TRACTABLE MODEL OF DYNAMIC MANY-TO-MANY MATCHING

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Abstract. We develop a tractable, dynamic, and strategic model of many-to-many matching with payoff externalities across links. The joint dynamic surplus or certain second properties of individual utilities, like payoff externalities, can typically be identified. We characterize a class of interior equilibria as solutions to an optimization problem with an objective function that consists of welfare minus an inefficiency loss term. In equilibrium, too few matches are formed. We compare transferable and non-transferable versions of the model; the equilibria of the two versions are equivalent up to a re-scaling of parameters. We describe the asymptotic limits of disappearing frictions.

1. Introduction

This paper develops a tractable, search-based, dynamic model of many-to many matching, with simple identification and welfare analysis. Our model is very general and it allows for a wide variety of applications including marriage search, job search with multiple jobs per worker and multiple workers per firm, co-authorship or friendship networks. The main difficulty is that the decisions to form a match typically confound many considerations: the current and future direct payoff consequences of the two matched partners, the payoff consequences from other past and future matches, the equilibrium conditions that determine the likelihood of future matches, etc. To give a very simple example, consider a (small) clinic that hires two types of workers: doctors and nurses. The workers care only about whether they are employed or not. Hospital payoffs depend on the size and composition of its labor force, including the possibility

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of externalities between doctors and nurses. When hiring a doctor, the hospital needs to consider not only doctor quality, but also her compatibility with its current labor force, andthe likelihood of hiring a complementary nurse.In our model, all these considerations can be disentangled. Closed-form formulas are derived to identify various parameters of the model. Among others, we give an explicit formula to identify the level and the sign of complementarity in the hospital example.

In the model, decisions to form matches are made sequentially, as best responses to the strategies of the other players. Agents meet each other randomly, and they simultaneously decide whether to accept the match. Each agent is characterized by a type, which may contain information about permanent characteristics of the agent, and/or the agent's history. The type can stochastically change between periods, possibly depending on the type of the matching partner. The agents can simultaneously hold multiple matches. Each decision balances both present payoffs and expected future utility consequences (including the potential for future matches).

As in the discrete choice literature, we assume that the payoffs include both systematic utility and a random idiosyncratic payoff shock. The systematic utility depends on the agent's own type, whether the match is formed and, if so, the partner's type. The utility may depend on the entire history of past matches through the type, and we allow for general payoff externalities between matches formed in different periods. In particular, we allow for positive complementarities between the workers in the hospital example. The payoff shocks are exponentially distributed. This assumption ensures tractability of the model and it plays the same role as the extreme value type I errors in the discrete choice literature.

The first class of results concerns identification. Contrary to the dynamic discrete choice (DDC) literature on dynamic games (Aguirregabiria and Mira (2007) and Bajari *et al.* (2007)), the individual systematic utility is typically not identified. The main reason is that, as is typical in the matching literature, we assume that individual decisions about whether to form a match are *not* observed, and data contain only information about outcomes, i.e., whether the match is formed. As in the rest of the matching literature (Dagsvik (2000), Choo and Siow (2006), and Choo (2015)), we can only hope to identify joint surplus of the match. Additionally, we cannot identify the respective utility levels, but only match surpluses: differences between the matched and unmatched payoffs. We define a reduced form parameter - *joint dynamic surplus* -

as the sum of the match surpluses and their dynamic consequences experienced by two matching partners. The joint dynamic surplus can be identified from data on the mass of current and future matches through an explicit formula. Our tool for identification, the matching function, generalizes the result from Choo (2015) to models of strategic and/or many-to-many matching.

A new result that is specific to the many-to-many matching aspect of the model is that certain *second-order properties* of individual utilities can be identified even without observing individual match decisions. These properties are discrete counterparts of second-order cross-derivatives of the utility function and they measure the degree of payoff externalities between matches. For instance, in the hospital example, the payoff complementarity between doctors and nurses can be identified from the differential hiring rates across hospitals with different employment composition. We present a general theory for the identification of second-order properties. The idea is to use the above identification of the joint dynamic surplus and the fact the utility of the current matching partner does not depend on the agent's past matches; by taking the difference of the matching function, we can identify the impact of past matches on the individual surplus from the current match.

The next class of results concerns welfare comparisons. We show that average welfare can be represented as a function of the distributions over outcomes, i.e., formed matches and unmatched agents. The function consists of easy to interpret terms: average systematic utility, a measure of the selectivity of individual acceptance strategies, a measure of search externalities present in the model, and an impact of market frictions. Further, if equilibrium strategies are interior (i.e., matches can be rejected with a positive probability) and market frictions have separable form, we show that equilibrium distribution can be found as a critical point of an optimization problem. The objective function is equal to average welfare minus an efficiency loss term. The efficiency loss is due to the fact that the agents do not internalize the positive externality of forming a match on their partner's utility. In effect, too few matches are formed in equilibrium. Welfare can be improved by an easy-to-calculate match subsidy. Apart from welfare analysis, the result proposes a method of finding an equilibrium that is less computationally demanding that solving for a fixed point.

