SMALL GROUP COORDINATION

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Abstract. This paper studies a coordination process of a large population. We assume that subgroups of the population differ in the intensity of their coordination problems. Specifically, small (but not large) groups are able to coordinate and simultaneously take a coalitional best response – an action, which improves the welfare of all the members of the group. When agents interact on a network, in the long-run they always play an action which is either payoff or risk-dominant. The outcome depends on the network. In general, the ability of small groups to coordinate may decrease the chances of selecting the efficient outcome. When players interact on many networks at the same time, the dynamics may uniquely select an action, which is neither payoff nor risk dominant. This happens when such an action is sufficiently "attractive" for small groups. We characterize situations when efficiency is helped or obstructed by the presence of small groups. As an application, we discuss how networks of interactions may affect the probability of self-segregation and polarization.

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1. Introduction

Consider a large population of players in a coordination game. Such a game has multiple equilibria; one of them may be payoff dominant. If the population is able to coordinate effectively, then one may expect them to choose the efficient outcome. Whether or not it happens, depends on how effective the coordination is. For example, the evolutionary literature studies the decentralized adjustment process, in which players choose their best responses independently from each other. This leads players to play the risk-dominant rather than the payoff-dominant action.

Suppose that there is a group of players with a coalitional best response – a behavior or action which, if taken jointly, improves the situation of all the members of the group. Then to take this action, the members of the group have to coordinate their behavior, as a coalitional best response requires many people changing their behavior at the same time. In this paper we assume that the ability of a group to coordinate effectively depends on its size and in particular, the larger the group the less probable it is that they will undertake their coalitional best response. So if a small group has a coalitional best response, then it often succeeds in performing it, while the coalitional best response of a large group is taken very rarely.

Given this assumption that groups of agents can coordinate and the smaller the group, the easier the coordination, we check if the details of interactions within the population (social structure) matters for the coordination process. The interplay between small group coordination and social structure is the main focus of this paper. Broadly speaking, we ask – Is the result of the coordination process affected by the way that society is organized? Secondly, is it affected by the details of the correlation between social structure and the payoffs from behavior? We give an affirmative answer to both these questions.

In response to the first question, social structure does matter. In particular, a society divided into small isolated entities, tends to coordinate successfully on the efficient action. When players' interactions are spread uniformly across the population, then the risk-dominant outcome prevails in the long-run. The second question allows for payoffs from interactions in society to depend on the form that social structure takes. In this case, surprisingly, in the long-run the population may uniquely choose a behavior which is payoff and risk dominated by the alternative. This occurs when there is an action, which is neither risk dominant nor efficient, nevertheless it is sufficiently "attractive" for small groups.

It is a common understanding that social networks play an important role in many aspects of economic life: exchange of information, goods, insurance, strengthening social cohesion and making certain cooperative outcomes possible. This paper finds that they also matter for equilibrium selection in simple coordination games. Understanding exactly how different networks influence equilibrium selection is useful. For instance, one might be interested in implementing a desired equilibrium outcome. The usual method considered in the literature is manipulating with payoffs. From this paper, one could learn how to affect the equilibrium played by redesigning the network. In some situations "social engineering" is easier and a more natural option than a transfer scheme. Finally, the specific examples of networks discussed seem to have some correlates in the real world. In the last section, we discuss how different networks of interactions may affect the probability of polarization and racial self-segregation, which were the original motivations for this paper. In terms of the literature, this is one of the very few papers that lead to network-dependent results, but this will be discussed more thoroughly after a quick overview of the model and results.

1.1. **Model and results.** The model is based on the evolutionary framework of [Kandori, Mailath, and Rob 1993] (further KMR) and [Ellison 1993]. Consider a large population of players located on the nodes of a network. Each player chooses one of two actions and the same action is used in interactions with all his neighbors. The payoffs in each interaction are given by the same symmetric 2×2 coordination game. The KMR dynamics consist of two main elements: Players change their actions through *individual best responses* and, with small probability, through *individual mistakes.* These elements lead to coordination on the risk-dominant equilibrium in (essentially) any network.

We keep these two elements and, in addition, we allow groups of players to change their behavior collectively through *coalitional best responses.* A coalitional best response is an action, which if taken by all members of the group, would simultaneously improve the payoffs of all of them. We assume that the size of the group determines the probability with which a coalitional best response, if available, can be adopted. Effectively, only coalitional best responses of small groups are adopted. Large groups can change their behavior only through individual best responses, mistakes, and, possibly, coalitional best responses of smaller sub-coalitions. We call the modification *small group dynamics,* since only sufficiently small groups of players benefit from it.

If the same action is payoff- and risk-dominant, then, we show, it is chosen by the small group dynamics in (essentially) any network. It turns out that the reason for this is exactly the same as the reason why the KMR dynamics select the risk-dominant action. However, when one action is risk-dominant and the other payoff dominant, then there is a tension between the KMR and small group dynamics. On one hand, the KMR dynamics push towards the risk-dominant action. On the other, coalitional best responses help small groups to coordinate on the efficient action. Which one of two effects prevails, depends on the network, leading to our network dependent results.

Before we go on, a short explanation of the relationship between the form of the network and coalitional best responses is in order. A coalitional best response is attractive for players because it allows them to coordinate their behavior with others. However, not all coalitions have a reason to coordinate. Coalitions are formed at random from all the members of the population. The members of the coalition may or may not be connected in the network. For example, consider a coalition in which all members are not connected, and hence do not interact with any other members in a payoff related way. This means that the behavior of any member of the coalition does not affect anyone else's incentives in the coalition. If a coalitional best response exists, then the change in the behavior it prescribes must already be an individual best response for each member. This is why the network on which players are located matters. Depending on the way that interactions are arranged, coalitions with a proper coalitional best response (that is not a consequence of individual best responses) may be more or less frequent.

We consider the following three networks:

(a) *Network of small groups* – the population is divided into many small groups of equal size. All players interact with all members of their own group and with nobody outside;

(b) *Random network* – players interact with a finite number of neighbors which are uniformly spread across the whole population. The connections are realizations in the random graph model [Bollobas 2001];

(c) *Large groups* – the population is divided into a few large groups. Players interact only with members of their own group and their neighbors are uniformly spread across the group, in the random graph model used above.

In the network of small groups, the payoff dominant action is a coalitional best response for each group, no matter what the rest of society is doing. Our dynamics lead very quickly to the payoff dominant selection. This is an immediate consequence of the main assumption. The situation is different in the random network (and also in the network of large groups). There, players' interactions are spread uniformly across the population and players' payoffs depend on the average action played by the whole population. Small coalitions do not affect the population average. A small coalition has a coalitional best response only when the action is already an individual best response. Hence, any coalitional best response can be replaced by a sequence of individual best responses and the dynamics of the population occur mostly through two individual elements of the benchmark KMR process. The KMR and the small group dynamics select the risk-dominant action in the long-run.

Till now, the payoff in each interaction of any two players was given by the same payoff function. In the second part, we extend the basic model to allow for the interplay between payoffs and social structure. We assume that there is more than one type of interaction. For example, individuals i and j may be connected with one type of interaction, but not with the other. The payoff depends on the behavior of both interacting players *and* also on the type of the interaction.

An illustrative example in this category is a *network selection game*. A player has to choose the network on which he would like to be active, from two possible choices A and B. These networks correspond to the types of interactions discussed above. He knows his place on each network and who his neighbours on each network will be. The payoff from each network is proportional to the number of his neighbors in that network who make the same decision. We say that a given network is payoff dominant if all the players would prefer everybody to choose this network to any other outcome. As a benchmark, we show that the KMR dynamics always selects the payoff dominant network (it turns out that in network selection games, payoff and risk dominance coincide). Introducing small group dynamics, on the contrary, may lead to a unique choice of a network which is payoff (and risk) dominated by the alternative.

To see how introducing small group dynamics leads to different results, consider the following example. The networks that a player has to choose between are the following: network A is a random network and B is a network of small groups. Introducing small group dynamics may select network B, even if A is payoff and risk dominant (unless the payoff and risk dominance of A are too strong). To find the long run outcome, we need to compare different ways in which the evolutionary process moves between both coordinations. We claim that transitions towards network A are purely individual: best responses or mistakes. Notice that the small coalitions are effective only when they can change the payoff for members of the coalition. However, transitions in network A are spread uniformly and the payoff from choosing this network depends on the average in the whole population. A small coalition cannot substantially affect the average. Hence, small coalitional best responses towards network A occur only when this network is already an individual best response. On the other hand, there are many coalitional transitions towards B. The payoff from choosing the small group network depends only on the behavior of the members of one's own small group. Even when choosing to live in the small group network is not an individual best response, it may be a coalitional best response for the whole group, precisely because all the members of the group are affected by the group action. In the end, both elements of the KMR process, individual best responses and mistakes, push the population towards the risk-dominant action, choosing network A. Coalitional best responses push the system towards coordination on the small group network. Unless the risk-dominance of A is not too strong, the latter effect dominates and the small group dynamics chooses network B.

This example shows that society is likely to divide into small groups even when such a division is risk and payoff dominated. However, not all divisions are equally possible. Suppose that interactions in A are still given by the random network and let B be a network of large groups. The internal structure of a large group is given by uniformly spread interactions and the payoff in network B depends on the average behavior of large group members. In this case, small group dynamics always choose whichever network is payoff dominant. This is in contrast to the previous example: Since now network B consists of large entities, small coalitions cannot affect the payoff from choosing network B. In this case, small group dynamics is dominated by the elements of the benchmark KMR process, which leads to coordination on the risk dominant outcome.

1.2. **Interpretation.** In the last section, we present two applications: self-segregation and polarization. In both cases the members of the community choose between two actions: one, which opens the community to the rest of society, and the other, which concentrates their activity within the community. We show that the network of interactions inside the community may affect the equilibrium choice.

In the self-segregation case, players choose between investing their time in education or developing a cultural identity. The payoff from education depends one one's own investment and the average investment in the community. The payoff from the cultural identity action depends on the number of interacting partners one has that also choose cultural identity. When interactions in the community are spread uniformly, then the community coordinates on whichever action is risk-dominant. When they form small groups, the community may coordinate on developing a cultural identity, even if it is risk- and payoff-dominated by education.

In the polarization case, players belong to one of two large groups. They choose whether they should interact with all their neighbors, or only with their neighbors who belong to the same large group. We argue that the internal structure of the group may affect the probability of polarization. If the interactions inside a large group are spread uniformly across the group, polarization is less probable, than if the interactions form small isolated groups.

1.3. **Related literature.** The central assumption of the paper, that small groups find it easier to overcome coordination problems, has long been recognized by the experimental literature. For example, in a coordination experiment of [Van Huyck, Battalio, and Beil 1990] subjects paired in repeated interactions quickly learn to play the payoff dominant equilibrium. However, when the same subjects interact with a large group of players (14 players in the experiment), they tend to move quickly to the risk dominant (and payoff-dominated) outcome. This finding is further confirmed by [Berninghaus, Ehrhart, and Keser

2002], who analyze different interaction structures. In a group of 3 players, the payoff dominant equilibrium prevails. When the interaction structure is that of a circle and players interact only with their neighbors, the risk dominant equilibrium arises more frequently.[1]

The model of the paper is built on the evolutionary model of KMR (see also [Young 1993]). There are three main assumptions in that model: myopia (players do not understand the dynamics), inertia (players act assuming that others stand by status quo) and experimentation (players make mistakes with a small probability). This paper modifies the assumption of inertia: we assume that coalitions of players can take each others behavior into account while computing one's payoff.

In the KMR, interactions are global. Local interaction, where players reside on the nodes of a network, was introduced in [Ellison 1993]. [Ellison 1993] and [Ellison 2000] prove that the risk dominant convention is selected on certain networks (like a circle or torus). This is further generalized in [Peski 2003], who shows that the risk dominant action is selected on any network satisfying a local density condition: the number of neighbors every player has must be either even or higher than a payoff-related (in particular, independent from the size of the population) constant. The risk-dominant selection on every network is also shown in [Blume 1993] and [Young 1998] in the context of different dynamics. Hence, the network does not affect the equilibrium selection. (Although [Ellison 1993] points out that it may substantially affect the speed with which the population coordinates.)

A rich literature shows that the payoff rather than the risk dominant selection occurs when the KMR dynamics are modified. [Ely 2002] shows that when people interact with other players at a given location *and* are able to choose a location, then the efficient outcome prevails. In a similar vein, [Canals and Vega-Redondo 1998] obtain the efficient selection when the evolutionary process acts on two levels: one, choosing individuals with the best survival rate, and on the other, choosing the most successful population. In [Robson and Vega Redondo 1996] the frequency with which players adjust their actions and match with another players is equalized, again leading to the efficient selection.

[1]One explaination for this phenomenon is that players in repeated interactions with a small number of opponents are involved in some sophisticated learning behavior (see [Fudenberg and Levine. 1998]). For example, players may signal their willingness to play the payoff-dominant action, in order to induce a proper response from their opponents. [Cooper, Dejong, Forsythe, and Ross 1992] suggests that preplay communication helps players to achieve the payoff-dominant equilibrium. If we assume that only a small groups of players may communicate successfully, it leads to small group ability to coordinate.

In this paper we remain agnostic about the reasons why small but not large groups can coordinate. We simply assume that it is the case and analyze its consequences. (However, it seems that the communication argument fits nicely with examples in the section 6.)

[Eshel, Samuelson, and Shaked 1998] present an evolutionary process, which, like this paper, has network-dependent predictions. Suppose that players choose actions in a Prisoner's Dilemma. If they were to take best responses, then only "defect" would survive. The authors assume that players imitate the behavior of the most successful player in the population rather than play the best-response. By itself, imitation is not sufficient for the selection of "cooperate" in the long-run. It is however sufficient when players live on a circle (as in [Ellison 1993]). Then, "cooperate" played by a group of players brings them the highest possible payoff and their success is further imitated by other players.

The idea of considering a coalitional deviation in the definition of equilibrium began with [Aumann 1959] and was also studied by others, including [Bernheim, Peleg, and Whinston 1987]. The efficient coordination is the only strong and the only coalition-proof equilibrium in our model.

1.4. **Structure of the paper.** The paper proceeds as follows. The next section introduces small group dynamics and discusses some of its features. Section 3 analyzes small group dynamics in symmetric games on networks. Section 4 presents results on network selection games. Section 5 combines and extends some of the results of both previous sections in one model. Section 6 interprets the model in the context of self-segregation and polarization.

2. Model and small-group dynamics

This section contains three parts: model of the coordination game, definition of the dynamics and description of networks used further in the paper.

2.1. **Model.** Each player in a population of size N chooses an action from a two-element set $S = \{a, b\}$. The action chosen is used in all interactions with other members of population. Denote the vector of choices of all players by $\eta \in \Sigma = S^N$; the set Σ is called the space of population states. The payoff of agent i is a function of actions used by all agents $-$ we represent it as function $u_i : \Sigma \to R$. We also use $u_i(x, \eta_{-i})$ to denote payoff of agent i when she plays x and the rest of the population plays according to η_{-i}. We say that action x is a *best response* for player i in state η, if $u_i(x, \eta_{-i}) \geq u_i(y, \eta_{-i})$, where $y \in S$ is the other action, $y \neq x$. We say that state η is a *steady state* (of the best response dynamics) if $\eta(i)$ is strict best response in state η for any agent i.

The model is specified in two settings. In the first one, we assume that there is a *single interaction.*[2] We assume that players are located on network $g_{ij} \in \{0, 1\}$, where $g_{ij} = g_{ji}$ denotes a connection (or lack of it) between players i and j. The payoff in each interaction is

[2]This is the standard case considered in the literature ([Ellison 1993], [Ellison 2000], [Lee, Szeidl, and Valentinyi 2003])

given by a payoff function in a symmetric coordination game, $u : S \times S \to R$, $u(a,a) \geq u(b,a)$ and $u(b,b) \geq u(a,b)$. Then the payoff in state η is equal to

$$u_i(\eta) = \sum_{j \neq i} g_{ij} u(\eta(i), \eta(j)).$$

The second case is of *multiple interactions*. There are potentially many types of interactions in the real world. The same behavior used in different interactions bears different consequences. For example, whether one speaks with slang may be treated differently depending on whether one is at business meeting or meeting with friends; it may be difficult to adjust the way of speaking to the circumstances. As another example, the clothes one puts on in the morning are going to be worn on a bus to a workplace, at the workplace, at a parents' meeting in a school and at a family dinner. In each of these circumstances, one's clothes will be perceived and judged differently.[3]

Suppose that there are two different interactions, each one associated with different payoff functions $u^k : S \times S \to R$, where $k = 1, 2$ (all functions u^k are payoff functions in some symmetric coordination games). Player uses the same action in all types of interaction. The interactions of type k form a network g^k. Then payoff from all types interactions in state η is equal to

$$u_i(\eta) = \sum_{j \neq i} g^1_{ij} u^1(\eta(i), \eta(j)) + \sum_{j \neq i} g^2_{ij} u^2(\eta(i), \eta(j)). \tag{2.1}$$

For both $x \in S$, a state $\mathbf{x} \in \Sigma$, such that $\mathbf{x}(i) = x$ for every agent i is called a *convention* $\mathbf{x}$. Conventions are steady states of best response dynamics.

