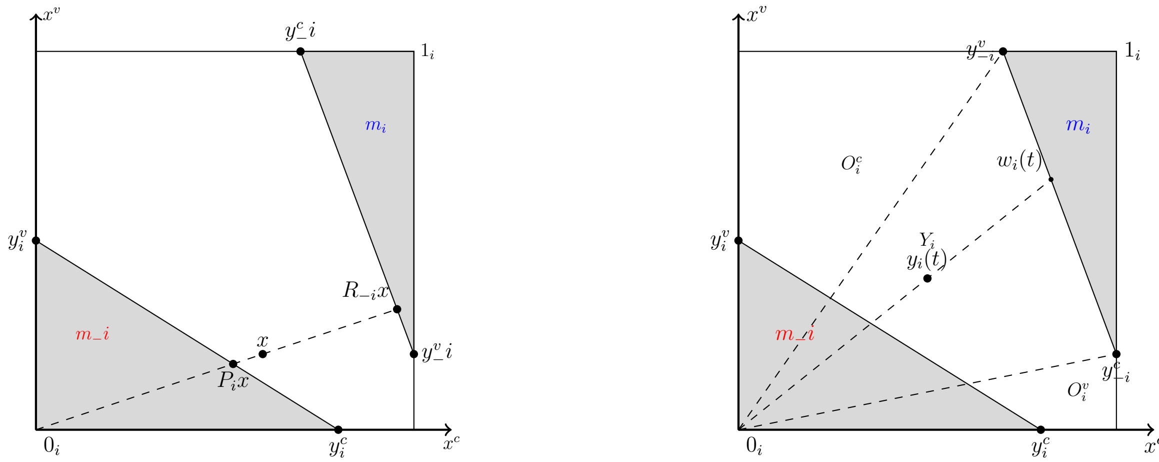


FIGURE 3.2

Let

$$P_ix = \sum_k \left(P_i^k x\right) y_i^k \text{ and } R_{-i}x = \sum_k \left(R_{-i}^k x\right) y_{-i}^k.$$

Then, $P_i^k x$ is the "k " th coordinate of the projection of x on $\text{bd}m_{-i}$; $R_i^k x$ is the "k " th coordinate of the projection of x on the line containing $\text{bd}m_i$. See the left panel of Figure 3.2

3.2.2. *Sorting.* Next, we show that the equilibrium types can be partially sorted. Let

$$\mathcal{U}_i^k = \left\{u \in \mathcal{U}_i : \arg\max_{x \in m_{-i}} u \cdot x = \left\{y_i^k\right\}\right\}$$

be the set of player i types for whom y_i^k is their optimal choice. Then, $\mathcal{U}_i = \mathcal{U}_i^1 \cup \{\psi_i\} \cup \mathcal{U}_i^2$.

The next result says that any equilibrium can be sorted on each side separately. In the Lemma below as well as the rest of this proof, we take $(-1)^c = -1$ and $(-1)^v = 2$.

Lemma 1. *For each equilibrium σ', there exists an equilibrium σ with exactly the same payoffs, $T_i^{*,\sigma} = T_i^{*,\sigma'}$ for each i, and such that for each player i and each $k = 1, 2$, there exists monotonic sequences $(-1)^k \eta_i^k(t) \geq (-1)^k \eta_i^k(t+2), t \in T_i$, such that $\eta_i^k(T_i^{*,\sigma}) = 0$ and for each u,*

$$\sigma(u) = t \text{ and } \sigma_i^M = y_i^k \text{ and } \text{ iff } (u^v - \psi_i^v) \in \left[(-1)^k \eta_i^k(t+2), (-1)^k \eta_i^k(t)\right).$$

From now on, we assume that the equilibrium satisfies the thesis of the Lemma.

Define a vector

$$\gamma = (-1_{c\text{th coordinate}}, 1_{v\text{th coordinate}}) \in \mathbb{R}^2. \tag{3.2}$$

Then, $u_i^k(t) := \psi_i + \eta_i^k(t)\gamma \in \mathcal{U}_i^k$ is the unique type u such that $\eta_i^k(t) = u^v - \psi_i^v$. By the Lemma, $u_i^k(t)$ is the "highest" type to concede in period t among all types in $\mathcal{U}_i^k$. We also take $\eta_i^k(t) = 0$ for each $t > T^*$.

Let $F(\eta) = \pi\{u : (u^v - \psi_i^v) \leq \eta\}$. Let

$$f^* = \frac{dF(0)}{d\eta} > 0. \tag{3.3}$$

3.2.3. *Notation: Concession rates.* For each player i, each side k, and each $t \in T_i$, let

$$F_i^k(t) = \begin{cases} \pi\left\{u : 0 \geq u^v - \psi_i^v \geq \eta_i^k(t)\right\} & \text{if } k = c, \\ \pi\left\{u : 0 \leq u^v - \psi_i^v \leq \eta_i^k(t)\right\} & \text{if } k = v. \end{cases}$$

be the mass of the types of player i on side k that have not conceded before period t. By assumptions, each F_i^k is differentiable, and its derivative is Lipschitz continuous with constant $K < \infty$.

We use the sorting properties to rewrite the definitions from Appendix A.1. For each function $h : T_i \to \mathbb{R}$,we write

$$\Delta h(t) = h(t) - h(t+2).$$

For each player i, each $t \in T_i, t \leq T_i^{*,\sigma}$, each k, let

$$Q_i^k(t) = \frac{\Delta F_i^k(t)}{\sum_l \Delta F_i^l(t)},$$

be the conditional probability of the concession on side k given the concession. The concession rate is equal to

$$p_i(t) = \frac{\sum_k \Delta F_i^k(t)}{\sum_k F_i^k(t) + \lambda},$$

and, for each $t \in T_{-i}$, the win outcome of player i and the weighted win allocation are equal to

$$w_i(t) = \sum_k Q_{-i}^k(t)\, y_{-i}^k, \text{ and}$$

$$y_i(t) = \frac{\mathrm{e}^{\cdot -\Delta} p_{-i}(t)}{\mathrm{e}^{-2\Delta} p_{-i}(t) + (1 - \mathrm{e}^{-2\Delta})} \sum_k Q_{-i}^k(t)\, y_{-i}^k.$$

3.2.4. *Best responses and indifference condition.* Next, we provide a characterization of the best response concession thresholds. For each $t \in T_i, t \leq T^{*,\sigma}$ define $x_i(t) \in X$ to be the unique allocation such that, if offered in period t,it would make each of the types $u_i^k(t)$ indifferent to conceding:

$$u_i^k(t) \cdot \left(x_i(t) - y_i^k\right) = 0 \text{ for each } k. \tag{3.4}$$

We refer to $x_i(t)$ as the *indifference point* of player i.

We say that *player i is active on side k in period $t \in T_i$* if $\eta_i^k(t) > \eta_i^k(t+2)$. Because of Lemma 7, in each period before T^*(with a possible exception of the first one), each player must be active on at least one side.

Lemma 2. *If player i is active on side k in period t, then, it must be that*

$$u_i^k(t+2) \cdot \left(y_i(t+1) - y_i^k\right) \leq 0, \text{ and}$$
$$u_i^k(t) \cdot \left(y_i(t-1) - y_i^k\right) \geq 0.$$

If player i is active on both sides in periods t and $t-2$, then

$$y_i(t-1) = x_i(t). \tag{3.5}$$

Proof. A straightforward corollary to Lemma 6. □

3.2.5. *Late game estimates.* We being the analysis of the late game. For each i and each η, define

$$T_i^\eta = \max\left\{t : \sum_k \eta_i^k(t) \geq \eta\right\}$$

and let $T^\eta = \max_i T_i^\eta$. We refer to periods $t > T$ as the *late game*. If η is small, all the remaining types in the late game are very close to ψ_i. The equilibrium behavior has many natural approximations. For each $t \in T_i$, let

$$\overline{P_i^k}(t) = \frac{(-1)^k \eta_i^k(t)}{\sum_l (-1)^l \eta_i^l(t)} \text{ and } \overline{Q_i^k}(t) = \frac{(-1)^k \Delta\eta_i^k(t)}{\sum_l (-1)^l \Delta\eta_i^l(t)}.$$

It is straightforward to verify that for each $t \in T_i$,

$$\Delta\overline{P_i^k}(t) = \frac{\sum_l (-1)^l \Delta\eta_i^l(t)}{\sum_l (-1)^l \eta_i^l(t)} \left(\overline{Q_i^k}(t) - \overline{P_i^k}(t+2)\right). \tag{3.6}$$

Additionally, the regularity of the density implies the following approximations:

Lemma 3. *There exists constant C that is independent from λ and β such that if $\eta \leq \frac{1}{4K}$, then for each i, k, each $t \in T_i$ and $t > T^{\eta}$,*

$$\left|P_i^k x_i(t) - \overline{P_i^k}(t)\right| \leq C\left(\sum_l F_i^l(t)\right),$$
$$\left|\Delta P_i^k x_i(t) - \Delta\overline{P_i^k}(t)\right| \leq C\left(\sum_l \Delta F_i^l(t)\right),$$
$$\left|Q_i^k(t) - \overline{Q_i^k}(t)\right| \leq C\left(\sum_l F_i^l(t)\right),$$
$$\left|\frac{\sum_l \Delta\eta_i^l(t)}{\sum_l \eta_i^l(t)} - \frac{\sum_l \Delta F_i^l(t)}{\sum_l F_i^l(t)}\right| \leq C\left(\sum_l \Delta F_i^l(t)\right).$$

3.2.6. *Late game: both sides active.* We use the above estimates to establish the key technical property of the late game:

Lemma 4. *There exists $n > 0$ that is independent from λ and Δ and such that, if $\lambda < \frac{1}{4}n$, then, for each $t > T^n$, each player is active on each side.*

Together with Lemma 2, the result implies that for $\eta > 0$ sufficiently small, each $t > T^{\eta}$, each player i such that $t \in T_i$, (3.5) holds.

Finally, we show that the win outcome must remain close to the diagonal. Recall that $\gamma = (-1, 1)$ is defined in (3.2). Then, $|\gamma \cdot x|$ measures the distance of allocation x from the diagonal.

Lemma 5. *There exists $\Delta^*, \lambda^* > 0$ such that for each $\delta > 0$, there exists $\eta_\delta \leq n$ such that for each $\Delta \leq \Delta^*$, $\lambda \leq \lambda^*$ for each $t > T^{\eta_\delta}, t \in T_i$,*

$$\left\|w_i(t) - a_{-i}^*\right\| \leq \delta.$$

3.2.7. *Proof of Theorem 1.* Let $\xi = \frac{1}{3}\left(S_j^* - S_{-j}^*\right) > 0$ and let

$$x = \frac{S_{-j}^* + \xi - 1}{S_j^* - \xi - 1} < 1.$$

As in the proof of Lemma 1 of the main paper, let $S_i(t)$ be defined as the maximum strength of the type conceding in period t. Then, for each player j, $t > T^{\eta_\delta}, t \in T_{-i}$, we have

$$S_i(t) = \max_k \max_{\eta \in \left[\eta_i^k(t+2), \eta_i^k(t)\right)} \frac{(\psi_i + \eta\gamma) \cdot (w_i(t))_i}{(\psi_i + \eta\gamma) \cdot y_i^k}$$
$$= \max_k \max_{\eta \in \left[\eta_i^k(t+2), \eta_i^k(t)\right)} \frac{1 - \alpha_{-i}^* + \left((\psi_i + \eta\gamma) \cdot \left(w_i(t) - a_{-i}^*\right)\right)_i}{\alpha_i^* + \eta\gamma \cdot y_i^k}.$$

Hence, by Lemma 5, there exists $\eta^*, \lambda^*, \Delta^* > 0$ such that for each $\eta \leq \eta^*, \eta \leq \lambda^*, \Delta \leq \Delta^*$, and each $t \in T_i$we have

$$S_j(t) \geq S_j^* - \xi,$$
$$S_{-j}(t) \leq S_{-j}^* + \xi.$$

The rest of the proof follows the same three-zone strategy as the proof of Lemma 1 of the main paper. We omit the details.