We compare our basic non-transferable utility (NTU) model with two transferable utility (TU) version, where transfers are negotiated through either take-it-or-leave-it or

Nash bargaining. We show that the TU versions are equivalent to the original model with modified parameters. In particular, all the above results (including identification) carry over to the TU model. Also, it is impossible to use choice data to distinguish between the NTU and TU models. (This latter fact coincides with the characterization of testable implications of stability in Echenique *et al.* (2013). A similar relation holds between the static cooperative NTU model of Dagsvik (2000) and the TU model of Choo and Siow (2006)).

Finally, we illustrate some applications of the general model to the special case of a dynamic model of marriage. We show that if search frictions disappear and the weight of the payoff shocks is reduced relative to the systematic utility, the stationary equilibrium converges to the distribution of outcomes under stable matching with transferable utility. We compare this result to Lauermann and Nöldeke (2014) which consider a model without payoff shocks and show that, under disappearing search frictions, stable NTU matching is an equilibrium. We also show that, if the types are one-dimensional, the supermodularity of the payoff function is sufficient to guarantee assortative matching. The last property distinguishes our model from Shimer and Smith (2000). We explain that the difference is due to the role played by exponential payoff shocks.

1.1. **Literature.** Single-agent dynamic discrete choice models were first introduced in Wolpin (1984) and Rust (1987) (for a recent survey, see Aguirregabiria and Mira (2010)). Aguirregabiria and Mira (2007) and Bajari *et al.* (2007) extend the DDC literature to games. Our approach differs in a few ways. Most importantly, we assume that only the match outcome (formation of lack thereof) is observed, and not individual decisions. Thus, we can identify the joint surplus, but not the individual payoffs. Further, we assume that the payoff shocks are distributed exponentially. This assumption is analogous to the extreme value type I error distributions in logit models in that it allows for explicit computations and model tractability. We also assume that the choice is always binary: accept or reject the match. Despite these differences, our single-agent results are related. The single-agent's identification is related to results from Hotz and Miller (1993) and Arcidiacono and Miller (2011).

A typical empirical work on matching relies on static models with cooperative solution concepts from classic matching theory (for the NTU case, see Dagsvik (2000), Menzel (2015a); for TU, see Choo and Siow (2006), Galichon and Salanie (2012), Fox (2010) among others. Also, see Graham (2011) for a review). In the NTU case, the cooperative matching literature based on stable matching that originated in Gale and Shapley (1962) has developed powerful tools to study social and economic relationships. Nevertheless, the existence of stable matching is not guaranteed beyond the marriage market. To avoid this problem, the literature typically either makes studies approximate solutions (for instance, Tan (1991), Kojima *et al.* (2013), Che *et al.* (2015)) or it makes restrictive payoff assumptions (see Kelso and Crawford (1982), Chung (2000), Hatfield and Milgrom (2005), Echenique and Oviedo (2006) and Ostrovsky (2008), among others). For example, Hatfield and Milgrom (2005) assume that workers are substitutes, which precludes the possibility of complementarity between doctors and nurses as in our example.

Choo (2015) considers a dynamic marriage search model. In each period, an agent decides whether to marry or stay single in the next period. There are two main differences with our paper. First, Choo (2015) clears the markets through the cooperative TU model of Choo and Siow (2006), while we are consistently strategic and our paper allows for both NTU and TU interpretations. Second, we allow for multiple matches for each agent, whereas Choo (2015) focuses on (monogamous) marriage. A main result in Choo (2015) shows that the "joint marriage utility" is identified. Despite the differences, we show that that a single-match version of our model has a remarkably similar identification result.

A main contribution of our paper is that it allows for simple and explicit identification in search-based network formation applications with non-trivial payoff externalities between connections. Such models are recognized as being very important for studying various social phenomena (Jackson (2010), Jackson *et al.* (2017)). At the same time, the econometric analysis of such models is typically very complex (see, for instance Menzel (2015b)). Mele (2017) considers a non-equilibrium model of network formation. While the utility allows for some dependence on network properties beyond the composition of individual links, the linear payoff assumption does not allow for payoff externalities between links that play an important role in many situations.

Our model is related to a substantial literature on search models in macroeconomics, labor, and theory (for instance, Burdett and Coles (1997), Wright and Burdett (1998); for search with transferable utility, see Shimer and Smith (2000) and Atakan (2006)). In particular, we assume random meeting of candidates for a match. Additionally, the TU model of take-it-or-leave-it bargaining is very similar to an unemployment search literature where firms "post" wages, but search remains undirected (for instance, Burdett and Judd (1983), Albrecht and Axell (1984), or Burdett and Mortensen (1998)). An important difference is that, unlike the above papers, we are not restricted to the one-dimensional case, and our analysis does not rely on the assortativeness of the matching.