We say that action x is *payoff dominant* iff convention $\mathbf{x}$ yields the highest possible payoff to all players: for any state η, any player i, $u_i(\eta) \leq u_i(\mathbf{x})$. We say that action x is *risk dominant* if it is a strict best-response to all other players mixing $\frac{1}{2} - \frac{1}{2}$ between both actions: Define a state $\mathbf{m} \in \Sigma$, such that for all i, $\mathbf{m}(i) = \frac{1}{2}a + \frac{1}{2}b$. Then, x is risk-dominant if for all players i, action $y \neq x$

$$u_i(x, \mathbf{m}) > u_i(y, \mathbf{m}).$$

This definition extends the standard definition of risk-dominance from simple 2-players coordination games.[4]

[3]As another example, consider wearing beards in Russia after Peter the Great issued laws against them. The beards were regarded by peers and lower folk as a sign of status and power. However, when a nobleman ventured to participate in State activities (went to public office, Court in Moscow), he was taxed and maybe even worse.

[4]This extends the definition of [Harsanyi and Selten 1988] to multiple interactions model.

2.2. **Dynamics.** The stochastic dynamics analyzed in this paper has three elements:

- *individual best responses*: each period an individual is drawn with a positive probability to change his strategy into the best-response,
- *individual mistakes*: after possibly updating his strategy, the individual with probability $\varepsilon > 0$ makes a mistake and switches his strategy into the other action,
- *coalitional best responses:* after all these, a coalition of players is drawn randomly (each coalition has a positive probability). Suppose that there is a coalitional best response - action, which if taken jointly, would improve the situation of all members of the coalition. We assume that the coalition switches to this action with probability $\varepsilon^{f(n)}$, where n is a size of the coalition and $f(.)$ is a convex and increasing function, such that $f(1) = 0$ and $\lim_{n\to\infty} \frac{f(n)}{n} = \infty$.[5]

We call this process as a *small group dynamics.* Since individual best responses occur with positive probability, small group dynamics is a stochastic approximation to the best-response dynamics: Any long-run equilibrium outcome must be also steady state of the best response dynamics.

Without the third element, this is exactly the dynamics of KMR. In particular, when $f(2) \geq 2$, then the probability of a two-player coalitional best response is dominated by the probability of two consecutive individual mistakes. In such case, our dynamics are equivalent to KMR.

The exact shape of $f(.)$ determines the relative frequency of coalitional best responses versus individual best responses and individual mistakes. Since it is increasing, the larger the coalition, the less likely the coalitional best responses is to occur. Very large coalitional best responses are highly improbable: Define constant

$$n_f = \max(n : f(n) < n) \tag{2.2}$$

- assumption $\lim_{n\to\infty} \frac{f(n)}{n} = \infty$ guarantees that n_f is well-defined and finite. Any coalitional best response involving $n > n_f$ players is less probable than if all participants changed their behavior through a sequence of consecutive individual mistakes, $\varepsilon^{f(n)} << \varepsilon^n$. This justifies the name of the dynamics: only small groups of players are able to efficiently take part in collective changes. Large groups change their behavior through the stochastic drift

[5]One can be much more flexible with the description of the dynamics: players may make mistakes when joining coalitions, different members of a coalition may switch into different actions in the same time and so on. None of this changes any of the results.

Notice also that in the class of games considered in this paper, single and multiple interactions, all coalitional best responses towards the same action are resistant to subcoalitional deviations. Precisely, if an action is a coalitional best response for a group of players, it is also a coalitional best response for any of its subgroup, given that the rest of the coalition is going to play this action.

of individual best responses, mistakes and, possibly, coalitional best responses of smaller subcoalitions.

Formally, the small group dynamics is given by a sequence of Markov and ergodic transition probabilities p^{ε}, where $p^{\varepsilon}(\eta_1, \eta_2)$ denotes the probability that the process will reach state η_2 in the next period given that in this period it is in state η_1. For each $\varepsilon > 0$, there is an unique ergodic stationary distribution $\mu^{\varepsilon} \in \Delta S$, which describes the long-run probabilities of different states. We say that *convention* $\mathbf{x}$ *is played in the long-run*, if the probability of state $\mathbf{x}$ converges to 1 when $\varepsilon \to 0$:

$$\lim_{\varepsilon \to 0} \mu^{\varepsilon}(\mathbf{x}) = 1.$$

In some cases we are able to show only that most of players coordinate on certain action. For any $\gamma > 0$, we say that *γ-neighborhood of convention* $\mathbf{x}$ *is played in the long-run,* if the probability of all states with at least $(1-\gamma)N$ players playing action x converges to 1 when $\varepsilon \to 0$:

$$\lim_{\varepsilon \to 0} \mu^{\varepsilon}(\{\eta : \#\{i : \eta(i) \neq x\} \leq \gamma N\}) = 1.$$

2.3. **Networks.** Two results in this paper are stated for all possible networks (Theorem 1 and Proposition 5). The other results are stated for particular networks - network of small groups, random network and large groups.

All three networks are parametrized with a single constant d. This is the average number of neighbors of any player. For player i, define the number of i's neighbors of player i as $d_i = \sum_{j \neq i} g_{ij}$. In the network of small groups, for any player, $d_i = d$. In two other networks, d_i is a random variable with an average equal to d. We choose such a parametrization, because this allows us for a a meaningful comparison between networks. (The total payoff from interactions increases with the number of one's neighbors, so naturally networks with a higher number of neighbors would lead to higher payoffs.)[6]

2.3.1. *Small groups.* Suppose that a population is divided into $N/(d+1)$ groups, where i and i belong to the same group if $\lfloor \frac{i}{d+1} \rfloor = \lfloor \frac{j}{d+1} \rfloor$. Players i and j are connected, $g_{ij} = 1$, if and only if they belong to the same group. In other words, each group is a small global interaction network of [Young 1993]) without any connections outside the group.

The examples of small group networks include: friend groups or cliques in high schools (nerds, losers, jocks, popular crowd and so on); alumni networks, which connect people who graduated from the same college; network of neighborhoods, which connect inhabitants

[6]None of the results changes if players are allowed to have different number of neighbors in different networks - proper scaling of payoffs might be necessary.

of easy to define inner city localities and civil society, which consists of numerous small organizations.

2.3.2. *Random network.* Small closed groups is an extremal way in which we can describe social interactions. This is because all neighbors of a player are also the only neighbors of each other. In the opposite extremum, interactions are spread across the whole population and player's neighbors do not overlap with neighbors of his neighbors. For example, sociologists analyze networks of acquaintances with people connected through a relation "I am personally acquainted with". An average person has around 500 acquaintances and these are widely spread across the population. In particular, according to a famous estimate, on average only 5 people separate me from any other randomly chosen person in the US ([Morris 2000] calls such a phenomenon as a exponential neighborhood growth). The widely spread interactions can be also found in Internet: for instance, consider a network of sites connected with a relation "contains a link to".

Interactions in a global interaction model, where all the players are connected with each other, are trivially spread uniformly across the population. Such a model however assumes that the number of neighbors of a player is equal to the size of the population. When the population is large, such an assumption becomes unrealistic and a different model is needed.

A *random network* is a very simple probabilistic model which on one hand has, approximately, uniformly distributed interactions, and, on the other hand, players have a limited number of neighbors.[7] Suppose that the network of interactions is chosen before players start to play a game. We assume that each pair of players gets connected with probability $\frac{d}{N}$ and all connections are chosen independently. The expected number of neighbors of any player is equal

$$\frac{d}{N}(N-1) \approx d,$$

which for large N does not depend on N. This probabilistic model is denoted with $G\left(N, \frac{d}{N}\right)$.[8]

We are interested in the probability that a random network realization has a particular property. Consider a property Q_N of a network with N players. We say that property Q_N occurs *in almost any network*, if

$$\lim_{N\to\infty} P_{G\left(N,\frac{d}{N}\right)}(\text{network with } N \text{ nodes has } Q_N) = 1$$

[7]All the results in this paper hold when we replace the random network with a global interaction network.

[8]This is a classic model in the theory of random graphs as developed by Erdos, Renyi - see [Bollobas 2001]. Random graphs in game theory were studied in [Kirman, Oddou, and Weber 1986] in a model of coalition formation through a faulty communication.

where the probability is calculated according to model $G\left(N, \frac{d}{N}\right)$.[9]

For example, we may define the property that the interactions are spread approximately uniformly across the whole population. Take any subset of players $U \subseteq \{1, ..., N\}$. If interactions are distributed uniformly, then almost every player i has approximately the same proportion of neighbors in set U, specifically equal to $\frac{|U|}{N}d$. To formalize this, for any $\delta > 0$, define

$$Q_N^\delta : \text{ For any subset of players } U \subseteq \{1, ..., N\}, \text{ there is at most } \delta N \text{ players}$$
$$\text{who have less than } \left(\frac{|U|}{N} - \delta\right) d \text{ or more than } \left(\frac{|U|}{N} + \delta\right) d \text{ neighbors in set } U.$$

We show in the appendix (part A.5), that for any $\delta > 0$, there is d_0, such that for any $d \geq d_0$, $\lim_{N\to\infty} P_{G\left(N,\frac{d}{N}\right)}\left(Q_N^\delta\right) = 1$. Hence, in almost any network, for any subset of players U, most of the players have approximately $\frac{|U|}{N}d$ neighbors in set U.

The random network is a probabilistic model and the uniform distribution holds only approximately, for sufficiently high d and when $N \to \infty$. For any finite N, any network has a positive probability - for example, network of small groups, global interaction network and a network without any connections between players. When N is large such networks become highly improbable and most players have uniformly spread connections. We say that they have a "typical" structure of interactions. Even when N is high, almost all networks have a fringe of players with "atypical" structure of connections. In particular, in almost any network there is an isolated group of d players, such that its members are connected with each other but not with anybody else.

One might consider models of uniformly spread interactions other than the random network. For example, one could require that every player in the network would have exactly the same number of neighbors and exactly uniformly distributed interactions. Such models, potentially deterministic, would not allow for any fringe. However, the definition of the random network is much simpler than the proper definition of the other models. Also, it seems that the existence of a fringe is not a drawback of the model. In real life, some people have more connections (some websites are linked to more websites) than others. Some people live lives of hermits and some other want to know as much people as possible. The majority, however, have similar, "typical" structure of interactions.[10]

[9]Note that the phrase "almost any" is used here in a different meaning than in measure theory. In particular, for finte N, any network has a positive probability and "for almost any" in the measure theoretic sense would mean "for any".

[10]The recent literature argues that some statistical properties of real-life networks are poorly reproduced by the random graph model of Erdos-Renyi. For example, the real life networks have usually much thicker tales in the distribution of the number of neighbors. The literature proposes another probabilistic model as

2.3.3. *Large groups.* In a small group network, players are divided into many small isolated entities without any connections between groups. One may be also interested in a network, where (a) players are divided into few large groups without connections between the groups and (b) interactions within a group are spread uniformly. For example, consider a network of acquaintances in two large cities. It is possible that some people have acquaintances in the other city, however most connections are contained inside each city. As another example, consider a network of websites which are designed for speakers of different languages. Sites in one language point most often to other sites in the same language and only very rarely to sites with a different language.

We assume that there are K large groups of equal size $\frac{1}{K}N$ each. Connections are restricted to one's own large group, no player is connected with any member of a different group. The internal structure of each large group looks like a random network. Formally, connections in group $k = 1, ..., K$ are realizations of a model $G\left(\frac{1}{K}N, \frac{Kd}{N}\right)$, where d is the average number of neighbors. Each realization, $g^1, ..., g^K$ is drawn independently.

3. Single interaction

Suppose that players participate in interactions of single type. The payoffs from each interaction are given by a function $u(.,.)$. We assume that action a is risk dominant - denote a measure of risk-dominance of action a as

$$\rho = \frac{u(b,b) - u(a,b)}{u(a,a) + u(b,b) - u(a,b) - u(b,a)}. \tag{3.1}$$

Parameter ρ says what proportion of neighbors playing a makes a player indifferent between both actions. Thus, if a is risk dominant, then $\rho < \frac{1}{2}$.

We begin with a general equilibrium selection result. Define the constant

$$d_{\min} = \frac{u(a,a) + u(b,b) - u(a,b) - u(b,a)}{\min\left(2\left(u(a,a) - u(b,b)\right), u(a,a) + u(a,b) - u(b,b) - u(b,a)\right)}. \tag{3.2}$$

Theorem 1. *Suppose that network of interactions is such that every player has at least $d_{\min}$ neighbors: for every i*

$$d_i = \sum_{j \neq i} g_{ij} \geq d_{\min}.$$

If action a is risk- and payoff dominant, then, convention **a** *is played in the long-run.*[11]

---

a better candidate - a *scale-free* network. For a discussion of properties of real-life networks and probabilistic models, see [Albert and Barabási 2002].

[11] The proof is sketched in the appendix. Under weaker conditions on network (number of neighbors of any player has to be either even or bigger than $d'_{\min} = \frac{u(a,a)+u(b,b)-u(a,b)-u(b,a)}{u(a,a)+u(a,b)-u(b,b)-u(b,a)}$), [Peski 2003] shows that KMR dynamics, i.e. with no coalitional deviations, always selects the risk-dominant outcome (regardless whether it is efficient or not).

If the same action is payoff and risk dominant, it is going to be the unique outcome in the long-run, regardless of the network. When action a is risk dominant and action b is payoff dominant, then a tension appears. Next, we present next two examples, network of small groups and random network, where, for large populations, small group dynamics chooses different conventions.

3.1. **Global interaction.** Before we analyze many small groups, it is easier to start with one group. Consider a global interaction network where all players are connected with each other. There are only two steady states, conventions $\mathbf{a}$ and $\mathbf{b}$. For any other state $\eta \in \Sigma$, there is a best response path leading to one of these two conventions. We have a simple result:

Proposition 1. *Suppose that action a is risk dominant and action b is efficient. There are: a threshold N_0 and a constant $c > 0$, such that, when $N \leq N_0$, only $\mathbf{b}$ is played in the long-run and for $N \geq N_0 + c$, only $\mathbf{a}$ is played in the long-run.*[12]

Hence, small groups tend to coordinate on the payoff-dominant and large groups on the risk-dominant action. Moreover, there is a threshold which divides small from large groups. In what follows, we present a brief argument why there is a qualitative difference in the behavior of large and small groups. The details and the reasoning why the threshold arises is left for the appendix B.

The baseline argument is standard. One may associate transitions between different conventions with a cost: the number of mistakes necessary to reach the basin of attraction of a convention, where the basin of attraction is a set of states, from which given convention can be reached through a path of individual best responses. These costs can be compared. If cost of transition $\mathbf{b} \rightarrow \mathbf{a}$ is smaller than $\mathbf{a} \rightarrow \mathbf{b}$, then it is more difficult to reach state $\mathbf{b}$ than $\mathbf{a}$ and convention $\mathbf{a}$ is a more probable outcome.

Suppose first that the size of the population, N, is so small that $f(N) < 1$ (recall that $f(.)$ is increasing and $f(1) = 0$). The payoff dominant action is always a coalitional best response for the grand coalition. The probability that it occurs is not smaller than $\varepsilon^{f(N)}$. Thus, the cost of transition $\mathbf{a} \rightarrow \mathbf{b}$ is not higher than $f(N)$. On the other hand, one needs at least one mistake to reach the basin of attraction of $\mathbf{a}$ starting from $\mathbf{b}$. The cost of transition $\mathbf{b} \rightarrow \mathbf{a}$ is not lower than 1. Since $f(N) < 1$, transition $\mathbf{a} \rightarrow \mathbf{b}$ is more probable and convention $\mathbf{b}$ is a unique stochastically stable outcome.

[12] The treshold depends on the shape of $f(.)$ and payoff function $u(.,.)$ and the constant depends only on the payoff function.

Suppose now that the population is large. The cost of transition $\mathbf{b} \to \mathbf{a}$ is not higher than ρN, where ρ is a measure of risk-dominance of action a defined in (3.1). When ρN players commit mistakes, then the remaining players prefer weakly to play a rather than b.