Appendix A. Preliminary analysis of the war-of-attrition

In this Appendix, we perform a preliminary analysis of the model from Section **??**. The notations the results that can be found here are used in all the remaining parts of the Appendix.

A.1. **Notations.** For each player $i = 1, 2$, let $t_i^0 = i$ be the first decision period for player i.

For each player i and each $t \in T_i$, each measurable set $U \subseteq \mathcal{U}_i$, define the probability that player i with preferences in U yield in period t as

$$f_i^\sigma(U|t) = (1 - \lambda) \int_U \sigma_i^T(t|u)\, d\pi_i(u).$$

Let $f_i^\sigma(t) = f_i^\sigma(\mathcal{U}|t)$ be the probability of concession in period t. Let

$$F_i^\sigma(t) = \lambda + \sum_{s \in T_i : s \geq t} f_i^\sigma(t), \text{ and}$$
$$p_i^\sigma(t) = \frac{1}{F_i^\sigma(t)} f_i^\sigma(t),$$

be, respectively, the probability that player i has not conceded before period t and the concession rate in period t.

For each $t \in T_{-i}$, let

$$w_i^\sigma(t) = \int \sigma_{-i}^M(u_{-i}) \frac{1}{f^\sigma(t)} df^\sigma(u_{-i}|t) \in X, \tag{A.1}$$

$$y_i^\sigma(t) = \frac{e^{-\Delta} p_{-i}^\sigma(t)}{e^{-2\Delta} p_{-i}^\sigma(t) + (1 - e^{-2\Delta})} w_i^\sigma(t) \in X. \tag{A.2}$$

Here, $w_i^\sigma(t)$ denotes the allocation that player i obtains in period t, conditionally on the opponent's concession in that period t; $y_i^\sigma(t)$ is the winning allocation weighted by the concession probability. Further, for each type $u \in \mathcal{U}_i$ of player i, let

$$L_i(u) = \max_{x \in m_i} u(x), \text{ and } S_i^\sigma(u,t) = \frac{u(w_i^\sigma(t))}{L_i(u)}.$$

Here, $L_i(u)$ is the payoff received upon concession, and $S_i^\sigma(u,t)$ is the (endogenous) strength ratio.

The superscripts σ in the above notation denotes dependence on the strategy profile σ; the subscript i, on the player i. We drop the superscripts and/or the subscripts from the above notation whenever it does not lead to confusion.

A.2. **Best response characterization.** The expected payoff of player i type u_i from yielding in period $t \in T_i$ given opponent strategies (σ) is equal to

$$U_i^\sigma(u_i,t) = \sum_{s: s<t, s\in T_{-i}} e^{-s\Delta} f_{-i}^\sigma(s)(u_i(w_i^\sigma(s))) + e^{-t\Delta} F_{-i}^\sigma(t+1) L_i(u_i).$$

For each $t \in T_i$, we have

$$\begin{aligned} & e^{t\Delta}\left[U_i^\sigma(u_i,t+2) - U_i^\sigma(u_i,t)\right] \qquad \text{(A.3)}\\ =& e^{-\Delta} f_{-i}^\sigma(t+1)(u_i(w_i^\sigma(t+1))) + \left[e^{-2\Delta}\left(F_{-i}^\sigma(t+1) - f_{-i}^\sigma(t+1)\right) - F_{-i}^\sigma(t+1)\right] L_i(u_i)\\ =& F_{-i}^\sigma(t+1)\left[e^{-\Delta} p_{-i}^\sigma(t+1)(u_i(w_i^\sigma(t+1))) - \left(e^{-2\Delta} p_{-i}^\sigma(t+1) + 1 - e^{-2\Delta}\right) L_i(u_i)\right]\\ =& \left(f_{-i}^\sigma(t+1) + \left(1 - e^{-2\Delta}\right) F_{-i}^\sigma(t+3)\right)\left[u_i(y_i(t+1)) - L_i(u_i)\right]. \end{aligned}$$

We have the following corollary to the above calculations and definitions.

Lemma 6. *For each type u_{-i} of player $-i$, each $t \in T_i$, $U_{-i}^\sigma(u_{-i},t+1) \geq (\leq) U_{-i}^\sigma(u_{-i},t-1)$ if and only if*

$$u_{-i}\left(y_{-i}^\sigma(t)\right) \geq (\leq) L_{-i}(u_{-i}), \text{ or } p_i^\sigma(t) \geq (\leq) \left(e^{\Delta} - e^{-\Delta}\right) \frac{1}{S_{-i}^\sigma(u_{-i},t) - e^{-\Delta}}.$$

A.3. **End of the war of attrition.** Let $T_i^{*,\sigma} = \max\{t \in T_i : f_i^\sigma(t) > 0\}$ be the last period in which a strategic type of player i concedes. We have the following standard result.

Lemma 7. *Suppose that σ is an equilibrium.*

1. *For each $t \leq T_i^{*,\sigma}$, $f^\sigma(t) > 0$. Also, $|T_i^{*,\sigma} - T_{-i}^{*,\sigma}| = 1$.*
2. *For each $t < T_i^{*,\sigma}$, $y_i^\sigma(t) \notin intm_i$.*
3. *For each i, $T_i^{*,\sigma} < \infty$, and $F_i^\sigma(T_i^{*,\sigma} + 2) = \lambda$.*

Proof. By Lemma 6, if$f(t) = 0$ for some $t \in T_{-i}$, then it is a strictly better response for (almost any type u of player i to yield in period $t-1$ rather than to wait to period $t+1$. It follows that $f_i^\sigma(t+1) = 0$. An induction implies that $f_i^\sigma(t') > 0$ for each $t' > t$. The second claim follows from the same argument.

If $t < T_i^{*\sigma}$, then the part 1 of Lemma 7 implies that there is a type u_i of player i for whom period $t+1$ is a best response. By Lemma 6, $u_i(y_i^\sigma(t)) < L_i(u_i)$. However, the latter inequality cannot be satisfied if $y_i^\sigma(t) \in \mathrm{int}m_i$.

For each i, let $L_i^{\min} = \inf_{u_i \in \mathcal{U}_i} L_i(u_i)$. Because $f_i^\sigma(t) > 0$ for each $t \leq T_i^{*,\sigma}$, it must be that for each $t \in T_i$, if $t < T_i^{*,\sigma}$, there is a type $u \in \mathcal{U}_{-i}$ of player $-i$ such that $U_{-i}^\sigma(u_{-i}, t-1) \leq U_{-i}^\sigma(u_{-i}, t+1)$. It follows from Lemma 6 that for each $t < T_i^{*,\sigma}$,

$$p_i^\sigma(t) \geq \left(1 - \mathrm{e}^{-\Delta}\right) \frac{1 + \mathrm{e}^{-\Delta}}{\mathrm{e}^{-\Delta}} \frac{1}{\max_{u \in A_{-i}} S_{-i}^\sigma(u_{-i}, t) - \mathrm{e}^{-\Delta}} \geq \left(1 - \mathrm{e}^{-\Delta}\right) L_{-i}^{\min} > 0,$$

which implies for each $t \leq T_i^{*,\sigma}$

$$\begin{aligned} F_i^\sigma(t) &= (1 - p_i^\sigma(t-2)) F_i^\sigma(t-2) \leq \left(1 - \left(1 - \mathrm{e}^{-\Delta}\right) L_{-i}^{\min}\right) F_i^\sigma(t-2) \\ &\leq \left(1 - \left(1 - \mathrm{e}^{-\Delta}\right) L_{-i}^{\min}\right)^{\left(t - t_i^0\right)/2}. \end{aligned}$$

Because $F_i^\sigma(t) \geq \lambda$, it must be that $T_i^{*,\sigma} - t_i^0 \leq \frac{\log \lambda}{\log\left(1 - \left(1 - \mathrm{e}^{-\Delta}\right) L_{-i}^{\min}\right)}$. □

A.4. **Monotonicity.** Recall that for $A, B \subseteq \mathbb{R}$,$A$ is strongly dominated by B, we write $A \leq_S B$ if for each $a, \in A, b \in B$, $\min(a, b) \in A$ and $\max(a, b) \in B$.

Lemma 8. *(Monotonicity) Take two types $u_i, u_i' \in \mathcal{U}_i$, and suppose that $S_i^\sigma(u_i, s) \leq S_i^\sigma(u_i', s)$ for each $s \in T_{-i}$ such that $s < T_{-i}^{*,\sigma}$. Then,* $\arg\max U_i^\sigma(u_i, .) \leq_S \arg\max U_i^\sigma(u_i', .)$. *If $S_i^\sigma(u_i, s) < S_i^\sigma(u_i', s)$ for each $s \in T_{-i}$ such that $s < T_{-i}^{*,\sigma}$, then, if $U_i^\sigma(u_i, t) \leq U_i^\sigma(u_i, t')$ for some $t < t'$*, then $U_i^\sigma(u_i', t) < U_i^\sigma(u_i', t')$.

Proof. Notice that

$$\begin{aligned}
&\frac{1}{L_i(u_i)}\left(U_i^\sigma(u_i,t') - U_i^\sigma(u_i,t)\right)\\
=&\sum_{s:t<s<t',s\in T_{-i}} \mathrm{e}^{-s\Delta} f^\sigma(s)\, S_i^\sigma(u_i,s) + \mathrm{e}^{-t'\Delta}\left(1-\sum_{s:s<t',s\in T_{-i}} f^\sigma(s)\right) - \mathrm{e}^{-T\Delta}\left(1-\sum_{s:s<t,s\in T_{-i}} f^\sigma(s)\right)\\
=&\frac{1}{L_i(u_i')}\left(U_i^\sigma(u_i',t') - U_i^\sigma(u_i',t)\right) - \sum_{s:t<s<t',s\in T_{-i}} \mathrm{e}^{-s\Delta} f^\sigma(s)\left[S_i^\sigma(u_i',s) - S_i^\sigma(u_i,s)\right].
\end{aligned}$$

Thus, function $U_i^0(u_i,t) = \frac{1}{L_i(u_i)} U_i^\sigma(u_i,t)$ has increasing differences in the strength ratio and time. The result follows from the Topkis Theorem. □

A.5. **Early game.** The next result discusses the concession behavior when a player may still have very weak (i.e., with strength not much higher than 1) types. It says that, essentially, either player $-i$ concedes early with a probability arbitrarily close to 1, or all the weak types of player i concede early, where "early" here means with a small amount of discounting.