There are few search papers with multi-dimensional preferences. Lauermann and Nöldeke (2014) considers a search model of marriage and frictions due to discounting. They show that, even if frictions disappear, the distributions over matches in some equilibria are not necessarily stable. Another exception is Coles and Francesconi (2016) which studies search with multidimensional types and random shocks. The authors prove the existence of equilibrium and use computational methods and calibration to estimate the differential impact of female features on match incentives.

1.2. **Overview.** Section 2 describes the model. Section 3 derives the matching function and other identification results. Section 4 derives the welfare formula and provides a characterization of an equilibrium. Section 5 describes an extension to the TU bargaining model. Section 6 discusses the application to the special case of dynamic marriage matching. All proofs can be found in the Appendix.

2. Model

A continuum population of agents lives in discrete time $t = 1, 2, ...$ In each period, each agent is characterized by type $x \in X$, where X is a finite set. Agent type is typically not permanent and may change depending on agent behavior. In each period, a mass $Q_t(x)$ of agents are born.

In each period, any two agents may meet at random. The mass of meetings between agents of type x and y is given by meeting function $q_t\left(x, y; \mu^X\right)$ and depend on the agent types as well as the mass distribution $\mu^X \in \mathbb{R}_+^X$ of agents in the population. (We discuss the meeting rates in more detail below.) Each agent observes the type of the other agent as well as an i.i.d. exponentially distributed payoff shock ε. The shocks are independent across agents, matches, and periods. Each agent simultaneously decides whether to accept the match. The match is formed only if it is accepted by both agents.

If a type x agent forms a match with a type y agent, agent x receives payoff equal to the systematic utility term $v_t(x, y)$ plus the payoff shock. The agent's type in the next period becomes x' with probability $P_t(x'|x, y)$ that may depend on her own current type as well as the type of her match partner. If the match is not formed, agent x receives a payoff $\upsilon_t(x, \emptyset)$ (and no payoff shock) and her new type is drawn with probability $P_t(x'|x, \emptyset)$. The agent x dies with probability $1 - \sum_{x'} P_t(x'|x, a)$, where $a \in X \cup \{\emptyset\}$. Agents discount the future with a factor $\beta < 1$.

The model is completely characterized by the systematic utility υ, birth rates Q, transition probabilities P, and meeting function q_t. In general, the parameters of the model may depend on time.

2.1. **Discussion.** The model is quite general. We illustrate its scope with a few applications.

Example 1. (Marriage matching.) The set of types is a disjoint union $X = M \cup F$ of male types M and female types F. The types contain information about the personal characteristics, education level, or economic situation of an agent. The types do not change. Each unmatched agent receives utility $\upsilon(x, \emptyset) = 0$ and dies (i.e., leaves from the market) at exogenous rate $1 - \delta > 0$. An agent x matched with agent y receives utility $\upsilon(x, y)$ and leaves the market.

Example 2. (1-to-1 matching with age.) In a version of the above example, each agent has type $x \in \{0, ..., T\}$, interpreted as her age. In each period that an agent $x < T$ does not find a match, her type increases by 1. If an agent finds a match, or the type is equal to T, the agent exits the market.

The identification for situations as in Example 2 has been studied in Dagsvik (2000), Choo and Siow (2006), and Menzel (2015a). Example 2 is a non-cooperative version of the model analyzed in Choo (2015). As far as we know, our paper is the first to study identification in many-to-one matchings or network formation cases described in the next three examples.

Example 3. (Hospitals with doctors and nurses.) Hospitals search to hire doctors and nurses. Each worker is one of two types $x \in \{d, n\}$. The worker type does not change. In each period, a mass $Q(x)$ of workers is born. The workers die (i.e., leave the market) at rate $\delta < 1$ in each period. There is a continuum (mass 1) of infinitely-lived hospitals. At any period, the type of a hospital is defined as its employment size and composition $m = (m_d, m_n)$, where m_x is the number of type x workers. To respect the finiteness constraint, we assume that no hospital can hire more than $m_d+m_n \leq m^0 < \infty$ workers. The hospital type evolves with the hiring decisions and with the workers' death events. The worker's systematic payoff from a job is υ_x and it may depend on her type. The hospital's systematic payoff depends on its labor size and composition as follows

$$\upsilon(m_d, m_n) = \sum m_x \pi_x + \theta m_d m_n,$$

where π_x is the worker's systematic productivity, and parameter θ measures the increase in marginal productivity of a worker in one occupation due to the hiring of another worker in a different occupation. Workers remain employed with the hospital until their death. In particular, as long as the worker are alive, their employment affects the payoffs from integrating future workers. Workers and firms discount the future with factor $\beta < 1$.