Consider now transition $\mathbf{a} \to \mathbf{b}$. The basin of attraction of $\mathbf{b}$ contains all the states with at least $(1-\rho) N$ agents playing b. It can be reached through a sequence of individual mistakes and coalitional best responses. Notice, however, that there can be only one coalitional best response, exactly at the end of the sequence. This is because if a group of player prefers jointly to switch to b tather than remain by status quo, then action b must be individual best response of all players in the new state. Since b is an individual best response for a player only when at least $(1-\rho) N$ other players play b (this is a property of global interaction network), the new state must be in the basin of attraction of $\mathbf{b}$. Suppose that the number of initial mistakes is equal to n_m and the size of the coalition in the final coalitional best response is equal to n_c. Then the cost of transition $\mathbf{a} \to \mathbf{b}$ cannot be lower than

$$\min_{n_m + n_c \geq (1-\rho)N} n_m + f(n_c).$$

Using the fact that $\lim_{n\to\infty} \frac{f(n)}{n} = \infty$, one can show that this expression is relatively close to $(1-\rho) N$ when N is large. Since $\rho < \frac{1}{2}$, this is higher than ρN and, for large N, risk-dominant convention $\mathbf{a}$ prevails.

3.2. **Network of small groups.** We have a simple corollary to the Proposition:

Corollary 1. *When player's interactions are given by the small group network and $f(d) < 1$, then the efficient action is chosen in the long-run.*

The only difference between small groups network and global interaction is that coalitions in the former can include members of different groups. This is however immaterial: members of different groups are not connected with each other, so their presence in the coalition is not affecting the incentives of other players. Coalitional best response of one bigger coalition consisting of members of different groups can be divided into two smaller coalitions, each one consisting of members of only one group. Since function $f(n)$ is convex, the probability of the coalitional best response of the big coalition is not higher then the probability of two consecutive smaller coalitional best responses. If n_1 is the number of members of one group and n_2 is the number of members of the other group, then $\varepsilon^{f(n_1+n_2)} \leq \varepsilon^{f(n_1)}\varepsilon^{f(n_2)}$.

3.3. **Random network.** Next, we assume that interactions are given by a realization in the random network model. We show, that in almost any network, most of the players play risk-dominant action in the long-run.

Proposition 2. *Suppose that action a is risk dominant. For any $\gamma > 0$, there is d_0, such that for any $d \geq d_0$, in almost any network, γ-neighborhood of convention* **a** *is played in the long-run.*

The Proposition says that in almost all networks most of the players play most of the time the risk dominant action a. In particular, it is not true that all the players play risk-dominant action. Recall that almost any random network has a fringe of players with "atypical" connections (section 2.3.2). Some members of the fringe tend to play the payoff dominant action. For example, there is almost always a group of d players who are connected with each other, but nobody else. The long-run behavior of such a group can be analyzed separately from the behavior of the rest. By Proposition 1, when d is sufficiently small, its members play the payoff dominant action.

The choice of high d guarantees that that the fringe is sufficiently small. Players outside the fringe choose the risk-dominant action.

As we argued above, the interactions in the random network are spread approximately uniformly across the population. This means, that the average player best responds to the average action played in the population. Approximately, the same happens in the global interaction network. If the approximation is good, for large N, the risk-dominant coordination is chosen. The proof of the Proposition (appendix C.2) shows that the approximation is sufficiently good when N is large and d sufficiently high.

4. Network selection games

In this section, we start analysis of the case with two types of interactions. We focus on a special class of games, where players decide in which network they want to participate. Suppose that there are two networks of interactions: network A described by indices $g_{ij}^A \in \{0,1\}$, and network B with indices $g_{ij}^B \in \{0,1\}$. We assume that the average number of neighbors in both networks is approximately equal to d

$$\frac{1}{N}\sum_i d_i^X \approx d,$$

for both $X = A$,B, where d_i^X is the number of neighbors of player i in a network X.

The payoffs from both interactions are given by functions

$$u^A(.,.) = \begin{array}{c|cc} & a & b \\ \hline a & u^A & 0 \\ b & 0 & 0 \end{array} \qquad , u^B(.,.) = \begin{array}{c|cc} & a & b \\ \hline a & 0 & 0 \\ b & 0 & u^B \end{array} \quad , \tag{4.1}$$

where $u^A, u^B > 0$ are some constants. Given equation (2.1), the payoff from playing action x in state η can be written as

$$u_i(x,\eta) = u^X \sum_{i\neq j} g_{ij}^X \mathbf{1}_{\{\eta(j)=x\}}$$

- the payoff from x is equal to u^X times the number of player's neighbors in network X who also participate in this network.

The average payoff from coordination on network $X = A, B$ is equal on average to $u^X d$. We say that network A is *payoff dominant*, if, on average, players prefer convention **a** to convention **b**, $u^A > u^B$. We say that network A is *risk dominant* if the participation in network A is, on average, the best response when others randomize $\frac{1}{2} - \frac{1}{2}$. One may easily check that, for example, action a is a strict best response to others mixing equally if and only if $u^A > u^B$. Thus, in network selection games payoff- and risk-dominance are equivalent.

As a benchmark, we consider the KMR dynamics in network selection games. If the number of neighbors of each player in network $X = A, B$ is constant and equal always to d and d is sufficiently high (but does not need to grow with the size of the population), then the KMR dynamics select the risk dominant (hence also payoff dominant) outcome. (This follows directly from a result in [Peski 2003].)

We discuss below two examples of network selection. In the first example, players choose between the random network and the small group network. We show that small group dynamics favor the latter: in the long run, coordination on small group network may occur even when it is payoff- and risk-dominated. In the second example, we confront the random network with large groups. The division into large groups is not favored by the evolutionary dynamics, i.e. occurs only when it is risk dominant (hence also efficient). In the last part, we put a tight bound on the maximal possible inefficiency that can arise in the network selection games.

In this and next sections, we restrict attention to the special case of small group dynamics:

Assumption 1. $f(d) = 0$.

The assumption says that all coalitional best responses of size smaller than d occur with non-disappearing probability when $\varepsilon \to 0$. The assumption is not necessary for the results, however it makes the proofs much shorter.[13]

4.1. **Small groups in random network.** Let network A be a realization from the random network $G\left(N, \frac{d}{N}\right)$ (section 2.3.2). The choice of action a is interpreted as a decision to participate in broad society, where interactions are spread uniformly across the population. Network B is a network of small groups of size $d+1$. Action b divides the population into

[13] What is necessary, is that $\frac{f(d)}{d}$ is sufficiently small.

small isolated entities. The choice of b is interpreted as a decision to step out of society and focus one's activity in one's own neighborhood.

We say that a group is *mobilized* when all its members choose participation in a group network B. We say that a group is *passive* when all its members choose network A. It turns out that in (almost) any steady state, small groups are either mobilized or passive. Indeed, all members of an average group have (almost) the same incentives. Suppose that proportion α of population chooses action a and proportion β of some group plays B. Then, on average, each player can obtain payoff $\alpha u^A d$ from playing a and each member of this group obtains $\beta u^B d$ from playing b. If $\beta u^A d > \alpha u^A d$, then each member of the group prefers to play b, if not, then each one prefers to choose a. Hence, in equilibrium, either $\beta = 0$ or $\beta = 1$.

We assume that the participation in broad society, network A, is socially efficient, $u^A > u^B$. Thus, convention **a** (coordination on the network random network) is preferred by (almost) all the players to convention **b** (coordination on small groups). As discussed above, the payoff dominant convention is also risk dominant.

Proposition 3. *Suppose that*

$$\frac{u^A}{u^B} < 1 + e^{-1} \approx 1.368.$$

For any $\gamma > 0$, sufficiently large d, if assumption 1 holds, then in almost any network A, γ-neighborhood of convention **b** *is chosen in the long-run.*[14]

The Proposition says that if $u^A < (1 + e^{-1})\, u^B$ and population is large, then with large probability huge majority of players coordinate on small group network B. And it happens despite the random network is efficient and risk dominant. Constant $1 + e^{-1}$ is tight, i.e. one can prove that for $\frac{u^A}{u^B} > 1 + e^{-1}$ convention **a** becomes a long-run equilibrium.

We sketch the argument why an inefficient coordination may arise. Suppose that $u^A = (1 + \delta)\, u^B$ for some $0 < \delta < 0.05$. (The proof in appendix C.3 compares the costs of transitions for any $\delta < e^{-1}$.) Take first transition $\mathbf{a} \to \mathbf{b}$. One needs at most $\frac{\delta}{1+\delta} N$ individual mistakes to reach the basin of attraction of **b**. Suppose that a proportion of $\frac{\delta}{1+\delta}$ groups switch to playing b, potentially through individual mistakes. The payoff from playing a becomes equal to $\left(1 - \frac{\delta}{1+\delta}\right) u^A d = u^B d$. But this makes action b a (weak) coalitional best response for any small group. A path of coalitional best responses now leads the population to convention **b**.

More than δN mistakes are needed for transition $\mathbf{b} \to \mathbf{a}$. Suppose that less than δN players change their behavior and start to play a while the rest keep on playing b. The $a-$players

[14]Given the assumptions of the Proposition, it is not true that *all* the agents in the long-run play b. There is always a fringe of players who in the long-run play action a. It exists for the same reasons as in the random network analyzed in section 3.3.

can be distributed across different small groups in many ways. For example, there might be anomalous groups, which have more than $\sqrt{\delta}(d+1)$ members who switched to play a. However, at most a proportion of $\sqrt{\delta}$ of groups is anomalous; otherwise the total number of $a-$players in would be higher than $\sqrt{\delta}(d+1)\sqrt{\delta}\frac{N}{d+1}=\delta$. The number of a-players in normal groups is not higher than $\left(1-\sqrt{\delta}\right)\sqrt{\delta}N$. Even if a sequence of individual and coalitional best responses somehow led all members of anomalous groups to switch to a, this would not make the number of a-players in the population higher than $\left(1-\sqrt{\delta}\right)\sqrt{\delta}N+\sqrt{\delta}N\leq 2\sqrt{\delta}N$, where $\sqrt{\delta}N$ players come from anomalous groups. Thus, the payoff from playing a cannot be higher than $2\sqrt{\delta}u^{A}d$.

On the other hand, the payoff from playing b for any member of normal group is not lower than $\left(1-\sqrt{\delta}\right)u^{B}d$, since only at most $\sqrt{\delta}d$ members of a normal group play a. When $\delta<0.05$, action b remains a best response for any member of a normal group. After all members of normal groups switch back to b through a sequence of individual best responses, the remaining anomalous groups switch back to b through coalitional best responses. Since $\delta>\frac{\delta}{1+\delta}$, then convention **b** prevails in the long-run.

4.2. **Large groups in random network.** Here, we assume that network A is a realization in the random network $G\left(N,\frac{d}{N}\right)$ model and network B is a network of K large groups with the average number of neighbors equal to d (section 2.3.3).

Similarly to the above, we say that large group is mobilized, if all members of the group play action b and it is passive if all its members play a. In (almost) all steady states, large groups are either fully mobilized or passive. If proportion α of the whole society plays a and proportion β of one's group plays b, then any member of this group prefers to play a if $\alpha u^{A}d>\beta u^{B}d$ and prefers to play b if $\alpha u^{A}d<\beta u^{B}d$. Since any member of the group faces the same choice, their decisions must be the same and $\beta=0$ or $\beta=1$.

Proposition 4. *Suppose that $u^{A}>u^{B}$. For any $\gamma>0$, any K, sufficiently large d, if assumption 1 holds, in almost any network A, γ-neighborhood of convention* **a** *is chosen in the long-run. If $u^{A}<u^{B}$, then (given the same assumptions) γ-neighborhood of convention* **b** *is chosen in the long-run.*

The Proposition says that the efficient, risk dominant action is going to be chosen in the long-run. This is in contrast to the previous result, where the inefficient and risk-dominated action was played in the long-run.

The argument is fairly simple. Since the efficient outcome is also risk dominant, it is selected by the KMR dynamics. Small group dynamics introduce the possibility of coalitional best responses of small groups. However, proper coalitional best responses occur very rarely. The payoffs from both actions depend on the average behavior of the whole population or one's own large group, respectively. This is because interactions, both in the society and inside large groups, take the form of the random network. Small coalitions do not change the average, hence a player wants to participate in a coalitional best response only it is his individual best response.

4.3. **Maximal bounds on inefficiency.** We saw above, that small group dynamics may select the inefficient outcome. When players are choosing between the small group network and the random network, even if the efficient action yields 36% higher payoff than the inefficient one, the inefficient may be selected in the long-run. The inefficiency does not have to happen always. In the second example, inefficiency disappears. The goal of this section is to derive the maximal possible bounds on it. We show that the inefficiency cannot be too high. Specifically, when the payoff from the efficient convention is twice as high as the payoff from the inefficient one, then the former is chosen in the long-run.

Proposition 5. *Take any two networks A and B. Suppose that*

$$u^A > 2u^B$$

and the number of neighbors of each player is higher than $d \geq \frac{1}{u^A - 2u^B}$. Then, convention **a** *is chosen uniquely in the long-run.*

The proof of the Theorem is based on Theorem 2 in [Peski 2003]. To get some intuition, it is instructive to observe that this bound is tight. There are networks A and B, where $u^A \approx 2u^B$ and small group dynamics selects convention **b** in the long-run. The example illustrates phenomenon of small group contagion in a direct analogy to the contagion discussed in the literature [Morris 2000].

Suppose that $u^A = 1.98u^B$. Let the population be $\{0, 1, 2\} \times \{0, ..., N-1\}$ named with labels (z, n). Network A consists of three parallel circles: player (z, n) is connected with players $(z, n-1)$ and $(z, n+1)$ (modulo N, so that player $(z, 0)$ is connected with $(z, N-1)$ and $(z, 1)$). Network B is formed out of N triples: in each pair players $(0, n)$, $(1, n)$ and $(2, n)$ are connected. All players have two neighbors in both networks.

Only three individual mistakes are needed to reach the basin of attraction of **b** starting from convention **a**. Suppose that any triple switches to play action b. Then, each one of their neighbors gets a payoff $0.99u^B$ from playing action a (since only half of his respective neighbors are playing a). Action b becomes a coalitional best response for his triple - if all of them switch to play b, they will get a payoff of $u^B > 0.99u^B$. On the other hand, one needs to change behavior of at least $N-2$ players to get out of the basin of attraction of convention **b**. If $f(N-2) < 3$ (which must be true for N high enough) and $f(3) < 1$, then only action b survives in the long-run.

5. Two types of interactions: random network and small groups

Section 3 shows, in the case of single interaction, that small group dynamics exhibit a tension between the efficient and the risk dominant coordination. If interactions are formed into small groups, then the efficient convention is selected. If interactions are spread uniformly across the population, then the risk dominant behavior prevails. Since it is easier for small groups to coordinate on the efficient action, the natural conclusion would be that the small groups are good.

Such a conclusion is however drawn into question by the example of section 4.1. There, it is a network of small groups which leads to inefficiency. And this happens despite the fact that the random network is itself risk dominant. It seems that small groups find it easier to mobilize and play their "group action" rather than the efficient one.

The goal of this section is to reconcile these results in a general model with two types of interactions (equation (2.1)). Suppose that interactions A are realizations in the random network $G\left(N, \frac{d}{N}\right)$. Network B, g^B_{ij}, is a network of small groups of size $d+1$. The payoffs in both types of interactions are given by functions $u^X(.,.)$ for both $X = A, B$. Both functions $u^X(.,.)$ come from symmetric 2×2 coordination games. The payoff from action $x \in S$ in state $\eta \in \Sigma$ is equal to

$$u_i(x,\eta) = \tau \sum_{j \neq i} g^A_{ij} u^A(x, \eta(j)) + (1-\tau) \sum_{j \neq i} g^B_{ij} u^B(x, \eta(j)), \tag{5.1}$$

where parameter τ will allow us to measure relative impact of each network (the higher τ, the stronger is the random network vs. the network of small groups). For instance, the case of single interaction and the random network from section 3.3 is covered when $u^A(.,.) = u^B(.,.)$ and $\tau = 1$.

In two propositions, we characterize the long-run behavior of such a model.

Proposition 6. *Suppose that the following conditions are satisfied:*

1. *action a is risk dominant in the average game*:

$$\begin{aligned}&\tau\left(u^A(a,a) + u^A(a,b)\right) + (1-\tau)\left(u^B(a,a) + u^B(a,b)\right)\\ &> \tau\left(u^A(b,a) + u^A(b,b)\right) + (1-\tau)\left(u^B(b,a) + u^B(b,b)\right)\end{aligned}$$

2. *action a is risk dominant in network A* : $u^A(a,a) + u^A(a,b) > u^A(b,a) + u^A(b,b)$.
3. *action a is payoff dominant in network B* : $u^B(a,a) > u^B(b,b)$.

For any $\gamma > 0$, sufficiently large d, if assumption 1 holds, then in almost any network A, γ-neighborhood of convention **a** *is chosen in the long-run.*

The first condition says that action a is the best response to all the other players randomizing equally between both actions. In particular, it is essentially a sufficient and necessary condition for the equilibrium selection of action a by the KMR dynamics.