Lemma 9. *For each $\delta > 0$, there exists $\varepsilon > 0$ and $\Delta^* > 0$ such that if $\Delta \leq \Delta^*$, then there exists T_0 such that $\mathrm{e}^{-\Delta T_0} \geq 1-2\delta$ and for each equilibrium σ, either (a) $F_{-i}^\sigma(T_0) \leq \delta$, or (b) $\sigma_i^{T_0}(u_i) \leq T_0$ for all $u_i \in \mathcal{U}_i$ st. $\sup_{t\in T_{-i}} S_i^\sigma(u_i,t) \leq 1+\varepsilon$.*

Proof. Let $k^* = \lceil -\log_2 \delta \rceil \leq -\log_2 \delta + 1$. Find $\varepsilon > 0$ such that $1-2\varepsilon \geq (1-\delta)^{\frac{1}{k^*}}$. Fix $\Delta^* > 0$ so that $2\Delta^*(1-\log_2\delta) \leq \log\frac{1-\delta}{1-2\delta}$. For each $\Delta \leq \Delta^*$, let n_Δ be the smallest even integer such that $\mathrm{e}^{-\Delta n_\Delta} \leq 1-2\varepsilon$. Then, $\mathrm{e}^{-\Delta n_\Delta} \geq (1-2\varepsilon)\,\mathrm{e}^{-2\Delta}$. Take $T_0 = k^* n_\Delta$. Then,

$$\mathrm{e}^{-T_0\Delta} \geq (1-2\varepsilon)^{k^*}\mathrm{e}^{-2\Delta k^*} \geq (1-\delta)\,\mathrm{e}^{-2\Delta(1-\log_2\delta)} \geq 1-2\delta.$$

Suppose that there is a type $u_i \in \mathcal{U}_i$ such that $S_i^\sigma(u_i,t) \leq 1+\varepsilon$ for each $t \in T_{-i}$, and suppose that $T \geq T_0$ is a best response stopping time for such type u_i. Then, it must be that for each $t \in T_i, t < T$, type u_i prefers to continue waiting till period T rather than conceding in period t:

$$F_i(t)\,L_i(u_i) \leq \sum_{t<s<T:s\in T_{-i}} f_{-i}(s)\,\mathrm{e}^{-(s-t)\Delta}\left[S_i(u_i,s)\,L_i(u_i)\right] + F_i(T)\,\mathrm{e}^{-(T-t)\Delta} L_i(u_i).$$

After some algebra, and taking into account that $S_i(u_i,s) \leq 1+\varepsilon$, we get

$$0 \leq \sum_{s>t:s\in T_{-i}} f_{-i}(s)\left(\mathrm{e}^{-(s-t)\Delta}(1+\varepsilon) - 1\right).$$

Due to the choice of n_Δ, for each $t \leq T - n_\Delta$, the above is not larger than

$$\leq \sum_{t<s<t+n_\Delta : s\in T_{-i}} f_{-i}(s)\,\varepsilon + \sum_{s>t+n_\Delta : s\in T_{-i}} f_{-i}(s)\left(\mathrm{e}^{-n_\Delta \Delta}(1+\varepsilon)-1\right)$$
$$\leq \varepsilon\left(\sum_{t<s<t+n_\Delta : s\in T_{-i}} f_{-i}(s) - \sum_{s>t+n_\Delta : s\in T_{-i}} f_{-i}(s)\right).$$

In the second inequality, we used the fact that $\mathrm{e}^{-\Delta n_\Delta}(1+\varepsilon) \leq (1-2\varepsilon)(1+\varepsilon) \leq -\varepsilon - 2\varepsilon^2 \leq -\varepsilon$. Thus, for any such t,

$$\sum_{t<s<t+n_\Delta : s\in T_{-i}} f_{-i}(s) \geq \frac{1}{2}\left(\sum_{t<s<t+n_\Delta : s\in T_{-i}} f_{-i}(s) + \sum_{s>t+n_\Delta : s\in T_{-i}} f_{-i}(s)\right) = \frac{1}{2}\sum_{t<s<T : s\in T_{-i}} f_{-i}(s).$$

It follows that

$$1 - F_{-i}(s) = \sum_{s<T_0 : s\in T_{-i}} f_{-i}(s) \geq \sum_{l=1}^{k^*}\frac{1}{2^l} = 1 - \frac{1}{2^{k^*}} \geq 1-\delta.$$

□

Appendix B. Proof of Theorem 1

B.1. **Proof of Lemma 1.** We have a simple observation.

Lemma 10. *For each equilibrium σ, any player i, each side k, any two types $\psi_i + \eta\gamma, \psi_i + \eta'\gamma \in \mathcal{U}_i^k$ and such that $0 \leq (-1)^k \eta' \leq (-1)^k \eta$, we have*

$$S_i(\psi_i + \eta\gamma, t) \leq S_i(\psi_i + \eta'\gamma, t).$$

Proof. Fix i and k. We begin with a simple observation. Suppose that $0 \leq (-1)^k \eta' \leq (-1)^k \eta$ (the first inequality implies that $\psi_i+\eta\gamma, \psi_i+\eta'\gamma \in \mathcal{U}_i^k$). Then, y_i^k is the optimal choice for preferences $\psi_i + \eta\gamma$ from the set of all allocations that deliver at most y_i^k to player with preferences $\psi_i + \eta\prime\gamma$:

$$\left\{y_i^k\right\} = \arg\max_{x\in X:(\psi_i+\eta\prime\gamma)(x)\leq(\psi_i+\eta\prime\gamma)\left(y_i^k\right)} (\psi_i+\eta\gamma)(x).$$

It follows that if $(\psi_i+\eta\gamma)(x) = (\psi_i+\eta\gamma)\left(y_i^k\right)$ for some $x \in X$, then

$$(\psi_i+\eta\prime\gamma)(x) \geq (\psi_i+\eta\prime\gamma)\left(y_i^k\right).$$

Notice that for each $t \in T_{-i}$, each η so that $\psi_i + \eta\gamma \in \mathcal{U}_i^k$,

$$S_i(\psi_i + \eta\gamma, t) = \frac{(\psi_i + \eta\gamma)(w_i(t))}{(\psi_i + \eta\gamma)\left(y_i^k\right)} = \frac{1}{\alpha_\eta},$$

where, $w_i(t) \in X$. Then, by linearity

$$(\psi_i + \eta\gamma)\left(y_i^k\right) = (\psi_i + \eta\gamma)(\alpha_\eta w_i(t) + (1 - \alpha_\eta)\mathbf{0}_i).$$

For each $\eta\prime$ such that $0 \leq (-1)^k \eta' \leq (-1)^k \eta$, the above observation implies that

$$(\psi_i + \eta\prime\gamma)(\alpha_\eta w_i(t) + (1 - \alpha_\eta)\mathbf{0}_i) \geq (\psi_i + \eta\prime\gamma)\left(y_i^k\right) = (\psi_i + \eta\prime\gamma)(\alpha_{\eta'} w_i(t) + (1 - \alpha_{\eta'})\mathbf{0}_i),$$

which implies that $\alpha_\eta \geq \alpha_{\eta'}$, or $S_i(\psi_i + \eta\gamma, t) \leq S_i(\psi_i + \eta\prime\gamma, t)$. □

We proceed with the proof of Lemma 1. Fix an equilibrium σ. For each player i, each k, choose a monotonic sequence $(-1)^k \eta_i^k(t) \geq (-1)^k \eta_i^k(t+2), t \in T_i$, such that for each $t \in T_i$,

$$\pi\left\{\beta + \gamma\eta : (-1)^k \eta \in \left[\eta_i^k(t+2), \eta_i^k(t)\right)\right\} = \int_{\mathcal{U}_i^k} \sigma(u|t)\, d\pi(u).$$

is equal to the probability that a type in $\mathcal{U}_i^k$ concedes in period t in equilibrium σ. Consider a strategy

$$\sigma'(u) = t \text{ and } \sigma_i^M = y_i^k \text{ and } \text{ iff } (u^v - \psi_i^v) \in \left[\eta_i^k(t+2), \eta_i^k(t)\right).$$

We going to show that $\left(\sigma', \sigma^M\right)$ is an equilibrium with the same payoffs as σ.

First, notice that the strategy σ_i' of player i leads to the same probabilities of yielding by player i as well as the same outcomes. It follows that player $-i$ payoffs

are not affected by the modification.

Second, we are going to show that t is a best response for each type $u = \beta + \gamma\eta$ such that $(-1)^k \eta \in \left(\eta_i^k(t+2), \eta_i^k(t)\right)$. On the contrary, suppose that t is not a best response for u. Notice that if the interval is not empty, t is played with strictly positive probability under strategy σ. Hence, there is some type $u' = \beta + (-1)^k \eta'\gamma \in \mathcal{U}_i^k$ for which t is a best response, $u' \neq u$. Suppose that $(-1)^k \eta' > (-1)^k \eta$. By Lemma 10 and Lemma 8, the best response of all types $v = \beta + \eta''\gamma$ such that $(-1)^k \eta'' \leq (-1)^k \eta$

is strictly larger than t. But this implies that

$$\begin{aligned}\sum_{s\in T_i:s>t}\int_{\mathcal{U}_i^k}\sigma(u|s)\,d\pi(u) &\geq \pi\left\{\beta+\gamma\eta'':(-1)^k\eta''\leq(-1)^k\eta\right\}\\ &= \sum_{s\in T_i:s>t}\pi\left\{\beta+\gamma\eta'':(-1)^k\eta''(-1)^k\eta\in\left[\eta_i^k(s+2),\eta_i^k(s)\right)\right\}\\ &+\pi\left\{\beta+\gamma\eta'':(-1)^k\eta''(-1)^k\eta\in\left[\eta_i^k(t+2),\eta\right)\right\}\\ &\geq\sum_{s\in T_i:s>t}\int_{\mathcal{U}_i^k}\sigma(u|t)\,d\pi(u)+\pi\left\{\beta+\gamma\eta'':(-1)^k\eta''(-1)^k\eta\in\left[\eta_i^k(t+2),\eta\right)\right\}.\\ &>\sum_{s\in T_i:s>t}\int_{\mathcal{U}_i^k}\sigma(u|t)\,d\pi(u).\end{aligned}$$

But this leads to a contradiction. A similar contradiction can be found when $(-1)^k\eta'<(-1)^k\eta$. This concludes the proof of the Lemma.

B.2. **Proof of Lemma 3.** Let $\alpha_i^k(t)\geq 0$ be such that

$$x_i(t)=\sum_l\alpha_i^l(t)\,y_i^l+\left(1-\sum_l\alpha_i^l(t)\right)\mathbf{0}_i,$$

so that

$$P_i^k x_i(t)=\frac{\alpha_i^k(t)}{\sum_l\alpha_i^l(t)}.$$

Lemma 11. *There exists constants $c_i,d_i^k>0$ such that $\sum_l d_i^l=1$, such that if we define $\overline{\alpha}_i^j(t)=\alpha_i^j-d_i^j\left(\sum_l\alpha_i^l(t)-1\right)$ for each side j, then*

$$(-1)^k\eta_i^k(t)=\frac{v_i\left(\sum_l\alpha_i^l(t)-1\right)}{c_i\overline{\alpha}_i^{-k}(t)}\text{ and }\overline{P_i^k}(t)=\overline{\alpha}_i^{-k}(t).$$

Proof. For each k, we have

$$\begin{aligned}
0 &= u_i^k(t)\left(y_i^k\right) - u_i^k(t)\,x_i(t)\\
&= \left(\psi_i + \eta_i^k(t)\,\gamma\right)\cdot\left(\left(\alpha_i^k(t)-1\right)\left(y_i^k\right)_i + \alpha_i^{-k}(t)\left(y_i^{-k}\right)_i\right)\\
&= \left(\sum_l \alpha_i^l(t)-1\right)v_i + \eta_i^k(t)\,\gamma\cdot\left(\alpha_i^{-k}(t)\left(\left(y_i^{-k}\right)_i - \left(y_i^k\right)_i\right) + \left(\sum_l \alpha_i^l(t)-1\right)\left(y_i^k\right)_i\right)\\
&= \left(\sum_l \alpha_i^l(t)-1\right)v_i - (-1)^k\,\eta_i^k(t)\left(\alpha_i^{-k}(t)\left(\gamma\cdot\left(y_i^v-y_i^c\right)\right) - (-1)^k\left(\sum_l \alpha_i^l(t)-1\right)\left(\gamma\cdot y_i^k\right)\right)\\
&= \left(\sum_l \alpha_i^l(t)-1\right)v_i - (-1)^k\,\eta_i^k(t)\,c_i\left(\alpha_i^{-k}(t) - d_i^{-k}\left(\sum_l \alpha_i^l(t)-1\right)\right)
\end{aligned}$$

where we take $c_i = \gamma\cdot(y_i^v - y_i^c) > 0$, and $d_i^{-k} = \frac{(-1)^k\left(\gamma\cdot y_i^k\right)}{c_i} > 0$. The constants satisfy the required conditions. (To see that the constants are positive, notice that $(y_i^v)^v > (y_i^v)^c$ and that $(y_i^c)^v < (y_i^c)^c$. Also, notice that $d_i^k + d_i^{-k} = \frac{(-1)^k\left(\left(\gamma\cdot y_i^k\right)-\left(\gamma\cdot y_i^{-k}\right)\right)}{c_i} = \frac{c_i}{c_i} = 1$.) This implies the first equality. For the second one, observe that

$$\overline{P_i^k}(t) = \frac{(-1)^k\,\eta_i^k(t)}{\sum_l (-1)^l\,\eta_i^l(t)} = \frac{\frac{1}{\overline{\alpha}_i^{-k}(t)}}{\frac{1}{\overline{\alpha}_i^{-k}(t)} + \frac{1}{\overline{\alpha}_i^k(t)}} = \frac{\overline{\alpha}_i^k(t)}{\sum_l \overline{\alpha}_i^l(t)}.$$