Example 4. (Co-authorship network.) There is a continuum population of agents with (permanent) characteristics $x_0 \in X_0$ representing their research area, ability, etc. In each period, a mass $Q(x_0)$ of x_0 agents are born. The agents die after $T > 0$ periods. When two agents meet, they decide whether they want to co-author a paper together. We define agent type $x = (x_0, s, c) \in X = X_0 \times \{0, ...T\} \times \{0, ...T\}^{X_0}$ as a tuple of the agent's own permanent characteristics x_0, age $s \leq T$, and the count variable c, where for each $\xi \in X_0$, $c(\xi)$ is the number of past papers co-authored with agents of characteristics ξ. Agent age increases by 1 in each period, and the agent dies after age T. Whenever an agent of (x_0, s, c) matches with a type $(y_0, u, d) \in X$, the next period count c is replaced by $c + \mathbb{1}_{y_0}$. (Similarly, for the co-author, her count variable d is replaced by $d + \mathbb{1}_{x_0}$.) The utility from the match depends on the agent's permanent characteristics and the characteristics of all his match partners (but on their own past co-authorships) $\upsilon((x_0, s, c), a) = \upsilon^*(x_0, c + \mathbb{1}_a)$, where $\mathbb{1}_a(\xi) = \begin{cases} 1, & \text{if } \xi = a \\ 0 & \text{otherwise.} \end{cases}$

Example 5. (Friendship network.) There is a continuum population of agents. Agent type consists of a tuple (x_0, s) where x_0 is the agent's permanent characteristics, and $s : X_0 \to \{0, 1, 2, ...\}$ is the count variable of the number $s(y_0)$ of friends with characteristics y_0 that the agent has. The agents meet each other at random, at rates that depend on their population sizes; a friendship is formed only if both agents agree on it. Agents are born with no friends at rate $Q(x_0, \emptyset)$, sever friendships at rate s, and they die at rate $\delta < s$. The utility of each agent depends on her own type as well as the structure of friendship sets $v_t(x_0, s)$. [1]

Our model has some limitations. First, we assume that the only possible decisions are to accept or to reject a match with another agent.The scope of decisions can be expanded by including, for instance, the possibility of matchings between more than two agents (three-authored papers in Example 4), or allowing for single-agent decisions. Second, the current model does not allow for post-match interactions or decisions. Such decisions are clearly important in some cases (for example, fertility choice in the context of marriage). At some cost of extra notation, it is possible to add post-match decisions to the model without a loss of tractability, or the main results. We discuss such extensions in the online Appendix (Peski (2019)).

Another limitation is that the model does not allow for a separation of previously formed matches. An important difficulty with separation in many-to-many matching model is that the decision to form a match has to take into account the likelihood that the current partner may want to separate, and because her future decisions will typically depend on her future type, my own decision will depend on the expected evolution of the partner type. In turn, the latter depends on her partners, and, by extension, by her partners' partners, etc. In short, each decision depends on the entire network of matches. For any practical application, one would have to localize the network somehow, and we leave the question of how to do it for future research.

A simpler case is to consider separation in the many-to-one matching model (as in Example 3). In such a case, there is a natural local network that consists of one large agent with multiple small agents connected to her. We show in the online Appendix (Peski (2019)) that separations can be added in a way that is analogous to the one proposed in Goussé *et al.* (2017) for dynamic marriage matching.

[1] Currarini *et al.* (2009) build a search-based model of the formation of friendships to study the qualitative patterns of homophily and heterophily among high school students. Their continuous-time model does not have stochastic shocks and allows for the decision to exit from the market and stop searching. Only the latter difference is substantive. (The individual decisions like the decision to drop out from the market can be easily added to our model - see below and the the online Appendix (Peski (2019)).)

Finally, we describe two limitations in the applications to the network formation (as described in Example 4). First, the finiteness of X eliminates the possibility that the utility of an agent can depend on past matches of the partner, the past matches of the past matches, etc. Also, we do not allow for meetings between past co-authors (in our model such matches happen with probability 0). Both restrictions can be important for some applications.

2.2. **Single-agent behavior.** A (threshold) *strategy* is a mapping $\sigma : \mathbb{N}\times X\times X \to \mathbb{R}^{+}$, with the interpretation that agent x accepts the match with y iff $\varepsilon \geq \sigma_t(x, y)$. The restriction to threshold strategies is w.l.o.g.