Whether two subsequent conditions are satisfied, depends on the structure of payoffs in each networks. Condition 2 says that action a is risk dominant in the random network and condition 3 requires action a to be efficient in the network of small groups. Both of them together clarify the asymmetry with which small group dynamics treats different networks. In section 3 we learned that small group network leads to the efficient outcome and the random network tends to select the risk dominant one. The Proposition further strengthens this prediction: action which is efficient in the first one *and* risk dominant in the second one is favored.

There are examples when each one of conditions (1)-(3) is not satisfied and small groups dynamics selects action b in the long-run.

The next Proposition offers comparative statics result on parameter τ.

Proposition 7. *Suppose that action a is risk dominant in both networks and action b is efficient in network B. Then there is $\tau_0 \in (0,1)$, such that:*

- *when $\tau > \tau_0$, given the same assumptions as in Proposition 6, γ-neighborhood of convention* **a** *is chosen in the long-run,*
- *when $\tau < \tau_0$, given the same assumptions as in Proposition 6, γ-neighborhood of convention* **b** *is chosen in the long-run.*

We interpret varying τ as changing the relative strength of the networks. When τ decreases, the network of small groups becomes stronger and the only possible regime change is from the risk dominant action to the efficient one in the groups. When τ increases, small groups become weaker and the regime change may move only in the opposite direction.

Notice that the welfare consequences of the regime change caused by moving τ depend on whether the action which is efficient in the small groups is also efficient in average game. If the other action is efficient in the average game, then strengthening civil society decreases welfare.

Proposition 7 has bite if conditions 1 and 2 of Proposition 5.1 push in the opposite direction to condition 3. If, for example, the same action is risk dominant in the random network and efficient in the small groups, the other action can be risk dominant in the average game, violating in this way condition 1 of Proposition 6. It is possible in such situation that for very weak and very strong small groups conditions 2 and 3 prevail, but for medium values of τ condition 1 determines the long-run outcome and the comparative statics result does not hold.

6. Interpretation

In this section, we present two applications. We show how networks of interactions may affect the probabilities of self-segregation and polarization.

6.1. Self-segregation.

An important question in studying segregation in the United States concerns sources of the gap in the academic performance between minorities and the majority. Even after controlling for a whole set of explanatory variables, race remains an important determinant of test scores. One hypothesis suggests that the main reason is cultural (see [Fryer 2003] and references therein). Historical circumstances led to the development of a specific attitude being adopted by black teenagers. Striving to high achievements is regarded as "acting white" and is punished by exclusion from the peer group [Austen-Smith and Fryer 2003]. The threat of exclusion causes an individual to invest in skills useful only inside the community, like developing a cultural identity, rather than skills which yield benefits outside, like an education.[15] [Akerlof and Kranton 2000] also study the economic implications of conforming to the group ideal rather than developing one's individual skills.

We present a simple model, in which players choose between an investment into a cultural identity or an education. Both choices induce externalities, however the externalities take different forms. We assume that the payoff from education is equal to the wage received in the future. The wage depends on the levels of education of an individual and of the whole population. This happens due to statistical discrimination: an employer receives an imperfect signal about a job candidate and evaluate this signal using his prior information. The prior depends on the average behavior in the population ([Arrow 1973], [Coate and Loury 1993]). On the other hand, the externality in the identity investment is local. We assume that all players interact on average with d neighbors and the payoff from choosing identity is proportional to the identity investment made by the neighbors.

We show that the structure of interactions inside the population, g_{ij}, may affect the equilibrium choice. Specifically, we consider two scenarios:

Scenario 1: Interactions are realizations in the random network $G\left(N, \frac{d}{N}\right)$.

Scenario 2: Interactions form a network of small groups of size $d+1$.

We show that the educational investment is more probable when the minority community does not have any internal structure (scenario 1). A pre-existing internal structure helps the minority to coordinate on the identity investment (scenario 2). It seems that the community is stronger, when it is already divided into small isolated groups.

Suppose that a minority member i has $e^* < 1$ units of time (or effort), which can be devoted to education $e_i \in \{0, e^*\}$ or invested into cultural identity, $c_i = e^* - e_i$. Define a

[15][Fryer 2003] gives an example of a Barcelonian choosing between learning to speak Catalan or learning software programming.

convention **e** as a state in which everybody invests in education and convention **c** as a state in which everybody invests in cultural identity.

The payoff from the identity investment depends on the behavior of i's neighbors. We assume that player i's payoff from cultural identity is proportional to the investment in cultural identity done by his neighbors:

$$u^C c_i \sum_{j \neq i} g_{ij} c_j.$$

Since in both scenarios the average number of interacting neighbors is equal to d, the payoff of an individual in convention **c** is equal to $u^C d \left(e^*\right)^2$.

The investment in education affects the probability that i becomes either high ability θ_H or low ability θ_L type. Precisely, suppose that the probability of becoming a high type is equal to e_i. The competitive risk-neutral employers receive profits w from the high ability types and 0 from the low ability types. They observe neither the educational investment nor the ability, however they observe an imperfect signal about the ability. During an interview, a candidate type θ_H makes a good impression with probability $p > \frac{1}{2}$ and a bad impression with probability $1 - p$. Analogously, type θ_L makes a bad impression with probability p and a good impression with probability $1 - p$.

Denote the average investment into education with $\bar{e} = \frac{1}{N} \sum_i e_i$. Then, the expected wage of an individual i is equal to

$$w\left(e_i, \bar{e}\right) = \frac{\left(2p-1\right)^2 w}{\bar{e}\left(2p-1\right) + 1 - p} e_i \bar{e} + \frac{2\left(1-p\right) pw}{\bar{e}\left(2p-1\right) + 1 - p} \bar{e}.$$

(The straightforward computations are omitted.) The more others invest in education, the higher the prior an employer has about one's quality. An increase in the average investment $\bar{e}$ raises one's incentive to invest in education.

The model presented in this section differs in two aspects from the core model analyzed in this paper. First, the payoff from the investment in education depends on actions of all the members of the population and the payoff from identity investment depends on the actions of d neighbors. When parameter d increases, then the payoffs from the coordination on the cultural identity increase relative to the payoffs from coordination on education. Hence, the welfare comparison between both actions depends on the number of neighbors d changes. To allow for meaningful comparison of payoffs while varying d, let us define an auxiliary constant

$$U^C = u^C d.$$

Then $U^C \left(e^*\right)^2$ is equal to payoff from convention **c**. In what follows, we assume that U^C remains constant when the number of neighbors d varies. Second, more importantly, the payoff from education depends nonlinearly on $\bar{e}$. In particular, risk-dominance of educational investment implies, but is not equivalent to, its payoff-dominance.

Lemma 1. *The investment into education is risk-dominant (best response when all the others mix equally) iff*

$$w > w_1 = \frac{\frac{1}{2}e^*(2p-1) + 1 - p}{(2p-1)^2} U^C.$$

When N is large, it is payoff-dominant iff

$$w > w_2 = \frac{e^*(2p-1) + 1 - p}{(2p-1)^2} U^C.$$

Notice that $w_2 > w_1$, i.e. if education is payoff-dominant, it is also risk-dominant. One can also show that for $w < w_2$, there is no payoff-dominant state. (When $w < w_2$, any player prefers that he and his neighbors invest in the cultural identity, while the rest of the population choose education.)[16]

Proposition 8. *Under scenario 1, for any $\gamma > 0$, a sufficiently large d, if assumption 1 holds, in almost any network of interactions inside the minority group, γ-neighborhood of the risk-dominant convention is chosen in the long-run. Hence, the minority approximately coordinates on* $\mathbf{c}$*, if and only if $w < w_1$.*

Under scenario 2, there is a constant $\delta > 0$, such that if $w < (1+\delta) w_2$, then (given the same conditions) γ-neighborhood of convention $\mathbf{c}$ *is chosen in the long-run.*

If interactions are given as in scenario 1, the risk-dominant action is played in the long-run. Thus, when $w > w_1$, the population evolves to coordinate on the investment in education. This is no longer true under scenario 2. When the wage belongs to the interval $w \in [w_1, w_2(1+\delta)]$ for some positive δ, the small groups force coordination on the identity investment. This happens despite education being risk-dominant, and, for $w > w_2$, also payoff-dominant.

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[16]Since the payoff from education is nonlinear, we cannot apply the standard results regarding the long-run behavior of the KMR dynamics. The following is true:

(1) Under scenario 1, the KMR dynamics choose the risk-dominant coordination.
(2) Under scenario 2, the KMR dynamics may choose the risk-dominated convention: in particular, there is $\delta' > 0$, such that for any $w < (1+\delta') w_1$, the KMR dynamics lead to coordination on the identity investment. This shows that when we drop the linearity assumption, the KMR dynamics may produce network dependent results.
(3) In general, whatever is the network of interactions, when $w > w_2$, the KMR dynamics choose the investment in education. This follows from methods of Theorem 2 in [Peski 2003].

Details will be available in the online appendix.

Sketch of the proof. The proof in the first scenario follows the argument behind Proposition 2. Since interactions are distributed uniformly across the population, the payoffs from both actions depend approximately on the number of all players using particular actions. One needs less than $\frac{1}{2}N$ individual mistakes to reach the basin of attraction of the risk-dominant action and more than $\frac{1}{2}N$ mistakes to reach the basin of attraction of the risk-dominated action. The argument uses the fact that the uniform distribution is approximately true, when d is sufficiently large.

The proof in the second scenario follows the argument behind Proposition 3. On one hand, we may easily check that when $\frac{\delta}{1+\delta}$ proportion of small groups switches to cultural investment, then it becomes a coalitional best response for any small group. On the other hand, consider a situation when all players play the identity investment. Suppose that, through individual mistakes, δN players start investing in education. As previously, we say that a small group is anomalous if it has more than $\sqrt{\delta}\,(d+1)$ members investing in education; otherwise the group is normal. If all members of the anomalous groups switch to education, there would be in total not more than $2\sqrt{\delta}N$ educated players in the population ($\sqrt{\delta}N$ from anomalous and $\sqrt{\delta}N$ from normal groups). The payoff from education of any member of the normal group cannot be higher than

$$w\left(e^{*},2\sqrt{\delta}e^{*}\right)=\frac{(2p-1)^{2}\,w}{2\sqrt{\delta}e^{*}\,(2p-1)+1-p}2\sqrt{\delta}\,(e^{*})^{2}+\frac{2\,(1-p)\,pw}{2\sqrt{\delta}e^{*}\,(2p-1)+1-p}2\sqrt{\delta}e^{*}.$$

Her payoff from the identity investment is not lower than

$$\left(1-\sqrt{\delta}\right)U^{C}\,(e^{*})^{2}+w\left(0,2\sqrt{\delta}e^{*}\right)=\left(1-\sqrt{\delta}\right)U^{C}\,(e^{*})^{2}+\frac{2\,(1-p)\,pw}{2\sqrt{\delta}e^{*}\,(2p-1)+1-p}2\sqrt{\delta}e^{*}.$$

For small δ,

$$\left(1-\sqrt{\delta}\right)U^{C}>\frac{(2p-1)^{2}\,w}{2\sqrt{\delta}e^{*}\,(2p-1)+1-p}2\sqrt{\delta}$$

and the latter is smaller than the former. Thus, cultural identity remains an individual best response for any member of a normal group and, one can easily check, a coalitional best response for members of an anomalous group. □

6.2. **Polarization.** Human societies are rarely homogenous. People tend to define themselves using many different categories. These include ethnic origins, language used, religion professed, political views, moral codes, wealth, occupation and many others. For particular people some of these categories may be more important than others. Some people understand themselves more as professionals, bankers or lawyers; others prefer to think about themselves as Greeks or Poles; yet another person may prefer to avoid any categories when thinking about herself.

Some categories may be more important than other in a given society. For example, the branch of Islam professed may be the most important defining category in a Middle Eastern country; urban-rural origins in a country in Latin America; color of skin in an African postcolonial country. We say that a society is *polarized* if there is a single dimension along which people tend to define and distinguish themselves from others. In polarized societies, it is easy to single out two or more homogenous groups which are the foundation for divisions (for example, these may be ethnic or religious groups or economic classes).

Polarization may be modeled as a coordination game.[17] Suppose that a population of size $2N$ is divided into two equal groups, Males, $\{1, ..., N\}$ and Females, $\{N+1, ..., 2N\}$. They live on a network and each player has, on average, d neighbors in her own group and d neighbors in the opposite one. Players choose between two actions, a "stay united" or interact with the whole society, and b, "polarize" or restrict your interactions only to own group. The payoff from a is equal to the total number of one's neighbors playing a multiplied by $u^A > 0$. The payoff from b is equal to the number of one's neighbors in one's own group, multiplied by $u^B > 0$.

We consider two scenarios of connections between players:

Scenario 1: The interactions are spread uniformly across the population: the network of interactions is drawn from the random network $g \in G\left(2N, \frac{2d}{2N}\right)$.

Scenario 2: The interactions *within* a group form small groups and the interactions *between* groups are drawn from the random network. Precisely, take a realization $g' \in$

[17]The game of coordination as a model of ethnic polarization is used in [Kuran 1998a], [Kuran 1998b], [Laitin 1998]. T. Kuran describes a process of ethnic polarization as a dramatic and unexpected change of equilibrium in a coordination game. Suppose that some external event shifts, even insignificantly, private preferences for ethnicity. Small change in preferences may nevertheless start a chain reaction: First activists start displaying ethnic behavior, then less active people, finally, even people with mild or non-existent preferences for ethnicity feel compelled to take on ethnic behavior.

D. Laitin studies the process of identity formation among Russian minorites in post-Soviet republics of Pribaltika, Ukraine and Kazachstan. After the collapse of the Soviet Union, Russians suddenly lost their dominant position and became minorities facing the decision whether to assimilate in the new society, or remain isolated. Russians' decision about assimilation may be understood as a coordination game - the more Russians who decide to assimilate, the more difficult it is for the rest to isolate. Laitin shows that different history and relations between Russians and locals affected in a differing ways the focality of both equilibria in a polarization game.

$G\left(2N, \frac{2d}{2N}\right)$ and form a network g in the following way

$$g_{ij} = g'_{ij} \text{ whenever } i \text{ and } j \text{ belong to opposite groups,}$$
$$g_{ij} = 1 \text{ whenever } i \text{ and } j \text{ belong to the same group and } \left\lfloor \frac{i}{d+1} \right\rfloor = \left\lfloor \frac{j}{d+1} \right\rfloor,$$
$$g_{ij} = 0 \text{ otherwise.}$$

We show that both scenarios lead to different equilibrium predictions. This means that the probability of polarization depends on the internal structure of the group.

Observe that this is a network selection game. Each player has on average $2d$ interactions in network A, the united one, and d interactions in network B, the polarized one. The average payoff from convention $\mathbf{a}$ is equal to $2du^A$ and the average payoff from convention $\mathbf{b}$ is equal to $u^B d$. We assume that $u^A < u^B < 2u^A$ - one prefers the united convention, however each interaction under polarization yields higher payoff than interactions with members of the opposite group. The action which is payoff-dominant, a, is also risk-dominant. The benchmark KMR dynamics choose coordination on action a in both scenarios.

Proposition 9. *For any $\gamma > 0$, a sufficiently large d, if assumption 1 holds, in almost any network under scenario* 1, *γ-neighborhood of convention* $\mathbf{a}$ *is chosen in the long-run.*

Under scenario 2, there is $\delta > 0$, such that, if $2u^A < (1+\delta)\, u^B$, then (given the same conditions) γ-neighborhood of convention $\mathbf{b}$ *is chosen in the long-run.*

Thus, scenario 1 leads to the efficient convention. Under scenario 2, polarization is favored, unless its inefficiency is too high.

Sketch of the proof. The case of scenario 1 is a consequence of Proposition 4.

The argument in the case of scenario 2 is very similar to the proof of Proposition 3. First, notice that $\frac{\delta}{1+\delta}N$ individual mistakes are sufficient for the transition to the basin of attraction of the convention $\mathbf{b}$. If a proportion of $\frac{\delta}{1+\delta}$ of Males switches to play of b, then it immediately becomes a coalitional best response for any Female group to switch to b. After all Females decide to polarize, Males will follow.

Second, at least δN individual mistakes are necessary in order to reach the basin of attraction of convention $\mathbf{a}$. If δN players switch to a, then there is at most $\frac{1}{2}\sqrt{\delta}$ proportion of anomalous groups, where each anomalous group has more than $\sqrt{\delta}\,(d+1)$ a-players. Even if all anomalous group members switch to a, there would be no more than $2\sqrt{\delta}N$ players of a in the whole population. Each member of a non-anomalous group faces a choice between the payoff of at most $2u^A\sqrt{\delta}d = 2\,(1+\delta)\,\sqrt{\delta}u^B d$ from playing a and the payoff of at least $\left(1-\sqrt{\delta}\right) u^B$ from playing b. Clearly, b is the best response. □

The difference between both scenarios shows that intragroup relations may affect the probability of polarization. Specifically, small groups fully contained inside the large groups aid polarization. It is easier for them to coordinate on the polarizing action b. Small groups which consists of members of relatively homogenous groups form, as it is called, a *bonding social capital*. The examples of bonding social capital may include clans or tribes, religious communities, ethnic cultural associations.