The sum in the denominator of the last expression is equal to $\sum_l \overline{\alpha}_i^l(t) = \sum_l \alpha_i^l(t) - \left(\sum_l d_i^l\right)\left(\sum_l \alpha_i^l(t) - 1\right) = \sum_l \alpha_i^l(t) - \sum_l \alpha_i^l(t) + 1 = 1.$ □

Lemma 12. *For each i,k,t, $\sum_l \Delta\alpha_i^l(t) \geq 0$, and there is a constant $\alpha^* > 0$ such that $\alpha_i^k(t), \overline{\alpha}_i^k(t) \leq \alpha^*$, and*

$$\left|\sum_l \alpha_i^l(t) - 1\right|\left|\Delta\overline{\alpha}_i^k(t)\right| \leq \overline{\alpha}_i^k(t)\sum_l \Delta\alpha_i^l(t).$$

Proof. Because space X is compact, there is a constant $\alpha^* > 0$ such that $\alpha_i^k(t) \leq \alpha^*$ for each k, t. It follows that $\overline{\alpha}_i^k(t) \leq \alpha_i^k(t) \leq \alpha^*$.

Using Lemma 11, we observe that

$$\sum_l \frac{1}{(-1)^l\,\eta_i^l(t)} = \frac{c}{v_{-i}}\frac{\sum \alpha_i^l(t) - \left(\sum_l d_i^l\right)\left(\sum \alpha_i^l(t) - 1\right)}{\sum \alpha_i^l(t) - 1} = \frac{c}{v_{-i}}\left(\frac{\sum \alpha_i^l(t)}{\sum \alpha_i^l(t) - 1} - 1\right).$$

Because $(-1)^{-k}\,\eta_i^k(t)$ is increasing with t for each k, the left hand side is decreasing with t, which implies that the right hand side is decreasing with t, or that $\sum \alpha_i^l(t)$ is increasing in t.

By the first claim, for each i and t, there is k such that $\left|\Delta\overline{\alpha}_i^{-k}(t)\right| \leq \Delta\overline{\alpha}_i^k(t)$. Because $(-1)^{-k}\eta_i^{-k}(t) = \frac{v_{-i}}{c_i}\frac{\sum_l \alpha_i^l(t)-1}{\overline{\alpha}_i^{-k}(t)}$ is increasing in t, we have

$$\frac{\sum_l \Delta\alpha_i^l(t)}{\Delta\overline{\alpha}_i^k(t)} \geq \frac{\sum_l \alpha_i^l(t)-1}{\overline{\alpha}_i^k(t+2)},$$

which implies that

$$\left|\Delta\overline{\alpha}_i^{-k}(t)\right|\left|\sum_l \alpha_i^l(t)-1\right| \leq \left(\Delta\overline{\alpha}_i^k(t)\right)\left|\sum_l \alpha_i^l(t)-1\right| \leq \overline{\alpha}_i^k(t)\sum_l \Delta\alpha_i^l(t)$$

□

For the next result, recall that $f^* = \frac{dF}{d\eta}(0)$ (see equation (3.3)).

Lemma 13. *There exist constants $C<\infty$ (all independent of β and λ) such that for each $i,k,t \in T^i$ and $t > T^\eta$,*

$$\left|\sum_l \Delta\alpha_i^l(t)-1\right| \leq C\left|\sum_l (-1)^l\Delta\eta_i^l(t)\right| \leq C^2\left|\sum_l \Delta F_i^l(t)\right|,$$

$$\left|\sum_l \alpha_i^l(t)-1\right| \leq C\left|\sum_l (-1)^l\eta_i^l(t)\right| \leq C^2\left|\sum_l F_i^l(t)\right|,$$

$$\left|\frac{f^*(-1)^k\Delta\eta_i^k(t)}{\Delta F_i^k(t)}-1\right|, \left|\frac{f^*(-1)^k\eta_i^k(t)}{F_i^k(t)}-1\right| \leq C\left|\sum_l F_i^l(t)\right|,$$

$$\left|\frac{f^*\sum_l(-1)^l\Delta\eta_i^l(t)}{\sum_l \Delta F_i^l(t)}-1\right|, \left|\frac{f^*\sum_l(-1)^l\eta_i^l(t)}{\sum_l F_i^l(t)}-1\right| \leq C\left|\sum_l F_i^l(t)\right|.$$

Proof. By Lemma 11,

$$\frac{v_i}{c_i}\left(\sum_l \alpha_i^l(t)-1\right) = \frac{c_i}{v_i}\left(\sum_l \left((-1)^l\eta_i^l(t)\right)^{-1}\right)^{-1}.$$

Hence,

$$\frac{v_i}{c_i}\left|\sum_l \Delta\alpha_i^l(t)\right| = \left|\sum_l\left((-1)^l\eta_i^l(t)\right)^{-1}\right|^{-1}\left|\sum_l\left((-1)^l\eta_i^l(t+2)\right)^{-1}\right|^{-1}\left|\sum_l(-1)^l\frac{\Delta\eta_i^l(t)}{\eta_i^l(t)\,\eta_i^l(t+2)}\right|$$
$$= \sum_l(-1)^l\overline{\alpha}_i^l(t)\,\overline{\alpha}_i^l(t+2)\,\Delta\eta_i^l(t) \leq \sum_l(-1)^l\Delta\eta_i^l(t),$$

where the last inequality comes from the fact that $\overline{\alpha}_i^l(t) \leq 1$ for each $i,,l,t$. This shows the first inequality in the first line.

Because the increments in F are always positive, and $\sum_l \eta_i^l \left(T_i^{*,\sigma}+2\right)=0$, the first inequality in the second line inequality follows from the above.

The first inequality in the third line follows from the fact that the derivative is Lipschitz. All the remaining inequalities follow from the first. □

We can proceed with the proof of Lemma 3. The definition of $\alpha_i^k(t)$ as well as 11 Lemma imply that

$$P_i^k x_i(t)=\frac{\alpha_i^k}{\sum_l \alpha_i^l} \text{ and } \overline{P_i^k}(t)=\overline{\alpha_i^k}.$$

Taking into account that $\alpha_i^k(t)=\overline{\alpha}_i^k(t)+d_i^k(t)\left(\sum_l \alpha_i^l-1\right)$, we have

$$P_i^k x_i(t)-\overline{P_i^k}(t)=\frac{\alpha_i^k(t)}{\sum_l \alpha_i^l(t)}-\overline{\alpha_i^k}(t)=-\left(\overline{\alpha_i^k}(t)-d_i^k\right)\frac{\sum_l \alpha_i^l-1}{\sum_l \alpha_i^l}.$$

An application of Lemma 13 demonstrates the first estimate in the thesis of Lemma 3.

Second, by Lemma 12,

$$\begin{aligned}\left|\Delta\left(P_i^k x_i(t)-\overline{P_i^k}(t)\right)\right| \le &\left|\Delta\overline{\alpha_i^k}(t)\right|\frac{\sum_l \alpha_i^l(t)-1}{\sum_l \alpha_i^l(t)}+\left|\overline{\alpha_i^k}(t)-d_i^k\right|\frac{\left|\sum_l \Delta\alpha_i^l(t)\right|}{\sum_l \alpha_i^l(t)}\\ &+\left|\overline{\alpha_i^k}(t)-d_i^k\right|\frac{\left|\sum_l \alpha_i^l(t)-1\right|\left|\sum_l \Delta\alpha_i^l\right|}{\left(\sum_l \alpha_i^l(t)\right)\left(\sum_l \alpha_i^l(t+2)\right)}\\ \le &3\alpha^*\left|\sum_l \Delta\alpha_i^l(t)\right|.\end{aligned}$$

Another application of Lemma 13 shows the second estimate in the thesis of Lemma 3.

Third, observe that due to Lemma 13,

$$\begin{aligned}\left|\frac{\overline{Q_i^k}(t)}{Q_i^k(t)}-1\right| &= \left|\frac{\Delta F_i^k(t)}{f^*(-1)^k\Delta\eta_i^k(t)}\frac{f^*\sum_l \eta_i^l(t)}{\sum_l \Delta F_i^l(t)}-1\right| \le \left|\frac{1+K\sum_l(-1)^l\eta_i^l(t)}{1-K\sum_l(-1)^l\eta_i^l(t)}-1\right|\\ &\le 8K\left|\sum_l(-1)^l\eta_i^l(t)\right| \le C\left|\sum_l F_i^l(t)\right|\end{aligned}$$

for appropriately defined constant C.

The same calculations show that

$$\left|\frac{\sum_l \Delta\eta_i^l(t)}{\sum_l \eta_i^l(t)}/\left(\frac{\sum_l \Delta F_i^l(t)}{\sum_l F_i^l(t)}\right)-1\right|=\left|\frac{f^*\sum_l \Delta\eta_i^l(t)}{\sum_l \Delta F_i^l(t)}\frac{\sum_l F_i^l(t)}{f^*\sum_l \eta_i^l(t)}-1\right|\le C\left|\sum_l F_i^l(t)\right|.$$

B.3. **Proof of Lemma 4.** Define

$$T_i^O = \max\left\{t \in T_i, t \leq T_i^{*,\sigma} : x_i(t) \in \text{int}O_i^k \text{ for some } k\right\},$$
$$T_i^\eta = \max\left\{t \in T_i, t \leq T_i^{*,\sigma} : \eta_i^k(t) \geq \eta \text{ for some } k\right\} \text{ for each } \eta,$$
$$T_i^k = \max\left\{t \in T_i, t \leq T_i^{*,\sigma} : \text{player } i \text{ is only active on side } k\right\}.$$

For each $x = 0, \eta, k$,let $T^x = \max T_i^x$.

B.3.1. *Geometry.* For each player i, and each k, define

$$Y_i = \left\{x \in X \backslash \text{int}m_{-i} : P_i^k x \leq P_i^k\left(y_{-i}^{-k}\right) \text{ for each } k\right\} = \text{con}\left\{y_{-i}^1, y_{-i}^2, \mathbf{0}\right\} \backslash \text{int}m_{-i},$$
$$Y_i^k = \left\{x \in Y_i : P_i^k x = P_i^k\left(y_{-i}^{-k}\right)\right\} = B_i \cap \text{con}\left\{y_{-i}^{-k}, \mathbf{0}\right\} \backslash \text{int}m_{-i},$$
$$O_i^k = \left\{x : P_i^k x \geq P_i^k\left(y_{-i}^{-k}\right)\right\}.$$

To interpret the above sets, it is helpful to notice that $y(t) \in Y_i$ for each $t \in T_{-i}$. Additionally, the definition (A.2) implies that $y_i(t)$ belongs to the convex hull spanned by the allocation obtained from the optimal choices of the other player and the 0 allocation.) Thus, set Y_i contains all possible weighted winning allocations of player i (i.e., when $-i$ concedes). Its subset Y_i^k contains only those allocations that are obtained if $-i$ concedes and chooses y_{-i}^{-k} with (conditional) probability 1. (The reason for the notation is that $-k$ for player $-i$ faces side k for player i.) Sets O_i^k contain allocations that cannot be obtained as winning allocations. (See the right panel of Figure 3.2.)