Let $r_t(x, y)$ denote the probability that an agent x meets an agent y who wants to form a match in period t. This probability is endogenous and it is determined in equilibrium. Let $U_t(x;\sigma)$ denote the present value of an agent who begins period t in state x and uses strategy σ. Let

$$U_t(x, a; \sigma) := v_t(x, a) + \beta \sum_{x'} P_t(x'|x, a)\, U_{t+1}(x'; \sigma) \tag{1}$$

be the expected continuation of an agent who forms a match a (or stays unmatched, if $a = \emptyset$). Then,

$$\begin{aligned} U_t(x;\sigma) :=& \sum_{y\in X} r_t(x,y)\, e^{-\sigma_t(x,y)} \left[U_t(x,y;\sigma) + E(\varepsilon|\varepsilon \geq \sigma_t(x,y))\right] \\ &+ \left(1 - \sum_{y\in X} r_t(x,y)\, e^{-\sigma_t(x,y)}\right) U_t(x,\emptyset;\sigma) \\ =& U_t(x,\emptyset;\sigma) + \sum_{y\in X} r_t(x,y)\, e^{-\sigma_t(x,y)} \left[U_t(x,y;\sigma) - U_t(x,\emptyset;\sigma) + 1 + \sigma_t(x,y)\right]. \end{aligned} \tag{2}$$

The second equality comes from the key property of the exponential distribution: for any $q \geq 0$,

$$E(\varepsilon|\varepsilon \geq q) = q + 1. \tag{3}$$

Let $U_t(x) = \max_\sigma U_t(x;\sigma)$ and $U_t(x,a) = \max_\sigma U_t(x,a;\sigma)$ denote the continuation values. Optimization over t-period thresholds in the last line of (2) yields *best response* strategy

$$\sigma_t(x,y) = \max\left(0, U_t(x,\emptyset) - U_t(x,y)\right). \tag{4}$$

The hiring threshold is equal to either the lowest value of payoff shock, or to the loss in the continuation utility following a match, whichever one is higher. (Strictly speaking, the best response threshold is unique only if $r_t(x,y) > 0$. Equation (4) will hold in any equilibrium that satisfies a trembling hand refinement.)

2.3. **Population.** Let $\mu_t(x,y)$ denote the equilibrium mass of matches by agents x with agents y in period t, and let $\mu_t(x,\emptyset)$ denote the mass of type x agents who are unmatched in period t. Let $\mu_t^X(x) = \sum_{a\in X\cup\{\emptyset\}} \mu_t(x,a)$ be the mass of all agents type x.

Potential match partners are randomly chosen from a continuum population. In each period, the mass of meetings between agents x and y depends on the mass of agents of each type and is described by function $q_t\left(x,y;\mu_t^X\right) = q_t\left(x,y;\mu_t^X\right)$. The meeting rate can be used to vary match frequencies, or even prohibit matches between agents of particular types. A natural example is when the mass of meetings between types x and y is proportional to the masses of agents x and y:

$$q_t\left(x,y;\mu_t^X\right) = q_t^0(x,y)\,\mu_t^X(x)\,\mu_t^X(y)\,, \tag{5}$$

where $q_t^0(x,y) > 0$ is a constant.

Because the best responses are unique in our model, we assume that all agents in the population use the same strategy σ. The probability that an agent x meets another agent y who wants to match with her is equal to

$$r_t(x,y) = \frac{1}{\mu_t^X(x)} q_t\left(x,y;\mu_t^X\right) e^{-\sigma_t(y,x)}, \tag{6}$$

and it is equal to the mass of meetings between xs and ys divided by the mass of x and multiplied by the probability that a y accepts the match with an x. The masses of formed matches as well as the agents who remain unmatched in period t satisfy the following equations:

$$\begin{aligned}\mu_t(x,y) &= \mu_t^X(x)\,r_t(x,y)\,e^{-\sigma_t(x,y)} &\text{(7a)}\\ &= q_t\left(x,y;\mu_t^X\right)\mathrm{e}^{-\sigma_t(x,y)}\mathrm{e}^{-\sigma_t(y,x)},\ \text{for each } x,y\in X, &\\ \mu_t(x,\emptyset) &= \mu_t^X(x) - \sum_y \mu_t(x,y)\ \text{ for each } x\in X. &\text{(7b)}\end{aligned}$$

The population dynamics are given by

$$\mu_t^X(x') = Q_t(x') + \sum_{x\in X, a\in X\cup\{\emptyset\}} \mu_{t-1}(x,a)\,P_{t-1}(x'|x,a)\ \text{ for each } x' \text{ and } t. \tag{7c}$$

2.4. **Equilibrium.** We employ the following standard equilibrium concept.

Definition 1. A tuple $(\mu_t, \sigma_t, U_t(.))_t$ is an *equilibrium* if the continuation values are determined through equations (1) and (2), the strategies satisfy (4) and if the masses evolve according to (7a)-(7c) for some initial distribution μ_0. The equilibrium is stationary if the equilibrium variables do not depend on time, with $\mu_0 = \mu_t$ determined in equilibrium.