Since small groups inside the large groups help polarization, one may expect that small groups crossing over the large group boundaries will obstruct polarization. The intuition is that such small groups might increase the probability of coalitional best responses towards the uniting action a, which speeds up the evolution in the direction of a. This may help the population to reach the efficient coordination on the uniting action. Small groups consisting of members of different social groups form a *bridging social capital.* A recent political science literature argues that bridging social capital indeed decreases the probability of conflict between large groups.[18]

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[18][Varshney 2002] documents evidences on the positive role of bridging social capital in containing communal violence in India. To certain degree, the authorities are able to exploit the group mechanisms in their policies. For example, Indian police stimulates and supervises creation of *mohalla (peace) committees* which associate respected members of neighboring Hindu and Muslim communities. Their role is to police communities during times of tensions, suppress gossips and react to any signs of rioting. The success of the committees (whenever they are properly used) comes from the fact that they are able to stop the riot in the neighborhood at its very birth. In the language of this paper, mohalla committee is a small group which manages to stand by the efficient action even when the rest of the society does something opposite. Through its role in the neighborhood, it further radiates the efficient behavior to the others.

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Appendix A. Preliminaries

This section of the appendix starts with the notation used in this paper. Later we discuss the stochastic dynamics and define the cost function and its properties. The next two parts contain the formal evolutionary argument behind all the proofs in this paper. The last part is concerned with some approximation results useful in the analysis of random graphs.

A.1. **Notation.** A distance between two states $\eta, \eta' \in \Sigma$ is measured with

$$\langle \eta, \eta' \rangle = \# \{ i : \eta (i) \neq \eta' (i) \} .$$

For any set of players C, we denote with $a_C (\eta)$ the proportion of players in C playing a in state η :

$$a_C (\eta) = \frac{1}{\# C} \# \{ i \in C : \eta (i) = a \} .$$

We also denote with $a (\eta)$ the proportion of the whole population playing a in state η. Similar notation is used for action b.

For any action $x \in S$, and any two states $\eta, \eta' \in \Sigma$, we say that $\eta \leq_x \eta$ iff for any i, $\eta (i) = x \Longrightarrow \eta' (i) = x$ (in other words, state η lies below state η' with respect to action x if all the players who play x in state η do the same in state η'). For any action $x \in S$ and any three states η, η', η'', if $\eta \geq_x \eta'$ and $\eta \geq_x \eta''$, then we write $\eta \geq \eta' \vee_x \eta''$. Similarly, if $\eta \leq_x \eta'$ and $\eta \leq_x \eta''$, then we write $\eta \leq_x \eta' \wedge_x \eta''$.

A.2. **Cost function.** Consider a sequence of Markov ergodic probability transitions on a state space $\Sigma = S^N$, $p^\varepsilon : \Sigma \times \Sigma \to [0, 1]$, for $\varepsilon > 0$. Denote an ergodic distribution of such a chain with μ^ε. We say that the system (Σ, p^ε) is a *limit probability system* if there is a cost function $c : \Sigma \times \Sigma \to R \cup \{\infty\}$, such that

$$c (\eta, \eta') = \lim_{\varepsilon \to 0} \frac{\log p^\varepsilon (\eta, \eta')}{\log \varepsilon}. \tag{A.1}$$

The evolutionary dynamics considered in this paper (as defined in section 2) is a limit probability system. Given a l.p.s. (Σ, p^ε) , we define an ordered cost function $Oc : \Sigma \times \Sigma \to R \cup \{\infty\}$ with

$$Oc (\eta, \eta') = \min_{\eta = \eta_0, \ldots, \eta_k = \eta',} \sum_{i=0}^{k-1} c (\eta_i, \eta_{i+1}) .$$

It is very useful to define three properties of the cost function. We say that cost function $c : \Sigma \times \Sigma \to R$ is

(1) *supermodular* iff

- for any three elements $\eta_1 \leq_x \eta_1' \wedge_x \eta_2$, there is $\eta_2' \geq_x \eta_1' \vee_x \eta_2$, st. $c(\eta_1, \eta_2) \geq c(\eta_1', \eta_2')$,
- for any three elements $\eta_1 \geq_x \eta_1' \vee_x \eta_2$, there is $\eta_2' \leq_x \eta_1' \wedge_x \eta_2$, st. $c(\eta_1, \eta_2) \geq c(\eta_1', \eta_2')$;

(2) *polarized* iff for any two states η, η', there are states $\eta^* \geq_x \eta \vee_x \eta'$ and $\eta_* \leq_x \eta \wedge_x \eta_*$ and $\eta \vee_x \eta_* \leq_x \eta'$, such that

$$c(\eta, \eta') \geq c(\eta, \eta^*) + c(\eta, \eta_*);$$

(3) *risk dominant:* For any state $\eta \in \Sigma$, we may define an opposite state $-\eta$ as a "mirror reflection" of η: $-\eta(i) = a \Longleftrightarrow \eta(i) = b$. We say that the risk-dominance of action $x \in S$ is satisfied iff for every $\eta, \eta' \in S$, $\eta_1 >_x \eta_2$, there is $\eta_2' \leq_x \eta_2$, such that

$$c(\eta_1, \eta_2) \geq c(-\eta_1, -\eta_2').$$

It is easy to check that the cost function derived from the small group dynamics is supermodular (it is a consequence of the fact that all games considered in this paper are games of coordination). [Peski 2003] shows that the supermodularity and the risk-dominance of the cost function implies the supermodularity and the risk-dominance of the ordered cost function.

A.3. **Lemma.** The following lemma is useful in the next subsection. It allows us to replace a limit probability system by another one with a smaller state space.

Lemma 2. *Suppose that* (Σ, p^ε) *is an l.p.s. and* $W \subseteq \Sigma$. *Then there is a l.p.s.* $(\Sigma_W, p_W^\varepsilon)$, $\Sigma_W = \Sigma \backslash W \cup \{w\}$, *such that for any subset* $A \subseteq \Sigma \backslash W$

$$\lim_{\varepsilon \to 0} \mu^\varepsilon(A) = \lim_{\varepsilon \to 0} \mu_W^\varepsilon(A) \tag{A.2}$$

and for any $\eta, \eta' \in \Sigma \backslash W$

$$\begin{aligned} Oc(\eta, \eta') \leq c_W(\eta, \eta') &\leq c(\eta, \eta'), \\ \min_{\eta' \in W} Oc(\eta, \eta) \leq Oc_W(\eta, w) &\leq \max_{\eta' \in W} Oc(\eta, \eta'), \\ \min_{\eta \in W} Oc(\eta, \eta') \leq Oc_W(w, \eta') &\leq \max_{\eta \in W} Oc(\eta, \eta'). \end{aligned} \tag{A.3}$$

Proof. Define function $i : \Sigma \to \Sigma_W$ by $i(\eta) = \eta$ if $\eta \in \Sigma \backslash W$ and $i(\eta) = w$ if $\eta \in W$. We construct a Markov chain on Σ_W, $p_W^\varepsilon : \Sigma_W \times \Sigma_W \to [0,1]$: for any $s, s' \in \Sigma_W$, any subset $\Sigma' \subseteq \Sigma_W$

$$p_W^\varepsilon(s, s') = \sum_{\eta \in i^{-1}(s), \eta' \in i^{-1}(s')} \frac{\mu^\varepsilon(\eta)}{\mu^\varepsilon(i^{-1}(\eta))} p^\varepsilon\left(\eta, i^{-1}(s')\right).$$

Define a measure $\mu_w^\varepsilon \in \Delta\Sigma_W$ with $\mu^\varepsilon(A) = \mu_w^\varepsilon(i^{-1}(A))$ for any $A \subseteq \Sigma_W$. Direct calculations show that μ_w^ε is an ergodic measure for Markov process p_W^ε.

Using Freidlin-Wentzell tree formula[19], we check that the sequence of ergodic distributions μ^{ε} converges: for each $\eta \in \Sigma$, there are $\beta(\eta) \geq 0$ and $\psi(\eta) \geq 0$, such that $\lim_{\varepsilon \to 0} \frac{\mu^{\varepsilon}(\eta)}{\beta(\eta)\varepsilon^{\psi(\eta)}} = 1$. Substitutions and some computations show that kernels p_W^{ε} have a cost function. There exists $c_W : \Sigma_W \times \Sigma_W \to R_+ \cup \{\infty\}$, such that for any $s, s' \in \Sigma_W$ limit exists

$$\lim_{\varepsilon \to 0} \frac{\log p_W^{\varepsilon}(s, s')}{\log \varepsilon} = c_W(s, s').$$

Thus, $(\Sigma_W, p_W^{\varepsilon})$ is a limit probability system. The equality (A.2) holds trivially.

We need to show that inequalities on the ordered costs functions are satisfied. Observe first that for $\eta, \eta' \in \Sigma \backslash W$,

$$c_W(\eta, \eta') = c(\eta, \eta'), \; c_W(\eta, w) = \min_{\eta' \in W} c(\eta, \eta') \text{ and}$$
$$\min_{\eta \in W} c(\eta, \eta') \leq c(w, \eta') \leq \max_{\eta \in W} c(\eta, \eta').$$

This leads in a straightforward way to all inequalities above, except for the last one. Only the last inequality, $Oc_W(w, \eta') \leq \max_{\eta \in W} Oc(\eta, \eta')$, requires a separate proof. Take $\eta^* \in W$, such that $\lim_{\varepsilon \to 0} \frac{\mu^{\varepsilon}(\eta^*)}{\mu^{\varepsilon}(W)} > 0$. Find a path $\eta^* = \eta_0, ..., \eta_m = \eta'$ which minimizes the definition of $Oc(\eta^*, \eta')$. Suppose that $k \geq 0$ is the first element along the path, such that $\eta_{k+1} \in \Sigma \backslash W$. We show that

$$c_W(w, \eta_{k+1}) \leq \sum_{i=0}^{k} c(\eta_i, \eta_{i+1}).$$

This, together with some obvious computations, proves inequality (A.3).

Indeed, observe that for any $i \geq 1$

$$\mu^{\varepsilon}(\eta_i) \geq \mu^{\varepsilon}(\eta_{i-1}) p^{\varepsilon}(\eta_{i-1}, \eta_i)$$

Hence,

$$c_W(w, \eta_{k+1}) = \lim_{\varepsilon \to 0} \frac{\log p_W^{\varepsilon}(w, \eta_{k+1})}{\log \varepsilon} \leq \lim_{\varepsilon \to 0} \frac{1}{\log \varepsilon} \log \left(\frac{\mu^{\varepsilon}(\eta_k)}{\mu^{\varepsilon}(W)} p^{\varepsilon}(\eta_k, \eta_{k+1}) \right)$$
$$\leq \lim_{\varepsilon \to 0} \frac{1}{\log \varepsilon} \log \left(\frac{\mu^{\varepsilon}(\eta^*)}{\mu^{\varepsilon}(W)} \right) + \sum_{i=0}^{k} c(\eta_i, \eta_{i+1}) = \sum_{i=0}^{k} c(\eta_i, \eta_{i+1}).$$

□

[19] See for example [Fudenberg and Levine 1998], chapter .

A.4. **General argument.** In this subsection we present a general argument behind our examples.

Suppose that the state space Σ is divided into disjoint subsets $Y_0 \cup ...Y_M = \Sigma$ in an "ordered" way: there is an action $x \in S$, such that for any $m < m'$ set Y_m lies above, with respect to relation $\leq_x$, set $Y_{m'}$. Precisely, for any $0 \leq m^* < M$, any states $\eta_- \in \bigcup_{m=0}^{m^*-1} Y_m, \eta_+ \in \bigcup_{m=m^*}^{M} Y_m$, if $\eta_- \leq_x \eta_+$ or $\eta_- \geq_x \eta_+$, then $\eta_- \leq_x \eta_+$ (if two states can be compared using relation $\leq_x$, then state η_- must be below state η_+).

Additionally, suppose that there are sets $W_m \subseteq Y_m$, which contain all the steady states of the small group dynamics and

- for any m^*, and $\eta \in Y_{m^*}$, there is a 0-cost path leading from η to some state $\eta' \in \bigcup_{m=0}^{m^*} W_m$, $Oc(\eta, \eta') = 0$,
- for any m^*, and $\eta \in W_m$, any $\eta' \notin W_m$, $Oc(\eta, \eta') > 0$.

We have the following proposition:

Proposition 10. *Suppose that sets W_m, Y have properties as described above and that for any $m > m'$*

$$\max_{\eta \in W_m} \min_{\eta' \in \bigcup_{m'' \leq m'} W_{m''}} Oc(\eta, \eta') < \min_{\eta \in W_{M-m}} \min_{\eta' \in \bigcup_{m'' \geq M-m'} W_{m''}} Oc(\eta, \eta').$$

Then in the long run, only states from set W_0 occur,

$$\lim_{\epsilon \to 0} \mu^{\varepsilon}(W_0) = 1.$$

The Proposition says that under certain conditions, the dynamics spends most of the time in one or many states in set W_0. It does not specify which state in W_0 is visited most often. The strategy of the proof is following. First, we replace the original limit probability system with a new one, in which all the states in sets W_m are "collapsed" into a single new state w_m. We describe properties of this new l.p.s. (and its cost function) and show that the original question can be replaced by the question of the stability of state w_0. Second, we show that state w_0 is indeed stochastically stable.

Proof. By repeated application of Lemma 2, we can find a state space $\Sigma' = \{w_0, w_1, ..., w_M\} \cup \Sigma \backslash \bigcup W_m$ and l.p.s. $(\Sigma', p^{\varepsilon\prime})$ with a cost function c' such that $\lim_{\epsilon \to 0} \mu^{\varepsilon}(W_0) = \lim_{\epsilon \to 0} \mu^{\varepsilon\prime}(w_0)$ and:

(1) for any $\eta \in Y_m \backslash W_m$, there is $m' \leq m$, such that $Oc'(\eta, w_{m'}) = 0$,
(2) for any $\eta' \notin W_m$, $Oc'(w_m, \eta') > 0$,

(3) for any $m > m'$

$$\min_{m'' \leq m'} Oc'(w_m, w_{m''}) < \min_{m'' \geq M-m'} Oc'(w_{M-m}, w_{m''}).$$

The rest of the proof is an application of the argument by [Young 1993]. It is sufficient to show that the spanning tree on the network $[M] = \{0, ..., M\}$ with the cost function $c_M : [M] \times [M] \to R \cup \{\infty\}$ defined by $c_M(m, m') = Oc'(w_m, w_{m'})$ must have a root at w_0 (this is because all steady states of the original dynamics are contained in $\bigcup W_m$). It is a consequence of the stated above properties and Theorem 2 in [Peski 2003]. □

A.5. **Random graphs.** Let $b(k, n; p) = \binom{n}{k} p^k (1-p)^{n-k}$ be the probability of k successes in n draws in a Bernoulli with probability p. Let $B(k, n; p) = \sum_{k' \leq k} b(k', n; p)$ be a corresponding cdf. We use the following lemma, which gives the bound on the probability of the tail of Bernoulli distribution (the proof is analogous to Theorem 1.7 from [Bollobas 2001]):

Lemma 3. *For $u > 1$, $d > 1$, $c > 0$*

$$\lim_{n \to \infty} \frac{\log\left[1 - B\left(udcn, cn^2; \frac{d}{n}\right)\right]}{-udcn\left(\log u - 1 + \frac{1}{u}\right)} \leq 1.$$

We prove two results about random graphs. They characterize some bounds on the number of neighbors a player might have in a given large set.

Proposition 11. *For any $\varepsilon \in \left(0, \frac{1}{2}\right)$, there is a d_0 high enough, such that for any $d \geq d_0$, any $2\varepsilon \leq \theta \leq 1 - 2\varepsilon$, in almost any network in $G\left(N, \frac{d}{N}\right)$, any set of vertices $U \subseteq \{1, ..., N\}$, $|U| = \theta N$, contains at most εN vertices which have* more *than $d(1+\varepsilon)\theta$ neighbors in set U. Moreover, d_0 may be chosen as a continuous function of ε.*

Proposition 12. *For any $\varepsilon \in \left(0, \frac{1}{2}\right)$, there is a d_0 high enough, such that for any $d \geq d_0$, any $2\varepsilon \leq \theta \leq 1 - 2\varepsilon$, in almost any network in $G\left(N, \frac{d}{N}\right)$, any set of vertices $U \subseteq \{1, ..., N\}$, $|U| = \theta N$, contains at most εN vertices which have* less *than $d(1-\varepsilon)(1-\theta)$ neighbors in set $\{1, ..., N\} \backslash U$. Moreover, d_0 may be chosen as a continuous function of ε.*

The proof of the second proposition is analogous to the first one.