We say that *side k of player i is regular* if $\text{int}O_i^k \neq \emptyset$.

Lemma 14. *For each $x \notin m_{-i}$, each i and each p,*

$$0 < P_{-i}^k(P_i x) < 1.$$

Proof. The projection of a projection. □

B.3.2. *Best response properties.* The subsequent claims are illustrated on Figure B.1. We leave them without a proof.

Lemma 15. *For each k, l, each $t \in T_i, t \leq T_i^{*,\sigma}$, if $x_i(t+2) \notin \text{int}O_i^{-k}$, and player i is only active on side k in period t, then $x_i(t) \notin O_i^{-k}$.* If $x_i(t+2) \in O_i^k$, and *player i is only active on side k in period t, then $x_i(t) \in \text{int}O_i^k$.*

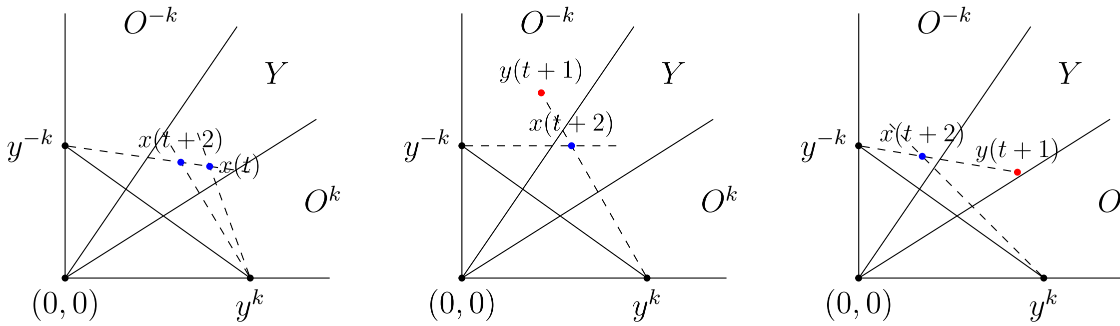


FIGURE B.1. Illustration of Lemma 15

Lemma 16. *For each k,l, each $t \in T_i, t \leq T_i^{*,\sigma}$, if $x_i(t+2) \notin O_{-i}^{-k}$, player i is active on side k in period $t+2$, and player $-i$ is only active on side k in period $t+1$, then player i is only active on k in period t.*

Lemma 17. *For each k,l, each $t \in T_i, t \leq T_i^{*,\sigma}$, if $x_i(t+2) \in intO_i^{-k}$, and player i is active on side $-k$ in period $t+2$, then player i is only active on $-k$ in period t.*

Lemma 18. *For each k,l, each $t \in T_i, t \leq T_i^{*,\sigma}$, if $P_i^k x_i(t+2) > P_i^k y_i(t+1)$, and player i is active on side k in period $t+2$, then the player is active only on side k in period t.*

B.3.3. *Approximations.* For each x, let $G_i^k(x) = \pi\left\{u \in \mathcal{U}_i^k : u(x) \leq u\left(y_i^k\right)\right\}$. For each player i, l, k, let

$$Q_{i,l}^{*,k} = P_i^k\left(y_{-i}^l\right).$$

Lemma 19. *There exists a constant $C < \infty$ and $\delta > 0$ such that for each $x \in X \backslash intm_{-i}$, if $\sum_l G_i^k(x) < \delta$, then*

$$\left|\frac{G_i^k(x)}{\sum G_i^l(x)} - \overline{P_i^k}(t)\right| \leq C \sum_l G_i^k(x)$$

Proof. Let $\alpha^k(x)$ be defined by $x = \sum_l \alpha_i^l(x) y_i^l$. Let $\overline{\alpha}_i^k(x) = \alpha_i^k(x) - d_i^k\left(\sum_l \alpha_i^l(x) - 1\right)$, where constants d_i^k are defined in Lemma 11. Then, Lemma 11 implies that

$$G_i^k(x) = (-1)^k\left(F\left(\frac{v_{-i}}{c_i}\frac{\sum_l \alpha_i^l(x) - 1}{\overline{\alpha}_i^{-k}(x)}\right) - F(0)\right).$$

Using the same arguments as in the proof of Lemma 13, we can show that there exists a constant $C<\infty$, such that

$$\left|\frac{G_i^k(x)}{f^*\frac{v_{-i}}{c_i}\frac{\sum_l \alpha_i^l(x)-1}{\overline{\alpha}_i^{-k}(x)}}\frac{f^*\sum_l \frac{v_{-i}}{c_i}\frac{\sum_l \alpha_i^l(x)-1}{\overline{\alpha}_i^l(x)}}{\sum_l G_i^l(x)}-1\right|\leq C\sum_l G_i^k(x).$$

But

$$\frac{f^*\frac{v_{-i}}{c_i}\frac{\sum_l \alpha_i^l(x)-1}{\overline{\alpha}_i^{-k}(x)}}{f^*\sum_l \frac{v_{-i}}{c_i}\frac{\sum_l \alpha_i^l(x)-1}{\overline{\alpha}_i^l(x)}}=\frac{\frac{1}{\overline{\alpha}_i^{-k}(x)}}{\sum_l \frac{1}{\overline{\alpha}_i^{-l}(x)}}=\frac{\overline{\alpha}_i^k(x)}{\sum_l \overline{\alpha}_i^l(x)}=\overline{\alpha}_i^k(x)=\overline{P_i^k}(t).$$

□

Lemma 20. *There exist constants $C<\infty$ and $\zeta_2>0,\zeta_2\leq\zeta_1$ (that do not depend on λ and β) such that for each $t\in T_i$* st. $T^{\zeta_2}<t<T^{*\sigma}$, if $x_i(t+2)\in Y_i^k$, then

$$Q_i^k(t)-C\left(\sum_k \Delta F_i^k(t)\right)\leq Q_{i,k}^{*k}\leq Q_i^k(t+2)+C\left(\sum_k \Delta F_i^k(t)\right).$$

Proof. We only show the first inequality; the proof of the second one is analogous. Assume that $x_i(t)\in Y_i$ and that $x_i(t+2)\in Y_i^k$.

By assumption, there exists $\alpha>0$ such that $x_i(t+2)=\alpha y_{-i}^{-k}+(1-\alpha)\mathbf{0}_i=:x\in Y_i^k$. Because $x_i(t)\notin \text{int}O_i^k$, we can find $x'=\alpha' y_{-i}^{-k}+(1-\alpha')\mathbf{0}_i$ such that

$$\sum_l G_i^l(x_i(t))-G_i^l(x_i(t+2))=\sum_l G_i^l(x')-G_i^l(x),\text{ and}$$
$$G_i^k(x_i(t+2))-G_i^k(x_1)\leq G_i^k(x')-G_i^k(x)$$

Then,

$$Q_i^k(t)=\frac{G_i^k(x_i(t))-G_i^k(x_i(t+2))}{\sum_l G_i^l(x_i(t))-G_i^l(x_i(t+2))}\leq\frac{G_i^k(x')-G_i^k(x)}{\sum_l G_i^l(x')-G_i^l(x)}. \tag{B.1}$$

For each $\alpha\geq 0$, let $H_i^k(\alpha):=G_i^k\left(\alpha y_{-i}^{-k}+(1-\alpha)\mathbf{0}_i\right)$. Let α^* be such that $\alpha^* y_{-i}^{-k}+(1-\alpha^*)\mathbf{0}_i=P_i\left(y_{-i}^{-k}\right)$. The assumptions on π imply that H_i^k has a Lipschitz continuous derivative h_i^k with a Lipschitz constant K. (To see it, notice that $\frac{u\cdot y_i^k}{u\cdot\left(\mathbf{1}-y_{-i}^{-k}\right)}$ is a continuous function of $u\in\mathcal{U}_i^k$.) Let $h^k=h_i^k(\alpha^*)$. As $\alpha\to\alpha^*$, $H_i^k(\alpha)\to 0$ and, by the L'Hospital's rule,

$$\frac{H_i^k(\alpha)}{\sum_l H_i^l(\alpha)}\to\frac{h^k}{\sum_l h^l}.$$

At the same time, Lemma 19 implies that

$$\frac{H_i^k(\alpha)}{\sum_l H_i^l(\alpha)} \to Q_{i,-k}^k.$$

Hence, the two limits are equal.

Then, for appropriately small η, $\alpha' \leq \frac{1}{2K}\sum_l h^l$, and the expression (B.1) is not larger than

$$\begin{aligned}
&\leq \frac{h^k(\alpha'-\alpha)+K(\alpha'-\alpha)^2}{\sum_l h^l(\alpha'-\alpha)-K(\alpha'-\alpha)^2} - Q_{i,-k}^k = \frac{h^k+K(\alpha'-\alpha)}{\sum_l h^l - K(\alpha'-\alpha)} - \frac{h^k}{\sum_l h^l}\\
&= K\frac{\left(h^k+\sum_l h^l\right)(\alpha'-\alpha)}{\left(\sum_l h^l - K(\alpha'-\alpha)\right)\left(\sum_l h^l\right)} \leq \frac{8K}{\left(\sum_l h^l\right)^2}\left[H_i^k(\alpha') - H_i^k(\alpha)\right]\\
&\leq C\left[\sum_l G_i^l(x_i(t)) - G_i^l(x_i(t+2))\right] = C\left(\sum_k \Delta F_i^k(t)\right)
\end{aligned}$$

for constant $C = 8K\left(\sum_l g^l\right)^{-2}$. □

Lemma 21. *There exist constants $\zeta_2' > 0$ and $C_0 > 0$ ((that does not depend on λ and β) such that for for each $t \in T_i$* st. $T^{\zeta_2'{}'} < t < T^{*\sigma}$, if $x_i(t+2), x_i(t) \in Y_i^k$ and player $-i$ is active on side $-k$ in periods $t+1$ and $t-1$, then

$$\beta_i \cdot \left(y_{-i}(t) - y_{-i}^k\right) \geq C_0(-1)^{-k}\eta_{-i}^{-k}(t+1).$$

Proof. Let $Q^* = \sum_l Q_{i,k}^{*l} y_i^l = P_i y_{-i}^k$, and let $q^* = \sum_p P_{-i}^p Q^* y_{-i}^p = P_{-i}\left(P_i y_{-i}^k\right)$. By definition, q^* belongs to the line that connects y_{-i}^k and y_{-i}^{-k}. Moreover, by Lemma 14, $P_{-i}^k q^* > 0$, and, because $(-1)^{-k}\gamma \cdot \left(y_{-i}^{-k} - y_{-i}^k\right) > 0$, we have

$$C_0 := \frac{1}{2}(-1)^{-k}\gamma \cdot \left(y_{-i}^{-k} - q^*\right) = \frac{1}{2}\left(P_{-i}^k q^*\right)(-1)^{-k}\gamma \cdot \left(y_{-i}^{-k} - y_{-i}^k\right) > 0.$$

Let $C < \infty$ be as in Lemma 20. Let $\zeta_2$be as in Lemma 20. We are going to fix $\zeta_2' \leq \zeta_2$ later. From now on assume that $t \geq T^{\zeta_2}$.