Assumption 1. *We assume that*

$$\sup_{t,x,a} |\upsilon_t(x,a)|, \sup_{t,x} Q_t(x) < \infty,$$

and for each x, y,

$$\frac{q_t\left(x, y; \mu_t^X\right)}{\mu_t^X(x)} \leq 1,$$

and the right-hand side is continuous in μ_t^X.

Theorem 1. *Given Assumption 1 , there exists an equilibrium for each initial state distribution μ_0. If the parameters of the model (i.e., υ, q, Q, P) do not depend on time, there is a stationary equilibrium.*

We say that μ are *equilibrium masses* if there is a strategy σ and continuation values $U_t(.)$ such that (μ, σ, U) is an equilibrium. In empirical applications, typically only the masses are observed. As we explain later, the masses provide a partial information about the strategy.

2.5. **Interior equilibrium.** A strategy σ is *interior* if $\sigma_t(x,y) > 0$ for all $x, y \in X$ and $t \geq 0$. An equilibrium is *interior* if σ is interior. By (4), the equilibrium strategies are interior if

$$\sigma_t(x,y) = U_t(x,\emptyset) - U_t(x,y) > 0, \tag{8}$$

i.e., if each match is costly and the lowest quality matches are always rejected. The cost can be exogenous, in the form of reduced systematic utility. At the end of Section 4, we establish a simple sufficient condition for the existence of an interior equilibrium that relies on such cost. The cost can also be endogenous, as a foregone opportunity of future matches. The latter kind of cost plays an important role in the one-to-one matching model analyzed in Section 6.

In the interior equilibrium, equation (2) implies that

$$
\begin{aligned}
U_t(x) =& U_t(x,\emptyset) + \sum_{y\in X} r_t(x,y)\, e^{-\sigma_t(x,y)} \\
=& U_t(x,\emptyset) + \frac{\sum_{y\in X}\mu_t(x,y)}{\mu_t^X(x)} = U_t(x,\emptyset) + \tilde{\mu}_t(x)\,, \qquad (9)
\end{aligned}
$$

where we denote $\tilde{\mu}_t(x) = \sum_{y\in X}\frac{\mu_t(x,y)}{\mu_t^X(x)} = 1 - \frac{\mu_t(x,\emptyset)}{\mu_t^X(x)}$ as the conditional probability that type x forms a match in period t. The above equation suggests that $\tilde{\mu}_t(x)$ can be interpreted as the (equilibrium) option value of being able to form a match in period t.

A recursive application of (1) and the above equation leads to the following formula for the continuation values given realized match $a \in X \cup \{\emptyset\}$:

$$
\begin{aligned}
U_t(x,a) =& \upsilon_t(x,a) + \beta\sum_{x_{t+1}} P_t(x_{t+1}|x,a)\left(U_{t+1}(x_{t+1},\emptyset) + \tilde{\mu}_{t+1}(x_{t+1})\right) \qquad (10)\\
=& \upsilon_t(x,a) + \beta\sum_{x_{t+1}} P_t(x_{t+1}|x,a)\left(\upsilon_{t+1}(x_{t+1},\emptyset) + \tilde{\mu}_{t+1}(x_{t+1})\right)\\
&+ \beta^2 \sum_{x_{t+1},x_{t+2}} P_t(x_{t+1}|x_t,a)\, P_t(x_{t+2}|x_{t+1},\emptyset)\left(U_{t+2}(x_{t+2},\emptyset) + \tilde{\mu}_{t+2}(x')\right)\\
=& \ldots\\
=& V_t^0(x,a) + \sum_{s>t}\beta^{s-t}\sum_{x'} P_t^s(x'|x,a)\,\tilde{\mu}_s(x')\,,
\end{aligned}
$$

where $P_t^{t+1} = P_t$ and, for each $s > t+1$, we recursively define (a) the s-period probability distribution of agents that are type x in period t, form a realized match a, and remain unmatched in any subsequent period before period s as

$$
P_t^s(x_s|x,a) := \sum_{x_{s-1}} P_t^{s-1}(x_{s-1}|x,a)\, P_s(x_s|x_{s-1},\emptyset)
$$

and (b) the present expected value of the stream of systematic utility generated by the strategy of never forming a match as

$$
V_t^0(x,a) := \upsilon_t(x,a) + \sum_{s>t}\beta^{s-t}\sum_{x'} P_t^s(x'|x,a)\,\upsilon_s(x',\emptyset) \qquad (11)
$$

In other words, the continuation value of an agent x with match a is equal to the present value of a strategy of never forming a match and the option value of being able to form matches in the future. [2]

2.6. **Reduced-form parameters.** For each $x \in X$ and $a \in X \cup \{\emptyset\}$, define *individual dynamic surplus* as

$$V_t(x,a) := V_t^0(x,a) - V_t^0(x,\emptyset). \tag{12}$$

The dynamic surplus is equal to the increase in the present value of never forming a match caused by match a. The above expression appears in the characterization of continuation values (10) and shall play an important role in the remainder of the paper. Importantly, $V_t(.)$ are defined using only the exogenous parameters. Moreover, it turns out that, in a well-defined sense, it is impossible to distinguish between an agent with systematic utilities $\upsilon_t(.,.)$ and an agent with utilities $V_t(.,.)$. To make this claim precise, let $U_t(x,a;\sigma,\upsilon)$ denote the present values defined above but where we explicitly refer to the utility function.