Proof of proposition 11. Take first sets $T^* = \{1, ..., \varepsilon N\}$ and $U^* = \{1, ..., \theta N\}$. We bound from above the probability that all vertices in T^* have more than $d(1+\varepsilon)\theta$ neighbors in set U^*. This probability is equal to the probability that in the random network $G\left(\theta N, \frac{d}{N}\right)$ vertices in set T^* have at least $d(1+\varepsilon)\theta$ neighbors,

$$P_d^{\theta N}\left(\forall_{i \in T^*} \sum_{j \in U^*} g_{ij} \geq d(1+\varepsilon)\theta\right),$$

This is smaller than

$$\leq P_d^{\theta N}\left(\sum_{i\in T^*,j\in U^*} g_{ij} \geq \varepsilon\theta d\left(1+\varepsilon\right)N\right)$$
$$\leq P_d^{\theta N}\left(2\sum_{i,j\in T^*} g_{ij} \geq \varepsilon^2\theta d\left(1+\varepsilon\right)N\right)+P_d^{\theta N}\left(\sum_{i\in T^*,j\in U^*\setminus T^*} g_{ij} \geq \varepsilon\left(1-\varepsilon\right)\theta d\left(1+\varepsilon\right)N\right).$$

The last inequality is a consequence of two observations. First, we can replace the statement about the number of neighbors of each vertex by the statement about the sum of all neighbors. Second, in the sum, each neighbor inside set T^* is counted twice. Further, this is equal to

$$\begin{aligned}
&= 1-B\left(\frac{\varepsilon^2\theta d\left(1+\varepsilon\right)N}{2},\frac{\varepsilon^2N^2}{2};\frac{d}{N}\right)\\
&+1-B\left(\varepsilon\left(1-\varepsilon\right)\theta d\left(1+\varepsilon\right)N,\varepsilon\left(1-\varepsilon\right)N;\frac{d}{N}\right)\\
&\leq \exp\left[-\frac{\varepsilon^2\theta d\left(1+\varepsilon\right)}{2}\left(\log\left(1+\varepsilon\right)+\frac{1}{1+\varepsilon}-1\right)N\right]\\
&+\exp\left[-\varepsilon\left(1-\varepsilon\right)\theta d\left(1+\varepsilon\right)\left(\log\left(1+\varepsilon\right)+\frac{1}{1+\varepsilon}-1\right)N\right],
\end{aligned}$$

where the last inequality holds asymptotically in the sense of Lemma 3. Denote

$$c_0^\varepsilon=\varepsilon^3\frac{1+\varepsilon}{2}\left[\log\left(1+\varepsilon\right)+\frac{1}{1+\varepsilon}-1\right]$$

- we can readily check that $c_0^\varepsilon>0$ and it is continuous in $\varepsilon\in\left(0,\frac{1}{2}\right)$. The last expression is asymptotically smaller than

$$\leq 2\exp\left[-dc_0^\varepsilon N\right].$$

We may finish the proof of the Proposition. The probability that there is a set of vertices U with at least εN vertices which have more than $d\left(1+\varepsilon\right)\theta$ neighbors in the set U is bounded by $\exp\left[-dc_0^\varepsilon N\right]$ times the number of possible ways we can choose sets T and U from the whole network:

$$\begin{aligned} & P_d^N \left(\begin{array}{c} \text{for any sets } T \subseteq U, |T| = \varepsilon N, |U| = \theta N, \\ \forall i \in T, \ \sum_{j \in \{1,\dots,\theta N\}} g_{ij} \geq d(1+\varepsilon)\theta \end{array} \right) \\ & \leq \binom{N}{\theta N} \binom{\theta N}{\varepsilon N} P_d^{\theta N} \left(\forall_{i \in T^*} \sum_{j \in U^*} g_{ij} \geq d(1+\varepsilon)\theta \right) \\ & \leq \exp\left(-N \left(d c_0^{\varepsilon} - \theta(1 - \log\theta) - \frac{\gamma}{\theta}\left(1 - \log\frac{\gamma}{\theta}\right)\right)\right) \\ & \leq \exp\left(-N\left(d c_0^{\varepsilon} - 4\right)\right). \end{aligned}$$

The second inequality comes from the Stirling's formula[20] and the last one from the fact that for $x < 1$, $x(1 - \log x) \leq 2$. When $d_0 > \frac{4}{c_0^{\varepsilon}}$, the last expression converges to 0. □

Appendix B. Equilibrium selection - first results

This part of the appendix contains proofs of Theorem 1 and Propositions 1 and 5.

Proof of theorem 1. The cost function is supermodular. Since we allow only for transitions in one direction. We show that it is also polarized and satisfies risk-dominance for action a.

For the former, note that any coalitional change may be decomposed into changes leading towards action a and changes leading towards action b. Both changes can be done separately, starting, in each case, from the original state. Sum of both costs cannot be higher than the cost of the original change.

To see the latter, denote with C the set of players i which change their actions in transition $\eta \to \eta' : C = \{i : \eta(i) \neq \eta'(i)\}$. Then, for any $i \in C$, she is better off in state η' than η :

$$\sum_{j \in C \setminus \{i\}} \left[u(b,b) - u(a,a)\right] g_{ij} + \sum_{j \in C \setminus \{i\}} \left[u(b, \eta(i)) - u(a, \eta(i))\right] g_{ij} \geq 0.$$

Due to the payoff- and risk-dominance of a,

$$u(a,a) > u(b,b) \text{ and } u(a,\eta(i)) + u(a,-\eta(i)) > u(b,\eta(i)) + u(b,-\eta(i)).$$

But this implies that every player $i \in C$ is (strictly) better off in state -η' playing a than in state -η playing b :

$$\sum_{j \in C \setminus \{i\}} \left[u(a,a) - u(b,b)\right] g_{ij} + \sum_{j \in C \setminus \{i\}} \left[u(a, -\eta(i)) - u(b, -\eta(i))\right] g_{ij} > 0.$$

[20]which leads to

$$\lim_{n \to \infty} \frac{\log \binom{n}{\gamma n}}{\gamma n (1 - \log \gamma)} = 1.$$

According to Theorem 2 of [Peski 2003], action a is stochastically stable. This, however, itself does not preclude the stochastic stability of b. In order to show that action a is uniquely stochastically stable, we need to follow steps described in section 4 of [Peski 2003] and check that the ordered cost function on the graph of steady states is *strictly* risk dominant. Details are available in the online appendix. □

Proof of proposition 1. Suppose that a is risk-dominant and b is payoff dominant. It is immediate to observe that there are only two steady states $\mathbf{a}, \mathbf{b}$. By the standard evolutionary argument of [Kandori, Mailath, and Rob 1993] and [Young 1993], we need to compare the costs of transitions from one state to another, $Oc(\mathbf{a}, \mathbf{b})$ and $Oc(\mathbf{b}, \mathbf{a})$. Suppose that the cost of transition from state $\mathbf{a}$ to $\mathbf{b}$ is given by the sum

$$Oc(\mathbf{a}, \mathbf{b}) = \sum_{k=0}^{m-1} c(\eta_k, \eta_{k+1}).$$

Let $\eta_{k^*} \rightarrow \eta_{k^*+1}$ be the first transition which is an individual or coalitional best response towards action b. Let $n = \langle \mathbf{a}, \eta_{k^*} \rangle$ and $n' = \langle \eta_{k^*}, \eta_{k^*+1} \rangle$. The cost of transition along the path from $\mathbf{a}$ to η_{k^*}, $\sum_{k=0}^{k^*-1} c(\eta_k, \eta_{k+1})$, cannot be smaller than n. The cost of the transition $\eta_{k^*} \rightarrow \eta_{k^*+1}$ is equal to $f(n')$. Moreover, by the definition of the coalitional best response, the participants must prefer playing b when $n + n'$ players play b rather than playing a when n players plays b :

$$nu(a, b) + (N - n - 1) u(a, a) \leq (n + n' - 1) u(b, b) + (N - n - n') u(b, a)$$

Define the smallest n' satisfying this inequality with $g_{ab}(N, n)$. Similarly, we define an anologuous function $g_{ab}(N, n)$. One may check that there are constants, c_{ab}, c_{ba} and $0 < s_{ab} < s_{ba}$, such that $s_{ab}(1 - \rho) < s_{ba}\rho$ and

$$g_{ab}(N, n) = \lceil c_{ab} + s_{ab}((1 - \rho) N - n) \rceil \text{ and } g_{ba}(N, n) = \lceil c_{ba} + s_{ba}(\rho N - n) \rceil,$$

where $\lceil x \rceil$ is the smallest natural number not smaller than x. Thus, for any N, the difference $g_{ab}(N, n) - g_{ba}(N, n)$ is weakly decreasing and approximately linear with n. Using both functions, we can compute the ordered costs

$$Oc_N(\mathbf{a}, \mathbf{b}) = \min_n n + f(g_{ab}(N, n)).$$

Similarly we may define $g_{ba}(N, n)$, such that

$$Oc_N(\mathbf{b}, \mathbf{a}) = \min_n n + f(g_{ba}(N, n)).$$

Define now N_0 as the lowest N, such that $Oc_N(\mathbf{b},\mathbf{a}) - Oc_N(\mathbf{a},\mathbf{b}) \geq 0$ and N_1 as the highest N, such that the difference is negative. We show that $N_1 - N_0$ is bounded by

$$N_1 - N_0 \leq c = \frac{2}{1-2\rho}\left(\frac{1}{s_{ab}} + \frac{1}{s_{ba}}\right), \tag{B.1}$$

where c is a payoff-related constant.

>From now on, we assume that $f(.)$ is a smooth convex function defined for all $x \in R$ (we can do it, w.l.o.g.). In order to similify the argument, we also assume that it is strictly convex (this assumption can be dropped). Observe that

$$\begin{aligned}
&Oc_{N_0}(\mathbf{b},\mathbf{a}) - Oc_{N_0}(\mathbf{a},\mathbf{b}) \\
&\leq \min_n n + f(g_{ba}(N_1,n)) - \min_n (n + f(g_{ab}(N_1,n))) \\
&\leq \min_x x + f(c_{ab} + s_{ab}((1-\rho)N_0 - x) + 1) - \min_{x'}(x' + f(c_{ba} + s_{ba}(\rho N_0 - x') - 1)).
\end{aligned}$$

Let x_0 and x_0' denote respectively solutions to the first and the second minimalization problems in the last line. Similarly,

$$\begin{aligned}
&Oc_{N_1}(\mathbf{b},\mathbf{a}) - Oc_{N_1}(\mathbf{a},\mathbf{b}) \\
&\geq \min_x x + f(c_{ab} + s_{ab}((1-\rho)N_1 - x) - 1) - \min_{x'}(x' + f(c_{ba} + s_{ba}(\rho N_1 - x) + 1))
\end{aligned}$$

and denote solutions to the minimalization problems with x_1 and x_1', respectively. Since $f(.)$ is strictly convex,

$$\begin{aligned}
c_{ab} + s_{ab}((1-\rho)N_0 - x_0) + 1 &= c_{ab} + s_{ab}((1-\rho)N_1 - x_1) - 1 \text{ and} \\
c_{ba} + s_{ba}(\rho N_0 - x_0') - 1 &= c_{ba} + s_{ba}(\rho N_1 - x_1') + 1.
\end{aligned}$$

Thus,

$$\begin{aligned}
s_{ab}(x_1 - x_0) &= s_{ab}(1-\rho)(N_1 - N_0) - 2 \\
s_{ba}(x_1' - x_0') &= s_{ba}\rho(N_1 - N_0) + 2,
\end{aligned}$$

We can bound

$$\begin{aligned}
&[Oc_{N_1}(\mathbf{b},\mathbf{a}) - Oc_{N_1}(\mathbf{a},\mathbf{b})] - [Oc_{N_0}(\mathbf{b},\mathbf{a}) - Oc_{N_0}(\mathbf{a},\mathbf{b})] \\
&\geq x_1 - x_1' - (x_0' - x_0') = (N_1 - N_0)(1-2\rho) - 2\left(\frac{1}{s_{ab}} + \frac{1}{s_{ba}}\right),
\end{aligned}$$

which leads to (B.1). □

Proof of proposition 5. The proof is very similar to the proof of Theorem 1. The cost function of the small group dynamics is, as before, supermodular and polarized. We verify only the risk-dominance of action a.

Suppose that there are $\eta' <_a \eta$, such that the transition $\eta \to \eta'$ is a coalitional best response towards action b. We show that for every player i, $\eta(i) \neq \eta'(i)$, it is a best response to play a in state $-\eta$. In other words, the composite transition $-\eta \to -\eta'$ towards action a can be done through a sequence of individual best responses. Indeed, the maximum payoff of player i in state η' cannot be higher than u^B. For him to be (even weakly) better off in state η', it must be that more than half of his neighbors in network A play b (otherwise, the payoff from a in state η would be higher than $\frac{1}{2}u^A > u^B$). But then, more than half of of player i's network A neighbors in state $-\eta$ play a, which makes action a a strict individual best response. According to Theorem 2 of [Peski 2003], action a is stochastically stable. Similarly as in the proof of Theorem 1, in order to show that action a is uniquely stochastically stable, we need to check that the ordered cost function on the graph of steady states is *strictly* risk dominant. Details are available in the online appendix. □

Appendix C. Equilibrium selection with two types of interactions: random network and small groups

In the first part of this section, we present a general result characterizing long-run behavior of the model presented in section 5. In the subsequent parts,we show how to apply the result to prove Propositions 2, 3, 6 and 7.

C.1. **Characterization.** A realization of the random network in model $G\left(N, \frac{d}{N}\right)$ is denoted with g_{ij}. For any player i, we denote the small group of which i is member by $s(i)$, where $s: \{1, ..., N\} \to \left\{1, ..., \frac{N}{d+1}\right\}$. With $a_s(\eta)$ we denote the proportion of members of group s, which play action a in state η,

$$a_s(\eta) = \frac{1}{d+1} \# \{i : s(i) = s \text{ and } \eta(i) = a\}$$

Assume that $f(d+1) = 0$ (assumption 1 holds) and that $n_f > d+1$ is defined with 2.2.

The payoffs of each player are defined by equation (5.1). Let $\pi_x(\alpha, \beta)$ denote an average payoff from action $x \in S$ when proportion α of the whole population and proportion β of one's own group plays action a :

$$\pi_x(\alpha, \beta) = \tau \left[\alpha u^A(x, a) + (1-\alpha) u^A(x, b)\right] + (1-\tau) \left[\beta u^B(x, a) + (1-\beta) u^B(x, b)\right].$$

Define constants γ_{ab} and γ_{ba} with

$$\pi_a(\gamma_{ba}, 1) = \pi_b(\gamma_{ba}, 0)$$

and $\gamma_{ab} = 1 - \gamma_{ba}$. These constants have a simple interpretation: Suppose that the proportion γ_{ba} of the population plays action a and the rest, proportion $\gamma_{ab} = 1 - \gamma_{ba}$, play b. Then action a is a coalitional (weak) best response for any small group playing b and action b is a coalitional (weak) best response for any small group playing a. We may compute that

$$\gamma_{ab} = \frac{(1-\tau)\left(u^B(a,a) - u^B(b,b)\right) + \tau\left(u^A(a,a) - u^A(b,a)\right)}{\tau\left(u^A(a,a) + u^A(b,b) - u^A(a,b) - u^A(b,a)\right)},$$
$$\gamma_{ba} = \frac{(1-\tau)\left(u^B(b,b) - u^B(a,a)\right) + \tau\left(u^A(b,b) - u^A(a,b)\right)}{\tau\left(u^A(a,a) + u^A(b,b) - u^A(a,b) - u^A(b,a)\right)}.$$

Next, define constants c_{ab} and c_{ba} with

$$\pi_a(0,1) = \pi_b(0, c_{ba}) \text{ and } \pi_b(1,0) = \pi_a(1, 1-c_{ab}).$$

Suppose that all groups but one's own play a and exactly proportion c_{ab} of one's own group plays b. Then, it is a coalitional best response for the remaining members to switch to b. Similarly, c_{ba} denotes the proportion of a group needed to play a, so that a becomes the coalitional best response for the rest of the group when all the other groups play b. We compute

$$c_{ab} = \frac{(1-\tau)\left(u^B(a,a) - u^B(b,b)\right) + \tau\left(u^A(a,a) - u^A(b,a)\right)}{(1-\tau)\left(u^B(a,a) - u^B(a,b)\right)},$$
$$c_{ba} = \frac{(1-\tau)\left(u^B(b,b) - u^B(a,a)\right) + \tau\left(u^A(b,b) - u^A(a,b)\right)}{(1-\tau)\left(u^B(b,b) - u^B(b,a)\right)}.$$

Finally, for $x = ab, ba$ define

$$T_x = \gamma_x\left(1 - \frac{1}{c_x} + \frac{1}{c_x}e^{-c_x}\right).$$

The rest of this part of the section is devoted to the proof of the Theorem:

Theorem 2. *Suppose that $T_{ab} > T_{ba}$. Then, (for any $\gamma > 0$,, d, d sufficiently large, with $(d+1)$-coalitional dynamics, assumption 1, holding, for almost any network) γ-neighborhood of convention* **a** *is chosen*[21].