Because $-i$ is active on side $-k$ in period $t+1$ and $t-1$, Lemma 2 implies that

$$u_{-i}^{-k}(t+1) \cdot \left(y_{-i}(t) - y_{-i}^{-k}\right)_{-i} = 0.$$

Recall that $u_{-i}^{-k}(t+1) = \psi_{-i} + \eta_{-i}^{-k}(t+1)\gamma$. Because $\psi_{-i} \cdot y_{-i}^k = \psi_{-i} \cdot y_{-i}^{-k}$, we have

$$\begin{aligned}
0 &= \left(\psi_{-i} + \eta_{-i}^{-k}(t+1)\gamma\right) \cdot \left(y_{-i}(t) - y_{-i}^{-k}\right)_{-i}\\
&= \psi_{-i} \cdot \left(y_{-i}(t) - y_{-i}^k\right)_{-i} + \eta_{-i}^{-k}(t+1)\gamma \cdot \left(y_{-i}(t) - y_{-i}^{-k}\right)_{-i}, \qquad \text{(B.2)}
\end{aligned}$$

and

$$\psi_{-i}\cdot\left(y_{-i}(t)-y_{-i}^{k}\right)_{-i}=(-1)^{-k}\eta_{-i}^{-k}(t+1)\left[(-1)^{-k}\gamma\cdot\left(y_{-i}^{-k}-y_{-i}(t)\right)_{-i}\right]$$
$$=(-1)^{-k}\eta_{-i}^{-k}(t+1)\,2C_0$$
$$+(-1)^{-k}\eta_{-i}^{-k}(t+1)\left[(-1)^{-k}\gamma\cdot\left(q^{*}-y_{-i}(t)\right)_{-i}\right].$$

We are going to show that the term in the square brackets of the last line is smaller than C_0. First, notice that

$$y_{-i}(t)=\alpha w_{-i}^{\sigma}(t)=\alpha^{*}w_{-i}^{\sigma}(t)+(\alpha-\alpha^{*})\,w_{-i}^{\sigma}(t),$$

where we denoted $\alpha=\frac{\mathrm{e}^{-\Delta}p^{\sigma}(t)}{\mathrm{e}^{-2\Delta}p^{\sigma}(t)+\left(1-\mathrm{e}^{-2\Delta}\right)}$ and α_0 is chosen so that $\alpha_0\psi_{-i}\cdot w_{-i}^{\sigma}(t)=v_i$. By (B.2),

$$(\alpha-\alpha_0)\,\psi_{-i}\cdot\left(w_{-i}^{\sigma}(t)-y_{-i}^{k}\right)_{-i}=-\eta_{-i}^{-k}(t+1)\,\gamma\cdot\left(y_{-i}(t)-y_{-i}^{-k}\right)_{-i},$$

and, using Lemma 13, we can find a constant $C'<\infty$ such that

$$|\alpha-\alpha_0|\leq C'\left|\sum_{l}F_{-i}^{l}(t)\right|\leq C'm_2'.$$

Additionally, notice that $w_{-i}^{\sigma}(t)=\sum_l Q_i^l(t)\left(\mathbf{1}-y_i^k\right)$, and, by Lemma 20,

$$Q_{i,k}^{*k}-C\left(\sum_{k}\Delta F_i^k(t)\right)\leq Q_i^k(t)\leq Q_{i,k}^{*k}+C\left(\sum_{k}\Delta F_i^k(t)\right). \tag{B.3}$$

Hence,

$$\left\|\alpha_0 w_{-i}^{\sigma}(t)-q^{*}\right\|\leq\left\|w_{-i}^{\sigma}(t)-Q^{*}\right\|\leq C''\left(\sum_{k}\Delta F_i^k(t)\right)\leq Cm_2'.$$

Thus,

$$\left[(-1)^{-k}\gamma\cdot\left(q^{*}-y_{-i}(t)\right)_{-i}\right]$$
$$\leq 2\left(\left\|(\alpha-\alpha_0)\,w_{-i}^{\sigma}(t)\right\|+\left\|\alpha_0 w_{-i}^{\sigma}(t)-q^{*}\right\|\right)$$
$$\leq(4C'+2C)\,m_2'.$$

Pick $\zeta_2'\leq\zeta_2$ such that $(4C'+2C)\,m_2'\leq\frac{1}{2}C_0$. □

We have the following useful bounds on the yielding probability. Let Δ_0 be such that for each $\Delta\leq\Delta_0$,

$$\mathrm{e}^{-2\Delta}\leq 1-\Delta\leq\mathrm{e}^{-\Delta}$$

Lemma 22. *Suppose that* $\Delta \leq \Delta_0$. *There are constants* $0 < p_{\min} \leq p_{\max} < \infty$ *such that for each equilibrium, each* $t \in T_i$ *st.* $t_i^0 < t \leq T_i^{*,\sigma}$,

$$\Delta p_{\min} \leq p_i(t) \leq \Delta p_{\max}.$$

Proof. By Lemma 7, for each each $t \in T_i$ st. $t_i^0 < t \leq T_i^{*,\sigma}$, there are types $u, u' \in \mathcal{U}_{-i}$ such that $t-1$ is a best response for type u and $t+1$ is a best response for type u'. By Lemma 6 ,

$$\left(\mathrm{e}^{\Delta} - \mathrm{e}^{-\Delta}\right) \frac{1}{S_{-i}^{\sigma}(u,t) - \mathrm{e}^{-\Delta}} \leq p_i(t) \leq \left(\mathrm{e}^{\Delta} - \mathrm{e}^{-\Delta}\right) \frac{1}{S_{-i}^{\sigma}(u',t) - \mathrm{e}^{-\Delta}}.$$

The claim follows from the fact that $S_{-i}^{\sigma}(u,t) \leq \frac{1}{v_i} =: S_{\max}$ and t $S_{-i}^{\sigma}(u,t) \geq \frac{\min_{x \in m_i} u(x)}{\max_{x \in m_{-i}} u(x)} =: S_{\min} > 1$, where the last inequality comes from Assumption 2. □

For each player i,define $T_i^F(\eta) = \max\{t : \sum_l F_i(t) + \lambda \geq \eta\}$.

Lemma 23. *There exist constants* $a \geq a' > 0$, *such that for each* $\Delta \leq \Delta_0$, *each* $\eta \in [0,1]$,

$$\eta^a \leq \beta^{T_i^F(\eta)} \leq \eta^{a'}, \text{ and } \eta^{\frac{a}{a'}} \leq \lambda + \sum_l F_{-i}^l\left(T_i^F(\eta)\right) \leq \eta^{\frac{a'}{a}}.$$

Proof. Notice that

$$\sum_l F_i(t) + \lambda = \prod_{s \in T_i : s < t} (1 - p^{\sigma}(t)).$$

Due to Lemma 22, and the choice of $\beta \geq \beta_0$ (which implies $\mathrm{e}^{-2\Delta} \leq 1 - \Delta \leq \mathrm{e}^{-\Delta}$), we have

$$\begin{aligned}
&\left(\mathrm{e}^{-\Delta T_i^F(\eta)}\right)^{p_{\max}} \leq (1 - \Delta p_{\min})^{T_i^F(\eta)/2}. \\
\leq &\eta = \prod_{s \in T_i : s < T_i^F(\eta)} (1 - p_I(t)) \leq \\
\leq &(1 - \Delta p_{\min})^{T_i^F(\eta)/2} \leq \left(\mathrm{e}^{-\Delta T_i^F(\eta)}\right)^{\frac{1}{2} p_{\min}}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\eta^{\frac{2}{p_{\min}}} &\leq \left(\mathrm{e}^{-\Delta T_i^F(\eta)}\right)^2 \\
\leq &\mathrm{e}^{-\Delta T_i^F(\eta)} = (1 - \Delta)^{T_i^F(\eta)} \\
\leq &\mathrm{e}^{-\Delta T_i^F(\eta)} \leq \eta^{\frac{1}{p_{\max}}}.
\end{aligned}$$

Take $a = \frac{2}{p\min}$ and $a' = \frac{1}{p_{\max}}$. The second claim follows from the first. □

B.3.4. *Late game properties.* Let $\zeta_1 = \min_{i,k} \max_{u \in \mathcal{U}_i^k} (-1)^k (u^v - \psi_i^v)$.

Lemma 24. *If $\eta \leq \zeta_1$, then $T^\eta \geq T^O$.*

Proof. Suppose that $T^O = T_i^O < T^\eta$. By definition there is k, such that $T_i^O = \max \left\{ t \in T_i, t \leq T_i^{*,\sigma} : x_i(t) \in \text{int} O_i^k \right\}$. By Lemma 15, it must be that player i is active on side k in period t. By Lemma 16, player i is only active on side k in period t. By another application of the first part, $x_i(t-2) \notin O_i^k$. A repetition of the same argument shows that player i is active only on side k for each $t < T^O, t \in T_i$. But this implies that $\eta_i^{-k}(t_i^0) = \eta_i^{-k}\left(T^O\right) < \eta \leq \zeta_1$, which contradicts the choice of $\eta \leq \zeta_1$. □

Lemma 25. *If $\eta \leq \zeta_1$, then, for each $t > T^\eta$* st. *$T^\eta < t < T^{*\sigma}$, if player i is active on side $-k$ in period $t+1$, and player $-i$ is only active on side k in period t, then $x_i(t+1) \in Y_i^{-k}$.*

Proof. Suppose that there is $t > T^\eta$ such that player i is active on side $-k$ in period $t+1$, and player $-i$ is only active on side k in period t. Lemma 24 implies that $x_i(t+1) \notin \text{int} O_i^k$ and $x_{-i}(t+2) \notin \text{int} O_{-i}^{-k}$. Suppose that $x_i(t+1) \notin Y_i^{-k}$, which implies that $x_i(t+1) \notin O_i^{-k}$. By Lemma 16, player i is only active on side k in period $t-1$. Another application of Lemma 15 shows that $x_{-i}(t) \notin O_{-i}^{-k}$ and player $-i$ is only active on side k in period t. A repetition of the same argument shows that each player is active only on side k for each $t' < t$. But this implies that $\eta_i^{-k}(t_i^0) = \eta_i^{-k}\left(T^O\right) < \eta \leq \zeta_1$, which contradicts the choice of ζ_1. □

Lemma 26. *If $T^\eta < T^{*,\sigma} - 2$, then, for each k,$T^k < T^{*,\sigma}$.*

Proof. On the contrary, suppose that $T_i^k = T_i^{*,\sigma}$ for some i. Let $\overline{T}_i = \max\{t : \text{player } i \text{ is active on side } -$ T_i^k. Because type ψ_i is the limit of types in $\mathcal{U}_i^{-k}$, the proof of Lemma 1 implies that type with preferences ψ_i must be indifferent between yielding in any period $t \in T_i$ and $T_i^{*,\sigma} \leq t \leq \overline{T}_i$. The calculations in Lemma 6 show that it must be that $\psi_i \cdot y_i(t) = L_i(\psi_i) = v_{-i}$, which implies that $y_i(t) \in \text{bd} m_{-i}$. Because there are types in $\mathcal{U}_i^k$ who weakly prefer to wait and yield only in period $T_i^{*,\delta}$, a similar argument shows that it must be $y_i(t) = y_i^k$ for each $t \in T_{-i}, T_i^{*,\sigma} \leq t \leq \overline{T}_i$. However, because $y_i(t)$ is a convex combination of $\mathbf{1} - y_{-i}^l$ for $l = 1, 2$ and the zero allocation, it must be that for each $t \in T_{-i}, T_i^{*,\sigma} \leq t \leq \overline{T}_i$, player $-i$ is only active on $-k$ side and that side k of player i is not regular. Note that it follows that side $-k$ of player $-i$ is regular.