Lemma 1. *For each t,x,y, each strategy σ (and each r),*

$$U_t(x,y;\sigma,\upsilon) - U_t(x,\emptyset;\sigma,\upsilon) = U_t(x,y;\sigma,V) - U_t(x,\emptyset;\sigma,V).$$

Recall that the best response behavior depends only on the difference between the continuation values if the match is formed and if in there is no match (see (8)). The Lemma implies that the best response behaviors of agents with utilities υ and V are identical. In other words, V_t play the role of structural parameters.

Henceforth, we shall assume that agents act *as if* they have utility $V_t(.,.)$. The Lemma holds for all strategies, including non-interior ones.

3. Identification

This section contains the identification results. The first result shows that the joint dynamic surplus of two match partners can be identified from the matching function. We illustrate with an application to Example 3. In the example, a certain second-order

[2] Formula (10) is closely related to Theorem 1 from Arcidiacono and Miller (2011), which is stated for an arbitrary benchmark strategy and an arbitrary distribution of payoff shocks with continuous density. Here, the natural benchmark strategy is to always reject the match, the distribution of payoff shocks is exponential for $a \neq \emptyset$, and it is equal to 0 for $a = \emptyset$.

property of the individual utility can be identified from data that without information on individual decisions. We present a general theory of such identification in the last part of this section.

For the identification results, we assume that the masses of matched agents $\mu_t(x,y)$ as well as unmatched agents $\mu_t(x,\emptyset)$ in each period are observed. Such data are typically available in the marriage or labor search context (if there is sufficient information about hirings and unemployment). Similarly, such data seem to be easily available in the co-authorship case. Additionally, to keep the results focused, we assume that the econometrician knows the non-utility parameters of the model: discount factor, transition probabilities, and meeting rate. Typically, these are not observed directly and must be identified from data. (Identification methods are discussed in the literature. For example, Mourifie (2017) discusses the identification of meeting rates; similarly, Goussé *et al.* (2017) describe the identification of meeting and death rates.)

3.1. **Matching function.** Substituting (8) and (10) into the evolution equations (7a) yields the *matching function*:

Theorem 2. *If* μ *are masses in the interior equilibrium, then*

$$\begin{aligned}\log\mu_t(x,y) = &\log q_t\left(x,y;\mu_t^X\right) + V_t(x,y) + V_t(y,x)\\ &+\sum_{s>t}\beta^{s-t}\sum_{x'}\left[P_t^s(x'|x,y) - P_t^s(x'|x,\emptyset)\right]\tilde{\mu}_s(x')\\ &+\sum_{s>t}\beta^{s-t}\sum_{y'}\left[P_t^s(y'|y,x) - P_t^s(y'|y,\emptyset)\right]\tilde{\mu}_s(y'). \end{aligned}\tag{13}$$

Equation (13) expresses the logarithm of the number of matches between types x and y in terms of the logarithm of the meeting rate, the surplus generated by the match, and the impact on the expected discounted probabilities of future matches. The last two terms measure the improvement in future matching prospects induced by the match. The larger the surplus generated by the match, and the larger the improvement in future matching prospects, the more matches are formed.

Matching function (13) is closely related to the dynamic marriage function in equation (3.1) of Choo (2015) (a great overview of different approaches to the matching function in the econometric and demographic literature can be found in Mourifie (2017)). Choo (2015) focuses on a one-to-one model of marriage with agents distinguished by age, whereas we look at many-to-many matching with general types. Additionally, our model includes the meeting function q_t that plays no role in demand-supply TU model of Choo (2015). The meeting function is an important component of search models, and it adds extra flexibility to modeling the impact of the masses of agents on the number of matches. Finally, as in Choo (2015), the future match probabilities can be used to compute the continuation payoffs. The details differ due to different assumptions on the payoff shock distribution, and a more general type dynamic in our paper.