[21] The proof below assumes that $c_{ab}, c_{ba} \geq 0$. Suppose not and, for example, $c_{ab} < 0$ - it is easy to see that, in such situation, $c_{ba} > 0$. It means that action b is group best response even if the whole society (including one's own group) play a. It is clear that in such situation only convention **b** is played in the long-run. Second, in order to define properly c_{ab} and c_{ba}we need to assume that

$$(1-\tau)\left(u^B(a,a) - u^B(a,b)\right) > 0 \text{ and } (1-\tau)\left(u^B(b,b) - u^B(b,a)\right) > 0$$

(in game of coordination we require only taht this inequalities are weak). The statement of the theorem remains almost the same if any of the inequalities is weak. For example, if $u^B(a,a) = u^B(a,b)$, then we set $c_{ab} = +\infty$ and $T_{ab} = \gamma_{ab}$ (which arises in result of taking the limit $c_{ab} \to +\infty$). The proof changes only slightly.

It turns out that only approximate conventions are stable with respect to the individual and coalitional best response process and they are the only candidates for the long-run outcome. In order to find out which one of them is stochastically stable, we use Proposition 10 and compare the costs of transition from one convention to another. As we show, $T_{ab}N$ ($T_{ba}N$) is equal (approximately) to the cost of the transition from convention $\mathbf{a}$ to convention $\mathbf{b}$ (from $\mathbf{b}$ to $\mathbf{a}$).

Choose now ε and δ small enough, $\delta > 10\sqrt{\varepsilon}$. Take d sufficiently high so that thesis of Proposition 3 holds for d and ε almost surely. Define sets

$$W_0 = \{\eta : (\eta, \mathbf{a}) \leq 2\delta N\} \text{ and } W_1 = \{\eta : (\eta, \mathbf{a}) \geq (\gamma_{ab} - 2\delta) N\}.$$

In three steps, we check the assumptions for Proposition 10:

(1) For any $\eta \notin W_1$, there is $\eta' \in W_0$, such that $Oc(\eta, \eta') = 0$ and for any $\eta'' \in \Sigma$, $Oc(\eta', \eta'') > 0$ (ordered cost function $Oc(.,.)$ was defined in the beginning of the appendix A.4).

(2) Transition from W_0 to W_1 costs at least $(T_{ab} - 6\delta) N$,

$$\min_{\eta \in W_0} \min_{\eta' \in W_1} Oc(\eta, \eta') \geq (T_{ab} - 6\delta) N \approx T_{ab}N.$$

(3) Transition from W_1 to W_0 costs at most $(T_{ba} + 3\delta) N$,

$$\max_{\eta \in W_1} \min_{\eta' \in W_0} Oc(\eta, \eta') \leq (T_{ba} + 3\delta) N \approx T_{ba}N.$$

Then, for δ sufficiently small, $(T_{ab} - 6\delta) N > (T_{ba} + 3\delta) N$. Proposition 10 ends the proof of the Theorem.

C.1.1. *Ad 1).* Take any state $\eta \in \Sigma$ such that $\delta N \leq (\eta, \mathbf{a}) \leq (\gamma_{ab} - 2\delta) N$.

First, we show that there is a small group k with at least one member playing b and such that it is a (strict) coalitional best response for the group to switch to a. In other words, there is a state $\eta' <_b \eta$, such that $Oc(\eta, \eta') = 0$ and $Oc(\eta', \eta) > 0$. By Proposition 11, there are at most εN players which have more than $(\gamma_{ab} - 2\delta)(1 + \varepsilon) d$ neighbors in network A playing a. For ε small enough, $(\gamma_{ab} - 2\delta)(1 + \varepsilon) < \gamma_{ab} - \delta$. There are at least $(1 - \gamma_{ab}) \frac{N}{d+1} - (d + 1) \varepsilon N$ small groups, which do not have any of the members inside the aforementioned set of εN players and at least one of their member plays b. For ε small enough, $(1 - \gamma_{ab}) \frac{N}{d+1} - (d + 1) \varepsilon N > 0$ and there is at least one small group like this. But then, by the definition of γ_{ab}, it is a (strict) coalitional best response for all members of the group who play b to switch to a.

Notice now that we can construct a 0-cost path leading from η to some $\eta_a \in W_0$, $(\eta_a, \mathbf{a}) < \delta N$, which consists of only strict coalitional best responses towards action a. η_a can be chosen in such way that there is no further 0-cost coalitional best response (note that $n^* =$

$\max\{n : f(n) = 0\} < \infty$) towards action a. Potentially, there might be a sequence of 0-cost coalitional best responses (of size at most n^*) towards action b leading from η_a to some state η'. It must be that $(\eta', \mathbf{a}) < \delta N$. Suppose otherwise and let η'_a be the first state along the sequence of n^*-coalitional best responses from η_a to η', such that $(\eta'_a, \mathbf{a}) \geq \delta N$. Then, when N is large enough (and $n^* << N$), it must be that $(\eta'_a, \mathbf{a}) < 2\delta N$ and there is a coalitional, 0-cost (strict) coalitional best response leading from η'_a towards action a (this is by the previous part of the argument). By the supermodularity of the cost function $c(.,.)$, it contradicts the choice of element η_a.

C.1.2. *Ad 2).* Suppose that $\eta = \eta_0, ..., \eta_k = \eta$ is a path which minimizes $Oc(\eta, \eta')$ for some $\eta \in W_0$ and $\eta' \in W_1$. By the strategic complementarity, we may assume that the path is increasing towards action b and that the first k' transitions along the path are done through the individual mistake process and the next $k - k'$ transitions occur purely through (individual or coalitional up to size n_f - see equation (2.2)) best responses. In other words, for all $l \leq k'$, there is only one player i_l such that $\eta_{l+1}(i_l) \neq \eta_l(i_l)$ and $u_{i_l}(b, \eta_l) < u_{i_l}(a, \eta_l)$ and for all other transitions, $l > k'$, if $\eta_{l+1}(i) \neq \eta_l(i)$, then $u_i(b, \eta_{l+1}) \geq u_i(a, \eta_l)$.

Notice that

$$Oc(\eta, \eta') \geq (\eta'_{k'}, \mathbf{a}).$$

This part of the proof would be over, if $(\eta'_{k'}, \mathbf{a}) > (T_{ab} - 4\delta) N$.

Indeed, suppose that there is $\eta \in \Sigma$, such that $(\eta, \mathbf{a}) \leq (T_{ab} - 4\delta) N$. We prove that there is no path of coalitional best responses towards b, which would lead from η to any state $\eta' \in W_1$, $(\eta', \mathbf{a}) \geq (\gamma_{ab} - 2\delta) N$. Assume w.l.o.g. that small groups are labeled according to the increasing number of players playing action a in state η : for any two groups $s < s'$, $a_s(\eta) \leq a_{s'}(\eta)$. We have a lemma:

Lemma 4. *Suppose that $(\eta, \mathbf{a}) \leq (T_{ab} - 4\delta) N$. There is a group $s^* < \gamma_{ab} \frac{N}{d+1}$, such that for all $s \geq s^*$*

$$\pi_b(\alpha(s^*), 0) \leq \pi_a(\alpha(s^*), a_s(\eta)) - \delta(1-\tau)\left(u^B(a,a) - u^B(a,b)\right)$$

where

$$\alpha(s) = \frac{d+1}{N} \sum_{s'=s+1}^{\frac{N}{d+1}} a_{s'}(\eta)$$

is the proportion of all players which belong to group $s' \geq s$ and play action a in state η.

Proof. Since $a_s(\eta)$ is increasing with s, it is enough to show the existence of group $s^* < \gamma_{ab} \frac{N}{d+1}$.

Suppose not and there is no group s^* like this. Then for all $s \leq \gamma_{ab} \frac{N}{d+1}$,

$$\pi_b(\alpha(s), 0) \geq \pi_a(\alpha(s), a_s(\eta)) - \delta(1-\tau)\left(u^B(a,a) - u^B(a,b)\right).$$

We may restate it after some manipulations with

$$a_s(\eta) \leq 1 - \frac{c_{ab}}{\gamma_{ab}}(\alpha(s) - \gamma_{ba}) + \delta \leq 1 - \frac{c_{ab}}{\gamma_{ab}} \frac{1}{N} \sum_{s'=s+1}^{\gamma_{ab}\frac{N}{d+1}} a_{s'}(\eta) + \delta.$$

Suppose that $y(x)$ for $x \in [0, \gamma_{ab}]$ is a solution to the differential equation

$$y' = \frac{c_{ab}}{\gamma_{ab}} y \text{ with an initial condition } y(\gamma_{ab}) = 1 + \delta.$$

Then, we observe that

$$a_s(\eta) \leq y\left(s\frac{d+1}{N}\right) = (1+\delta)\exp\left(-\frac{c_{ab}}{\gamma_{ab}}\left(\gamma_{ab}\frac{N}{d+1} - s\right)\right).$$

Further,

$$\begin{aligned}\frac{1}{N}(\eta, \mathbf{a}) &= 1 - \frac{(d+1)}{N}\sum_{s'=1}^{\frac{N}{d+1}} a_{s'}(\eta) \\ &\geq 1 - \gamma_{ba} - \int_0^{\gamma_{ab}} y(x)\,dx = \gamma_{ab} - \frac{\gamma_{ab}}{c_{ab}}(1+\delta)\left[1 - e^{-c_{ab}}\right] = T_{ab} - \delta.\end{aligned}$$

But this yields a contradiction. □

Construct now a state $\eta^* \geq_b \eta$, such that $\eta^*(i) = b$ for any player i, st. $s(i) < s^*$ and $\eta^*(i) = \eta(i)$ for any player i, st. $s(i) \geq s^*$. Notice that

$$\alpha(s^*) = \frac{(\eta^*, \mathbf{a})}{N}.$$

We show that there is no path of coalitional best responses leading from state η^* to $\eta' \in W_1$, such that each coalition on the path has a size smaller than n_f. By the strategic complementarity, there is no a similar path of coalitional best responses leading from η to state $\eta' \in W_1$.

Let

$$y_a = \frac{\delta}{10}, y_b = 2\frac{\varepsilon}{y_a}.$$

Suppose that $\eta^{**} \geq_b \eta^*$ is a state reached by the coalitional best response from η^* and such that $\frac{1}{N}(\eta^*, \eta^{**}) \in [y_a, 2y_a]$. Then, there is a group of at most $d+1$ players, all of whom play b in state η^{**} and a in state η^*, and for whom it is a strict coalitional best response to switch jointly to a in state η^{**}. By the strategic complementarity, this contradicts the choice of η^{**} as state reached by coalitional best response path from η^*.

Indeed, denote set of players $C = \{i : \eta^{**}(i) = b \text{ and } \eta^*(i) = a\}$ who play action a in state η^* and b in state η^*. There are at least $y_a N$ and at most $2y_a N$ players in set C. By Propositions 11 and 12, at most εN players have less than $d\left(\frac{(\eta^*, \mathbf{a})}{N} - 2y_a\right)(1-\varepsilon)$ neighbors in network A playing a in state η^{**}. Call the set of these players C_ε. There is a group $\hat{s}$, and a player $i \in C \backslash C_\varepsilon$, $s(i) = \hat{s}$, such that at most $y_b(d+1)$ of other members of group $\hat{s}$ belong to set C_ε (otherwise, one would need at least $y_a y_b N = 2\varepsilon N$ players in C_ε). Consider all members of group $\hat{s}$, which belong to $C \backslash C_\varepsilon$. If they continue playing b, then their payoff is bounded from above by

$$\pi_b\left(\left(\frac{(\eta^*, \mathbf{a})}{N} - 2y_a\right)(1-\varepsilon), 0\right)$$
$$\leq \pi_b(\alpha(s^*), 0) + \tau\left[u^A(b,b) - u^A(b,a)\right](2y_a + \varepsilon).$$

If all players in $C \backslash C_\varepsilon$ switched back to a, then their payoff afterwards would be bounded from below by

$$\pi_a\left(\left(1 - \frac{(\eta^*, \mathbf{a})}{N} - 2y_a\right)(1-\varepsilon), a_s(\eta) - y_b\right)$$
$$\geq \pi_a(\alpha(s^*), a_s(\eta)) - \tau(2y_a + \varepsilon)\left[u^A(a,a) - u^A(a,b)\right] - (1-\tau)\, y_b\left[u^B(a,a) - u^B(a,b)\right].$$

Thus, by the Lemma, action a is a strict coalitional best response for these players when

$$\delta(1-\tau)\left(u^B(a,a) - u^B(a,b)\right) > 6y_a\tau\left[u^A(a,a) - u^A(a,b)\right] + 2\frac{\varepsilon}{y_a}(1-\tau)\left[u^B(a,a) - u^B(a,b)\right].$$

But this is true by the choice of y_a, y_b, ε and δ.

C.1.3. *Ad 3).* We show that there is a path from $\mathbf{b}$ to $\mathbf{a}$ of cost at most $T_{ba} + 3\delta$. We start with a lemma:

Lemma 5. *There is state η^*, $(\eta^*, \mathbf{b}) \leq T_{ba} + 3\delta$, such that for all groups s,*

$$\pi_a(\alpha(s), 1) \geq \pi_b(\alpha(s), a_{\eta^*}(s)) + \delta(1-\tau)\left(u^B(a,a) - u^B(a,b)\right)$$

where

$$\alpha(s) = \frac{d+1}{N}\left(s - 1 + \sum_{s=s+1}^{\frac{N}{d+1}} a_s(\eta)\right).$$

was defined in Lemma 4.

This lemma and its proof is a direct counterpart of Lemma 4 - details are available on request.

Now, suppose that η^{**} is the last which can be reached from η^* through a path of coalitional best responses towards action a, where each coalition has a size not bigger than d. By the strategic complementarity, such a state exists, $\eta^{**} \geq_a \eta^*$ and there is no coalitional best response towards action a which would involve a coalition of size d or smaller. We show that it must be that $(\eta^{**}, \mathbf{a}) \leq 2\delta$.

Suppose not. Let $y_b = \frac{\delta}{10}$. All players, except for, possibly, set C_ε of at most εN agents, have at least $d\frac{(\eta^{**},\mathbf{a})}{N}(1-\varepsilon)$ neighbors who play action a. There is at most $\frac{\varepsilon}{y_b}$ proportion of the groups who have more than $y_b(d+1)$ members in set C_ε Let s^* be the first group, which has less than $y_b(d+1)$ members in set C_ε and contains a player who plays b :

$$s^* = \inf \left\{ \begin{array}{c} s : \#\{i : s(i) = s \text{ and } i \in C_\varepsilon\} \leq y_b(d+1) \\ \text{and there is } i^*,\ s(i^*) = i^* \text{ and } \eta^{**}(i) = b \end{array} \right\}.$$

Then

$$\alpha(s^*) \leq \frac{(\eta^{**},\mathbf{a})}{N} + \frac{\varepsilon}{y_b} + \frac{s^* N}{d+1} y_b.$$

The payoff of all players b in small group s^* in state η^{**} is bounded from above by

$$\begin{aligned} &\pi_b\left(\frac{(\eta^{**},\mathbf{a})}{N}(1-\varepsilon), a_{\eta^{**}}(s^*)\right) \\ &\leq \pi_b(\alpha(s^*), a_{\eta^*}(s^*)) + \left(\varepsilon + \frac{\varepsilon}{y_b} + y_b\right)\tau\left[u^A(b,b) - u^A(b,a)\right]. \end{aligned}$$

(notice that $a_{\eta^{**}}(s^*) \geq a_{\eta^*}(s^*)$). On the other hand, the payoff from the coalitional best response towards a (if all members of s^* who do not belong top C_ε switched to a) is bounded from below by

$$\begin{aligned} &\pi_a\left(\frac{(\eta^{**},\mathbf{a})}{N}(1-\varepsilon), 1-y_b\right) \\ &\geq \pi_a(\alpha(s^*), 1) - \tau\left(\varepsilon + \frac{\varepsilon}{y_b} + y_b\right)\left[u^A(a,a) - u^A(a,b)\right] - (1-\tau)\, y_b\left[u^B(a,a) - u^B(a,b)\right]. \end{aligned}$$

Thus, action a is a coalitional best response when

$$\delta > 2\left(\varepsilon + \frac{\varepsilon}{y_b} + y_b\right)\frac{\tau\left[u^A(b,b) - u^A(b,a)\right]}{(1-\tau)\left(u^B(a,a) - u^B(a,b)\right)} + y_b.$$

The last inequality holds by the choice of y_b, δ and ε.