If player i is the last player, i.e., $T_i^{*,\sigma} = T^{*,\sigma}$, then $x_{-i}(T^{*,\sigma}-1) = y_{-i}^{-k} \in \mathrm{int}O_{-i}^{-k}$. If player $-i$ is the last player, then Lemma 25 implies that it must be that $x_{-i}(T^{*,\sigma}) \in Y_{-i}^{-k}$. But because player $-i$ is only active on side $-k$ in period $T^{*,\sigma}-2$, then Lemma 15 implies that $x_{-i}(T^{*,\sigma}-2) \in \mathrm{int}O_{-i}^{-k}$. In any case, we obtain a contradiction with Lemma 24. □

Lemma 27. *There is $m_3 > 0, \eta^3 \leq \zeta_2, \zeta_1$ such that if $\eta \leq m_3$, then, for each $t > T^\eta$* st. *$T^\eta + 2 < t < T^{*\sigma}$, if player i is active on side $-k$ in period $t+1$, $y_{-i}(t+1) \in U_{-i}^l(t+2)$ for each l, and player $-i$ is only active on side k in period t, then, $y_{-i}(t-1) \in U_{-i}^l(t)$ for each l, player i is active on side $-k$ in period $t-1$, and player $-i$ is only active on side k in period $t-2$.*

Proof. Take period $t \in T_{-i}$ such that $T^\eta + 2 < t < T^{*\sigma}$, and such that player i is active on side $-k$ in period $t+1$, and player $-i$ is only active on side k in period t. By Lemma 25, $x_i(t+1) \in Y_i^{-k}$, and by Lemma 24, $x_i(t+3), x_i(t-1) \in Y_i$.

First, we are going to show that player $-i$ is only active on side k in period $t-2$. By Lemma 18, it is enough to show that

$$P_{-i}^k x_{-i}(t) > P_{-i}^k y_{-i}(t-1) \tag{B.4}$$

Because $y_{-i}(t+1) \in U_{-i}^l(t+2)$ for each l, and player $-i$ is active on side k in period t, it must be that $y_{-i}(t+1) \in I_{-i}^k(t+2)$ and that

$$P_{-i}^k x_{-i}(t+2) \geq P_{-i}^k y_{-i}(t+1). \tag{B.5}$$

By Lemma 20,

$$Q_{i,-k}^{*-k} \leq Q_i^{-k}(t+1) + C\left(\sum_l \Delta F_i^l(t)\right), \text{ and}$$
$$Q_{i,-k}^{*-k} \geq Q_i^{-k}(t-1) - C\left(\sum_k \Delta F_i^k(t)\right),$$

where $Q^{*l}_{i,k} = P^l_i y^k_{-i}$. Because $P^k_{-i}\left(y^{-k}_i\right) > P^k_{-i}\left(y^k_i\right)$ (due to the side $-k$ of player i facing the side k of player $-i$), inequality (B.5) implies that

$$\begin{aligned}
P^k_{-i}x_{-i}(t+2) &\geq P^k_{-i}y_{-i}(t+1)\\
&= \sum_l Q^l_i(t+1)\,P^k_{-i}y^l_i\\
&= P^k_{-i}y^k_i + Q^{-k}_i(t+1)\left[P^k_{-i}y^{-k}_i - P^k_{-i}y^k_i\right]\\
&\geq P^k_{-i}y^k_i + Q^{*-k}_{i,-k}\left[P^k_{-i}y^{-k}_i - P^k_{-i}y^k_i\right] - C\left(\sum_l \Delta F^l_i(t)\right)\\
&= A - C\left(\sum_l \Delta F^l_i(t)\right),
\end{aligned}$$

where we denoted $A = P^k_{-i}\left(\mathbf{1} - y^k_i\right) + Q^{*-k}_{i,-k}\left[P^k_{-i}\left(\mathbf{1} - y^{-k}_i\right) - P^k_{-i}\left(\mathbf{1} - y^k_i\right)\right]$. On the other hand, we have

$$P^k_{-i}y_{-i}(t-1) \leq A + C\left(\sum_l \Delta F^l_i(t)\right). \tag{B.6}$$

Because player $-i$ is only active on side k in period t, we have $\overline{Q^k_{-i}}(t) = 1$. By the equation (3.6) and Lemma 3, we have

$$\begin{aligned}
P^k_{-i}x_{-i}(t) &\geq P^k_{-i}x_{-i}(t+2) + \frac{\sum_l \Delta F^l_{-i}(t)}{\sum_l F^l_{-i}(t)}\left(1 - P^k_{-i}(t+2)\right) - C\sum_l \Delta F^l_i(t)\\
&\geq P^k_{-i}x_{-i}(t+2)\left(1 - \frac{\sum_l \Delta F^l_{-i}(t)}{\sum_l F^l_{-i}(t)}\right) + \frac{\sum_l \Delta F^l_{-i}(t)}{\sum_l F^l_{-i}(t)} - C\sum_l \Delta F^l_i(t)\\
&\geq A\left(1 - \frac{\sum_l \Delta F^l_{-i}(t)}{\sum_l F^l_{-i}(t)}\right) + \frac{\sum_l \Delta F^l_{-i}(t)}{\sum_l F^l_{-i}(t)} - 2C\left(\sum_l \Delta F^l_i(t)\right)\\
&= A + \frac{\sum_l \Delta F^l_{-i}(t)}{\sum_l F^l_{-i}(t)}(1-A) - 2C\left(\sum_l \Delta F^l_i(t)\right).
\end{aligned} \tag{B.7}$$

Note that $A < 1$. Find $m_3 \leq \zeta_1, \zeta_2$ small enough so that

$$m_3 \leq \frac{1-A}{12f^*C}.$$

Then, for each $\eta \leq m_3$ $t > T^\eta$, Lemma 3 implies that

$$\sum_l F^l_{-i}(t) \leq 2f^*\sum_l (-1)^l\,\eta^l_{-i}(t) \leq 4f^*\eta \leq \frac{1-A}{3C}.$$

The inequality (B.4) follows from the above bound, as well as the inequalities (B.6) and (B.7). □

B.3.5. *Proof of Lemma 4.* Let m_3 be as in Lemma 27.Let $a \geq a' > 0$ be constants from Lemma 23. Let

$$m_4 = \left(\frac{1}{2}\right)^{\frac{a}{a'}} (m_3)^{\left(\frac{a}{a'}\right)^2}.$$

Let $n > 0, n < m_3$ be a very small number to be fixed later and such that

$$n \leq \frac{1}{4} m_4^{\frac{a}{a'}}.$$

Suppose that $\lambda < \frac{1}{4}n$.

On the contrary, suppose that there is an equilibrium such that $T^k > T^n$. Using Lemma 23, we can show that $T^n > T^{*,\sigma}+2$. By Lemma 26, $T^k < T_i^{*,\sigma}$ for each i. Thus, if $T_{-i}^k = T^k$, then player $-i$ is active on both sides in period T^k+2. In particular, by Lemma 2,$y_{-i}\left(T^k+1\right) \in U_{-i}^l\left(T^k+2\right)$ for each l. Moreover, player i is active on both sides (including side $-k$) in period T^k+1. In such a situation, a repeated application of Lemma 27 shows that player $-i$ is active only on side k in each period t such that $T^k \leq t < T^{m_3}$.

Then, by Lemma 23, we have

$$\lambda + \sum_l F_i^l\left(T^{m_3}\right) \geq m_3^{\frac{a}{a'}},$$

$$\lambda + \sum_l F_i^l\left(T^{m_4}\right) \leq (m_4)^{\frac{a'}{a}} = \frac{1}{2} m_3^{\frac{a}{a'}},$$

$$\beta^{T^{m_4}} \geq m_4^a = \left(\frac{1}{2}\right)^{a^2(a')^{-1}} m_3^{a^3(a')^{-2}}.$$

Using the first two inequalities, we conclude that

$$\sum_l \left(F_i^l\left(T^{m_3}\right) - F_i^l\left(T^{m_4}\right)\right) \geq m_3^{\frac{a}{a'}} - (m_4)^{\frac{a'}{a}} = \frac{1}{2} m_3^{\frac{a}{a'}}.$$

Let $n_4 = \eta_{-i}^k\left(T^{m_4}\right)$. Then, an application of Lemmas 13 and 23 shows that

$$n_4 \geq \frac{1}{2} f^* F_{-i}^k\left(T^{m_4}\right) \geq \frac{1}{2} f^* \left(m_4^{\frac{a}{a'}} - \lambda - n\right) \geq \frac{1}{4} f^* \left(\frac{1}{2}\right)^{a^2(a')^{-2}} m_3^{a^3(a')^{-3}}.$$

Let $n_0 = \eta_{-i}^{-k}\left(T^k+2\right)$, where $(-1)^{-k} n_0 \leq n$ be the last type on side $-k$ to yield in the "late game". Let $T_0 = \max\left\{t < T^k : \text{player } -i \text{ is active on side } -k \text{ in period } t\right\}$. By the above, $T_0 \leq T^{m_3}$.Moreover, the continuity implies that the type n_0 must be indifferent between yielding in period T^k+2 and T^0 and weakly prefer it to yielding in any period in-between. However, we are going to show that if n is sufficiently small, then type n_0 strictly prefers to yield in period T^{m_4} rather than in period T_0. This will yield a contradiction, and conclude the proof of the Lemma.

For this purpose, let $u_0=\beta_i+n_0\gamma$ be the type that corresponds to n_0. Notice that formula (A.3) implies that

$$\begin{aligned}
&U^{\sigma}_{-i}\left(u_0 T^{m_4}\right)-U^{\sigma}_{-i}\left(u_0,T_0\right)\\
&=\sum_{t\in T_i:T_0<t\leq T^{m_4}}\mathrm{e}^{-s\Delta}\left(f^{\sigma}\left(t+1\right)+\left(1-\mathrm{e}^{-2\Delta}\right)\left(\sum_{s:s>t+1,z\in T_i}f^{\sigma}\left(z\right)\right)\right)\beta_i\cdot\left[y_{-i}\left(t+1\right)-y^{-k}_{-i}\right]\\
&\quad+n_0\sum_{t\in T_i:T_0<t\leq T^{m_4}}\mathrm{e}^{-s\Delta}\left(f^{\sigma}\left(t+1\right)+\left(1-\mathrm{e}^{-2\Delta}\right)\left(\sum_{s:s>t+1,z\in T_i}f^{\sigma}\left(z\right)\right)\right)\gamma\cdot\left[y_{-i}\left(t+1\right)-y^{-k}_{-i}\right].
\end{aligned}\tag{B.8}$$

Because $y_{-i}\left(t+1\right)\in X\backslash \mathrm{int}m_i$, we have $\beta_i\cdot\left[y_{-i}\left(t+1\right)-y^{-k}_{-i}\right]\geq 0$ for each t. Moreover, by Lemma 21, for each $t\geq T^{m_3}$,

$$\beta_i\cdot\left(y_{-i}\left(t\right)-y^{k}_{-i}\right)\geq C_0\left(-1\right)^k\eta^k_{-i}\left(t+1\right)\geq C_0 n_4.$$

Hence, the first term of (B.8) is not smaller than

$$\begin{aligned}
&\geq\sum_{t\in T_i:T^0<t\leq T^{m_4}}\mathrm{e}^{-s\Delta}\left(f^{\sigma}\left(t+1\right)\right)\beta_i\cdot\left[y_{-i}\left(t+1\right)-y^{-k}_{-i}\right]\\
&\geq\beta^{T^{m_4}}\sum_l\left(F^l_i\left(T^{m_3}\right)-F^l_i\left(T^{m_4}\right)\right)C_0 n_4\\
&\geq\frac{1}{4}C_0 f^*\left(\frac{1}{2}\right)^{a^2(a')^{-2}}m_3^{a^3(a')^{-3}}\frac{1}{2}m_3^{a(a')^{-1}}\left(\frac{1}{2}\right)^{a^2(a')^{-1}}m_3^{a^3(a')^{-2}}=:c_0.
\end{aligned}$$

Let $x^*=\max_{x\in X}\left|\gamma\cdot\left(x-y^{-k}_{-i}\right)\right|$ Then, the second term of (B.8) is not smaller than

$$\geq -nx^*.$$

The lemma is concluded by picking $n<\frac{c_0}{x}$.