All the proofs in the rest of the section are applications of Theorem 2.

C.2. **Proof of proposition 2.** Set $\tau = 1$, $u^A(.,.) = u^B(.,.)$ and suppose that action a is risk dominant. Then, $T_{ab} = 1-\rho$ and $T_{ba} = \rho$.

C.3. **Proof of proposition 3 .** Suppose that $\tau = \frac{1}{2}$ and the payoffs of the interactions are given in table (4.1). We compute that $T_{ab} = \frac{u^A - u^B}{u^A}$ and $T_{ba} = \frac{u^B}{u^A e}$.

C.4. **Proof of proposition 6.** Suppose that action a is risk-dominant in network A and in the average game and efficient in network B. We show that $T_{ba} < T_{ab}$. Observe first that if $c_{ba} < 0$, then action a is a group best response when all other groups are playing b and covention **a** is the unique stochastically stable state. Assume that $c_{ba} \geq 0$. Denote

$$\begin{aligned} r_A &= u^A(a,a) + u^A(a,b) - u^A(b,a) - u^A(b,b) \\ r_{av} &= \tau\left(u^A(a,a) + u^A(a,b) - u^A(b,a) - u^A(b,b)\right) \\ &\quad + (1-\tau)\left(u^B(a,a) + u^B(a,b) - u^B(b,a) - u^B(b,b)\right), \\ p &= u^B(a,a) - u^B(b,b) \\ c &= \tau\left(u^A(a,a) + u^A(b,b) - u^A(a,b) - u^A(b,a)\right). \end{aligned}$$

$r_A \geq 0$ is a measure of the risk-dominance of a in network A, $r_{av} \geq 0$ is a measure of the average risk-dominance of a, $p \geq 0$ is a measure of the efficiency of a in network B and c is an useful constant. Using the definition of the variables $\gamma_{\cdot}$ and $c_{\cdot}$ and the fact that $e^x \geq 1 + x$, we compute

$$\begin{aligned} T_{ab} - T_{ba} &= (\gamma_{ab} - \gamma_{ba}) - \left(\frac{\gamma_{ab}}{c_{ab}} - \frac{\gamma_{ba}}{c_{ba}}\right) + \frac{\gamma_{ab}}{c_{ab}}e^{-c_{ab}} - \frac{\gamma_{ba}}{c_{ba}}e^{-c_{ba}} \\ &= \frac{1}{c}\left(c\frac{\gamma_{ab}}{c_{ab}}e^{-c_{ab}} - c\frac{\gamma_{ba}}{c_{ba}}e^{-c_{ba}} + r_{av}\right) \\ &\geq \frac{e^{-c_{ba}}}{c}\left(c\frac{\gamma_{ab}}{c_{ab}}(c_{ba} - c_{ab}) + c\frac{\gamma_{ab}}{c_{ab}} - c\frac{\gamma_{ba}}{c_{ba}} + e^{c_{ba}}r_{av}\right) \\ &= \frac{e^{-c_{ba}}}{c}\left(c_{ba}\left(2(1-\tau)p + \tau r_A\right) + \left(e^{c_{ba}} - c_{ba} - 1\right)r_{av}\right). \end{aligned}$$

Since $c_{ba} \geq 0$, the last expression is positive.

C.5. **Proof of proposition 7.** Suppose that the assumptions of the Proposition are satisfied. We show that $T_{ab} - T_{ba}$ as a function of τ is single-crossing in 0. In other words, if, for some configuration of parameters $T_{ba} = T_{ab} = T$, then

$$\frac{d\left(T_{ab} - T_{ba}\right)}{d\tau} > 0.$$

We use the same notation as in the previous Proposition: $p < 0$ and $r_A > 0$ and $r_{av} > 0$. Additionally, we normalize the payoffs in such way that $u^A(a,a) + u^A(b,b) - u^A(a,b) -$

$u^A(b,a) = 1$ and $c = \tau$. Let us compute

$$\begin{aligned}
\frac{d\gamma_{ab}}{d\tau} &= -\frac{1}{\tau^2}p, \ \frac{d\gamma_{ba}}{d\tau} = \frac{1}{\tau^2}p, \\
\frac{\gamma_{ab}}{c_{ab}}\frac{dc_{ab}}{d\tau} &= \frac{1}{(1-\tau)}\frac{u^A(a,a) - u^A(b,a)}{\tau} = \frac{1+r_A}{2\tau(1-\tau)}, \\
\frac{\gamma_{ba}}{c_{ba}}\frac{dc_{ba}}{d\tau} &= \frac{1-r_A}{2\tau(1-\tau)}.
\end{aligned}$$

We may use the computations from the previous Proposition (note that $T_{ab} = T_{ba}$)

$$\begin{aligned}
&\frac{d\tau(T_{ab} - T_{ba})}{d\tau} \\
&= \frac{d}{d\tau}\left[c\frac{\gamma_{ab}}{c_{ab}}e^{-c_{ab}} - c\frac{\gamma_{ba}}{c_{ba}}e^{-c_{ba}} + r_{av}\right] \\
&= -\frac{1}{1-\tau}(T_{ab} - T_{ba}) + \frac{r_{av}}{(1-\tau)} + \frac{1+r_A}{2(1-\tau)}e^{-c_{ab}} - \frac{1-r_A}{2(1-\tau)}e^{-c_{ba}} + \frac{dr_{av}}{d\tau} \\
&= \frac{1+r_A}{2(1-\tau)}e^{-c_{ab}} - \frac{1-r_A}{2(1-\tau)}e^{-c_{ba}} + \frac{r_A}{(1-\tau)} \\
&= \frac{1}{2(1-\tau)}\left[\left(e^{-c_{ba}} - e^{-c_{ab}}\right) + r_A\left(2 - e^{-c_{ba}} - e^{-c_{ab}}\right)\right].
\end{aligned}$$

The second term is clearly positive. The first term is positive as long as $c_{ba} < c_{ab}$. Notice that $\gamma_{ab} - \gamma_{ba} = 2(1-\tau)p + \tau r_A < 0$. Indeed, if not, then $2(1-\tau)p + \tau r_A \geq 0$, $r_{av} > 0$ and, as the first section implies, action a is chosen in some neighborhood of the parameters. But this would mean a contradiction with $T_{ba} = T_{ab}$. Thus, $\gamma_{ba} > \gamma_{ab}$. Since

$$\gamma_{ba}\left(1 + \frac{1}{s_{ba}}\left(e^{-s_{ba}} - 1\right)\right) = \gamma_{ab}\left(1 + \frac{1}{s_{ab}}\left(e^{-s_{ab}} - 1\right)\right).$$

it is sufficient to show that function $f(x) = 1 + \frac{1}{x}(e^{-x} - 1)$ is increasing in $x \geq 0$. Let us compute

$$\begin{aligned}
f'(x) &= \frac{1}{x^2} - \frac{1}{x^2}e^{-x} - \frac{1}{x}e^{-x} \text{ or} \\
x^2 f'(x) &= 1 - e^{-x} - xe^{-x}.
\end{aligned}$$

When $x = 0$, then $\lim_{x\to 0} x^2 f'(x) = 0$. Similarly, $\lim_{x\to\infty} x^2 f(x) = 1$. Also, the derivative of $x^2 f'(x)$ is positive

$$\left(x^2 f'(x)\right)' = e^{-x} - e^{-x} + xe^{-x} = xe^{-x} > 0.$$

Appendix D. Equilibrium selection with large groups

This part of the appendix deals with the proof of Proposition 4. The argument presented here is for a case of $u^A > u^B$. The other case is analogous.

A proportion of of the whole population playing a is denoted with $a(\eta)$ and a proportion of members of group $k = 1, ..., K$, who play action a in state η is denoted with $a_k(\eta)$:

$$a_k(\eta) = \frac{1}{N/K} \# \{i : i \text{ belongs to large group } k \text{ and } \eta(i) = a\}.$$

Choose ε and δ small enough, $2\varepsilon < \delta$, $u^A > \frac{1+\varepsilon}{1-\varepsilon}\frac{1+2K^2\delta}{1-2K^2\delta}u^B$.. Take d sufficiently high, so that the thesis of Proposition 3 hold for ε and random graphs $G\left(N, \frac{d}{N}\right)$ and $G\left(\frac{1}{K}N, \frac{Kd^B}{N}\right)$. For any $0 \leq k_1 < k \leq k_2 \leq K$ find positive constants $\gamma_{k_1,k_2}(k) \in (0,1)$ as solutions to the system of equations

$$\frac{k}{K} + \sum_{k'=k+1}^{k_2} \frac{\gamma_{k_1,k_2}(k')}{K} = 1 - \gamma_{k_1,k_2}(k).$$

One may check that $\gamma_{k_1,k_2}(k)$ increases (weakly) with k.

For $k = 0, ..., K$ define sets

$$Y_k = \left\{ \begin{array}{c} \eta : \text{ there is set } C \subseteq \{0, ..., K\} \text{ of large groups, } |C| = K - k, \\ st. \, \forall k' \in C \;\, u^A a(\eta) \geq u^B (1 - a_{k'}(\eta)) + 2\delta \end{array} \right\} \backslash Y_{k-1},$$

$$W_k = \left\{ \begin{array}{c} \eta : \text{ there is set } C \subseteq \{0, ..., K\} \text{ of large groups, } |C| = K - k, \\ \text{for any group } k' \in C, \; a_k(\eta) \geq 1 - 2\delta\frac{N}{K} \end{array} \right\} \cap Y_k.$$

Set W_k consists of states which have $K-k$ groups coordinating on action a (passive) and the rest (approximately) prefer to play action b. In set W_0, all groups coordinate (approximately) on action a.

We show in three steps that assumptions of Proposition 10 are satisfied:

(1) For any $\eta \in Y_k$, there is a path of individual best responses leading to state in $\bigcup_{k' \leq k} W_{k'}$.

(2) For any $k_1 < k_2$, transition from W_{k_1} to W_{k_2} costs more than $\frac{1}{K}\sum_{k=k_1+1}^{k_2} \gamma_{k_1,k_2}(k) N$,

$$\min_{\eta \in W_{k_1}} \min_{\eta' \in \bigcup_{k' \geq k_2} W_{k'}} Oc(\eta, \eta') \geq \sum_{k=k_1+1}^{k_2} \frac{\gamma_{k_1,k_2}(k)}{K} N + \delta N.$$

(3) For any $k_1 > k_2$, transition from W_{k_1} to W_{k_2} costs less than $\sum_{k=K-k_1+1}^{K-k_2} \gamma_{K-k_1,K-k_2}(k) N$,

$$\max_{\eta \in W_{k_1}} \min_{\eta' \in \bigcup_{k' \leq k_2} W_{k'}} Oc(\eta, \eta') \leq \sum_{k=K-k_1+1}^{K-k_2} \frac{\gamma_{K-k_1,K-k_2}}{K}(k) N.$$

This ends the proof of the Proposition.

D.1. **Ad 1).** Suppose that there is a large group k, such that $a_k(\eta) < 1 - \delta N$ and $u^A a(\eta) \geq u^B(1 - a_k(\eta)) + 2\delta$. There are at most $2\varepsilon \frac{N}{K}$ members of group k, that have more than $(1 - a_k(\eta))(1 + \varepsilon)$ neighbors in network B playing B or less than $a(\eta)(1 - \varepsilon)$ neighbors in network A playing a. Since $2\varepsilon < \delta$, there is at least one player who plays b and strictly prefers to play a.

By the strategic complementarity, this implies the existence of a path of individual best responses leading from state η to some state η', such that $a_k(\eta') \geq 1 - \delta N$. But then, a similar argument to the one used in section C.1.1 shows that there is no (coalitional) best response path leading to state η'', such that $a_k(\eta'') < 1 - 2\delta N$.

D.2. **Ad 2).** The argument follows the one used in section C.1.2. Suppose that $\eta \in W_{k_1}$ and a state $\eta' \leq_a \eta$ is such that

$$(\eta, \eta') \leq \sum_{k=k_1+1}^{k_2} \frac{\gamma_{k_1,k_2}(k)}{K} N + \delta N.$$

W.l.o.g. we assume that $a_k(\eta')$ is weakly decreasing in k. There is a group $k_1 < k^*$, such that for any $k \geq k^*$

$$\frac{k^* - 1}{K} + \sum_{k=k^*}^{k_2} \frac{1 - a_k(\eta')}{K} < a_{k^*}(\eta) + K\delta.$$

This comes from the definition of constants $\gamma_{k_1,k_2}(k)$.

Consider a state η^*, such that all members of groups $1, ..., k^* - 1$ play b and all members of the group with a higher index play as in state η'. We show that there is no path of coalitional best responses (of coalitions of size n_f) leading from η^* to a state in $W_{k'}$. There is no group of players of size δN who play a in state η^*, but would be better off playing b if all of them switched jointly to b. Indeed, the payoff from playing a of all but at most εN players is not smaller than

$$(1-\varepsilon)\,u^{A}\left(\frac{K-k_{2}}{K}+\sum_{k=k^{*}}^{k_{2}}\frac{a_{k}\left(\eta'\right)}{K}-\delta\right)$$
$$=(1-\varepsilon)\,u^{A}\left(1-\frac{k^{*}-1}{K}-\sum_{k=k^{*}}^{k_{2}}\frac{1-a_{k}\left(\eta'\right)}{K}-\delta\right).$$

After the coalitional action of δN, payoff of all but at most εN from playing b cannot be higher than

$$(1+\varepsilon)\,u_{B}\left(1-a_{k^{*}}(\eta)+\delta\right).$$

By the choice of parameters, the latter is smaller than the former.

D.3. **Ad 3).** Take any state $\eta\in W_{k_1}$. W.l.o.g. we may assume that $a_k(\eta)$ is (weakly) decreasing. We construct a state η' in the following way. For any any group $K-k_1<k\leq K-k_2$, take $\gamma_{K-k_1,K-k_2}(k)\,\frac{N}{K}$ of its members who play action b in state η and switch their actions to a. For any other group, let its members play as in state η. Then

$$(\eta,\eta')=\sum_{k=K-k_{1}+1}^{K-k_{2}}\frac{\gamma_{K-k_{1},K-k_{2}}}{K}(k)\,N.$$

Let η^* be the last state reacheable from η' through a sequence of individual best responses towards action a. We show that at least $K-k_2$ groups have at least $1-2\delta\frac{N}{K}$ members playing action a.

Let $k^*\geq K-k_1+1$ be the lowest number of a large group, such that $a_{k^*}(\eta)\leq 1-2\delta\frac{N}{K}$. If $k^*>K-k_2$, then the claim is proven. Suppose not, $k^*\leq K-k_2$. We show that there is at least one individual in group k^* who plays b, but would prefer to play a. First observe that all but $2\delta\frac{N}{K}$ members of groups $k<k^*$ and at least $\gamma_{k_1,k_2}(k)\,\frac{N}{K}$ of members of groups $k^*\leq k\leq K-k_2$ play action a. Thus, for all but, possibly, $\varepsilon\frac{N}{K}$ members of group k^*, the payoff from playing a is not smaller than

$$u^{A}(1-\varepsilon)\left(\frac{k^{*}-1}{K}+\sum_{k=k^{*}}^{K-k_{2}}\frac{\gamma_{K-k_{1},K-k_{2}}(k)}{K}\right).$$

On the other hand, except for possibly $\varepsilon\frac{N}{K}$) members of group k^*, the payoff from playing b to all the other members of k^* is equal to at most

$$u^{B}(1+\varepsilon)\left(1-\gamma_{K-k_{1},K-k_{2}}\left(k^{*}\right)\right).$$

By the choice of parameters, the latter is smaller than the former.

Appendix E. Self-segregation

Proof of lemma 1. The first assertion is immediate, so we focus on the second. The payoff of every player from convention **e** is equal to

$$w\left(e^{*},e^{*}\right)=\frac{\left(2p-1\right)^{2}w}{e^{*}\left(2p-1\right)+1-p}\left(e^{*}\right)^{2}+\frac{2\left(1-p\right)pw}{\bar{e}\left(2p-1\right)+1-p}e^{*},$$

which is equal to the player's highest possible payoff from the investment into education. However, if player i and all of her neighbors switched to the identity investment: her payoff then would be equal to

$$U^{C}\left(e^{*}\right)^{2}+w\left(0,e^{*}\right)=U^{C}\left(e^{*}\right)^{2}+\frac{2\left(1-p\right)pw}{\bar{e}\left(2p-1\right)+1-p}e^{*},$$

which is also equal to the highest possible payoff from the identity investment.[22] When $w<w_2$, the latter is higher than the former and she (though not necessarily her neighbors) would prefer such a situation to convention **e**. When $w>w_2$, the latter is smaller than the former and convention **e** is payoff dominant. □

[22]When d is small relative to N, a change of behavior of $d+1$ players does not affect the average level of educational investment $\bar{e}=e^{*}$.