B.4. **Proof of Lemma 5.**

Lemma 28. *For each x, if $y\in bdm_i$ and $P_i x=P_i y$, then $R_{-i}x=y$ and $P_{-i}y=y$. Moreover, there exist constants $A_i>1$ for each i such that for each i, each $x\in bdm_i$,*

$$\gamma\cdot\left(R_{-i}x\right)_{-i}=-A_i\left(\gamma\cdot x\right).$$

Proof. See Figure 3.2. □

Let n be as in Lemma 4. Let

$$\overline{p_i}(t) = \gamma \cdot \left(\sum_k \overline{P_i^k} w_i(t) y_i^k \right),$$

$$p_i(t) = \gamma \cdot \left(\sum_k P_i^k x_i(t) y_i^k \right),$$

$$q_i(t) = \gamma \cdot \left(\sum_l Q_i^l(t) y_{-i}^l \right),$$

$$\overline{q_i}(t) = \gamma \cdot \left(\sum_l \overline{Q_i^l}(t) y_{-i}^l \right).$$

Using the projection notation from Section 3.2.1, we show (**??**) is equivalent to

$$P_i x_i(t) = P_i y_i(t-1) = P_i \left(\sum_l Q_{-i}^l(t-1) y_{-i}^l \right).$$

Because $\sum_l Q_{-i}^l(t-1) y_{-i}^l \in \text{bd} m_i$, Lemma 28 implies that

$$\gamma \cdot \sum_l Q_{-i}^l(t-1) y_{-i}^l = A_{-i}(\gamma \cdot P_i x_i(t)), \text{ or} \tag{B.9}$$

$$q_{-i}(t-1) = A_{-i} p_i(t).$$

Further, (3.6) implies that

$$\Delta \overline{p_i}(t) = c(t)(\overline{q_i}(t) - \overline{p_i}(t+2)), \tag{B.10}$$

for $c(t) = \frac{\sum_l \Delta \eta_i^l(t)}{\sum_l \eta_i^l(t)} \in [0,1]$.

Let $C < \infty$ be the constant from Lemma 3. Let $C = C' \max A_i$. Then, Lemma 3 implies that

$$\Delta \overline{p_i}(t-2) = c(t-2)(A_{-i}\overline{p_{-i}}(t-1) - \overline{p_i}(t) \pm C'\eta), \tag{B.11}$$

where $\pm C\eta$ is a bound on the error term of the expression in the brackets.

Lemma 29. *For each $\eta \leq n$, we have that for each $t > T^\eta$ and $t \in T_i$, either (a) $\overline{p_i}(t) \cdot \overline{p_{-i}}(t-1) > 0$, or (b) $|\overline{p_i}(t) - \overline{p_{-i}}(t-1)_i| \leq C'\eta$.*

Proof. We prove the Lemma by induction on $t \geq T^*$. If $t = T^*$, then $c(T^*-1) = 1$, and $\overline{p_i}(T^*-1) = \overline{q_i}(T^*-1) = A_i \overline{p}_i \pm C\eta$, where $\pm C\eta$ is a bound on the error term. Thus, the claim holds for $t = T^*$.

Suppose that the claim holds for some $t > T^{\eta}$ and $t \in T_i$. Suppose that $\overline{p_{-i}}(t-1) > 0$ (the proof in the other case is analogous). The inductive claim implies that $\overline{p_i}(t) \geq \overline{p_{-i}}(t-1) - C\eta$, and we need to show that $\overline{p_i}(t-2) \geq p_{-i}(t-1) - C\eta$. By Lemma 4 and the above discussion, equation (B.11) holds. Because $A_{-i} > 1$,

$$A_{-i}\overline{p_{-i}}(t-1) \pm C'\eta \geq \overline{p_{-i}}(t-1) - C'\eta,$$

and

$$\begin{aligned}\overline{p_i}(t-2) - \overline{p_{-i}}(t-1) &= \Delta\overline{p_i}(t-2) + \overline{p_i}(t) - \overline{p_{-i}}(t-1) \\ &\geq (1 - c(t-2))\left(\overline{p_i}(t) - \overline{p_{-i}}(t-1)\right) - c(t)\, C'\eta \\ &\geq -C'\eta.\end{aligned}$$

□

Lemma 30. *For each $\delta > 0$, there is $c_0 > 0$ and $\eta_\delta \leq n$, such that for each $t > T^{\eta_\delta}$ and $t \in T_i$, if $\overline{p_i}(t), \overline{p_{-i}}(t-1) \geq \delta$, then $p_{-i}(t-2) \geq \delta + c(t-2)\, c_0\delta$. (An analogous claim holds when $\overline{p_i}(t), \overline{p_{-i}}(t-1) \leq -\delta$.)*

Proof. Choose η_δ so that $(\min_i A_i - 1)\,\delta \geq 2C'\eta_\delta$. Let $c_0 = \frac{1}{2}(\min_i A_i - 1)$. By formula (B.11)

$$\begin{aligned}\overline{p_i}(t-2) - \delta \geq& \Delta\overline{p_i}(t-2) + \overline{p_i}(t) - \delta \\ \geq& c(t-2)\left((A_{-i} - 1)\,\overline{p_{-i}}(t-1) - \delta\right) \\ &+ c(t-2)\left(\overline{p_{-i}}(t-1) - \delta\right) \\ &+ (1 - c(t-2))\left(\overline{p_i}(t) - \delta\right) \\ \geq& c(t-2)\left((A_{-i} - 1)\,\delta - C'\eta\right) \geq c(t-2)\, C\delta.\end{aligned}$$

□

Lemma 31. *There exists a $D < \infty$ such that for each $\delta > 0$, there is $\eta_\delta \leq n$, such that if $\overline{p_i}(t) \geq D\delta$ for some t, then for each $t' > T^{\eta_\delta}, t' < t$, $t \in T'_j$, $\overline{p_j}(t') \geq \delta$. (An analogous claim holds when $p_i(t) \leq -D\delta$.)*

Proof. Let $D = 2\max_i A_i$. Let η'_δ be the constant from Lemma 30. Let $\eta_\delta \leq \eta'_\delta$ be such that $C\eta_\delta \leq \frac{1}{2}D\delta$. By Lemma 30, it is enough to show that if $t_0 = \max\{t : \overline{p_i}(t) \geq D\delta \text{ for } i \text{ st. } t \in T_i\}$, then $\overline{p_{-i}}(t_0 + 1) \geq \delta$. To see it, notice that

$\overline{p_i}(t_0+2) \leq D\delta \leq \overline{p_i}(t_0)$. Hence, formula (B.11) and the fact that $c(t_0) \leq 1$ imply that

$$\begin{aligned}0 \leq D\delta - \overline{p_i}(t_0+2) \leq & A_{-i}\overline{p_{-i}}(t_0+1) - \overline{p_i}(t_0+2) + C'\eta \\ \leq & A_{-i}\overline{p_{-i}}(t_0+1) + C'\eta - D\delta - (\overline{p_i}(t_0+2) - D\delta) \\ \leq & A_{-i}\overline{p_{-i}}(t_0+1) - \frac{1}{2}D\delta.\end{aligned}$$

The claim follows from the choice of constant D. □

Lemma 32. *There exists $\Delta^*, \eta^* > 0$ such that for each integer $A > 0$, for each $\eta \leq \frac{1}{2^{8A}}\eta^*$, there exists $\lambda^* > 0$ such that if $\Delta \leq \Delta^*, \lambda \leq \lambda^*$, then for each player i,*

$$\sum_{t \in T_i : T^{\eta} \leq t \leq T^{2^{8(A+1)}\eta}} \frac{\sum_l \Delta\eta_i^l(t)}{\sum_l \eta_i^l(t)} \geq A.$$

Proof. If $\Delta > 0$ is sufficiently small, then Assumption 2 implies that $p_i^{\sigma}(t) \leq \frac{1}{4}$ for each $t \in T_i$ and $t > 1$. Then, Lemma 3 implies that there exists $\eta^* > 0$ such that for each $\eta \leq \eta^*$, there exists $\lambda^* > 0$ such that

$$\frac{\sum_l \Delta\eta_i^l(t)}{\sum_l \eta_i^l(t)} \leq \frac{1}{2} \text{for each t} \in T_i \text{ and } T_i^{\eta} \leq \text{t} \leq T_i^{\eta^*}.$$

Hence, for each k, there exist t_k^1, t_k^2 such that $8^k\eta \leq \sum_l \eta_i^l(t_k^1) \leq 2 \cdot 8^k$ and $4 \cdot 8^k\eta \leq \sum_l \eta_i^l(t_k^2) \leq 8^{k+1}$. Then, we have

$$\begin{aligned}\sum_{t \in T_i : T^{\eta} \leq t \leq T^{8(A+1)\eta}} \frac{\sum_l \Delta\eta_i^l(t)}{\sum_l \eta_i^l(t)} \geq & \sum_{k=0}^{8A} \frac{1}{8^{k+1}\eta} \sum_{t \in T_i : T^{8^k\eta} \leq t < T^{8^{k+1}\eta}} \sum_l \Delta\eta_i^l(t) \\ \geq & \sum_{k=0}^{8A} \frac{1}{8^{k+1}\eta}\left(\sum_l \eta_i^l\left(t_k^2\right) - \sum_l \eta_i^l\left(t_k^1\right)\right) \geq \sum_{k=0}^{8A} \frac{1}{8^{k+1}\eta}\left(2 \cdot 8^k\eta\right) \\ \geq & 8A \cdot \frac{1}{4} > A.\end{aligned}$$

□

B.4.1. *Proof of Lemma 5.*

Proof. Let $P_{\max} = \left\lceil Dc_0^{-1} \max_{x \in X} |\gamma \cdot x| \right\rceil$, where C is the constant from Lemma 30. Let η'_{δ} be the constant from Lemma 31 (in the proof, we choose it so that it satisfies also Lemma 30). Let $\eta_{\delta} = \frac{1}{8^{2(P_{\max}+1)}}\eta'_{\frac{1}{D}\frac{1}{\max_i A_i}\delta}$, where D is the constant from Lemma 31 .

On the contrary, suppose that $|p_i(t)| \leq \delta$ for some $t > T^{\eta_\delta}, t \in T_i$. W.l.o.g. we assume that $\overline{p_i}(t) > 0$. Then, Lemma 31 implies that $\overline{p_{-i}}(t+1) \geq \frac{1}{D}\frac{1}{\max_i A_i}\delta$. By Lemma 30, for each $t \geq T^{\eta'_{D^{-1}\delta}}$,

$$p_{-i}(t-2) \geq \delta + c_0 D^{-1}\frac{1}{\max_i A_i}\delta c(t-2).$$

Hence, using the definition of $c(.)$, we have

$$\frac{1}{\delta}P_{\max} \geq \sum_{t\in T_i : T^{\eta_\delta} \leq t \leq T^{4\left\lceil \log\frac{1}{\delta}P_{\max}\right\rceil \eta_\delta}} \frac{\sum_l \Delta\eta_i^l(t)}{\sum_l \eta_i^l(t)}.$$

It follows from Lemma 32 that

$$|\overline{p_i}(t)| \leq \frac{1}{\max_i A_i}\delta.$$

Note that by Lemma 4 and equation (3.5), we have

$$w_i(t) = R_{-i}y_i(t) = R_{-i}x_i(t).$$

Hence, by Lemma 28,

$$|\gamma \cdot w_i(t)| \leq \delta.$$

The result follows from the fact that $w_i(t)$ belongs to the boundary of menu m_i, and that $\mathbf{1} - e^*_{-i}$ is the intersection of the boundary with the diagonal $\{x : \gamma \cdot x = 0\}$. □