3.2. **Joint surplus.** As in the above literature, the matching function is a basis for identification. Suppose that an econometrician has period-by-period data on the numbers of matches formed between different agent types (i.e., $\mu_t(x, y)$) as well as the number of agents who remain unmatched (i.e., $\mu_t(x, \emptyset)$). If the other parameters of the matching model: transition probabilities P_t, and meeting rate function $q_t(.)$ are observed as well, then the joint dynamic surplus

$$V_t(x, y) + V_t(y, x)$$

can be identified from observable data. We emphasize that $V_t(x, y)$ is an exogenous, reduced-form parameter of the model (see Lemma 1 and the following discussion). Identification of the joint dynamic surplus is a common feature of empirical matching models like (Dagsvik (2000), Choo and Siow (2006), and others).

Typically, individual utilities cannot be separately identified, in contrast to the literature that extends single-agent discrete choice models to dynamic games like Aguirregabiria and Mira (2007) and Bajari *et al.* (2007). In that literature, the actions of agents are typically observable. Here, we do not observe individual agent decisions, but only outcomes of their joint decisions. Thus, we cannot determine whether the lack of a match between x and y is due to lack of interest of one or the other or both agents. On the other hand, the lack of match is an unambiguous signal that the joint surplus is low.

As we explain below, given additional assumptions, certain properties of individual payoffs can be identified.

3.3. **Example: Payoff externalities.** We illustrate the identification using the special model from Example 3. We assume that the meeting function is equal to

$$q_t\left(x, (m_d, m_n); \mu^X\right) = \mu^X(x)\, \mu^X(m_d, m_n).$$

The increase in the worker's present value due to being hired is equal to

$$V\left(x,(m_d,m_n)\right)=\frac{1}{1-\beta\delta}\upsilon_x,$$

or the present value of the utility stream υ_x received before the worker dies. For a hospital with employment composition (m_d, m_n), the present value from a strategy of never hiring anyone else is equal to

$$\frac{1}{1-\beta\delta}\sum_x n_x\pi_x+\theta\frac{1}{1-\beta\delta^2}m_d m_n.$$

The increase in the present value due to hiring a doctor is equal to

$$V\left(\left(m_d,m_n\right),d\right)\frac{1}{1-\beta\delta}\pi_d+\theta\frac{1}{1-\beta\delta^2}m_n.$$

The joint surplus is equal to

$$V\left(x,(m_d,m_n)\right)+V\left(\left(m_d,m_n\right),x\right)=\frac{1}{1-\beta\delta}\left(\pi_x+\upsilon_x\right)+\theta\frac{1}{1-\beta\delta^2}m_{-x}.$$

The joint surplus is identified through the matching function (13):

$$\begin{aligned}&\frac{1}{1-\beta\delta}\left(\pi_x+\upsilon_x\right)+\theta\frac{1}{1-\beta\delta^2}m_{-x}\\ =&\log\mu_t\left(x,(m_d,m_n)\right)-\sum_{s>t}\left(\beta\delta\right)^{s-t}\tilde{\mu}_s\left(x'\right)+\sum_{s>t}\beta^{s-t}\left(\tilde{\mu}_s\left(m_x+1,m_{-x}\right)-\tilde{\mu}_s\left(m_x,m_{-x}\right)\right).\end{aligned}$$

The matching function holds for each x and each firm (m_d, m_n). In particular, one cannot separately identify the systematic productivity of a worker at the hospital π_x and her utility from employment υ_x.

On the other hand, we can separately identify the sum of job utilities versus parameter θ. Computing the difference of the above formula applied to two hospitals $m = (m_x, m_{-x})$ and $m' = (m_x, m_{-x} - 1)$ leads to

$$\begin{aligned}\theta\frac{1}{1-\beta\delta^2}=&\log\frac{\mu_t\left(x,(m_x,m_{-x})\right)}{\mu_t\left(x,(m_x,m_{-x}-1)\right)}\\ &+\sum_{s>t}\beta^{s-t}\left[\begin{array}{c}\left(\tilde{\mu}_s\left(m_x+1,m_{-x}\right)-\tilde{\mu}_s\left(m_x,m_{-x}\right)\right)\\ -\left(\tilde{\mu}_s\left(m_x+1,m_{-x}-1\right)-\tilde{\mu}_s\left(m_x,m_{-x}-1\right)\right)\end{array}\right].\end{aligned}$$

3.4. **Second-order properties.** In the above example, we are able to identify the complementarity parameter of the hospital's production function due to an implicit feature of the example: the fact that worker utility does not depend on the labor composition of the hospital. By taking the difference of the matching functions for hospitals with different composition, we can eliminate the worker utility from the equations.

We can generalize this observation. Suppose that agent type consists of two components $x = (x_1, x_2) \in X_1 \times X_2 = X$. Suppose both components affect the agent's own match utilities and transition probabilities; but only the first component affects the systematic match utility or transition probabilities of the partner in the match.

Corollary 1. *Suppose that for each $x = (x_1, x_2)$, each $y_1 \in Y_1$, and each $y_2, y_2' \in X_2$,*

$$V_t\left(\left(x_1, x_2\right), \left(y_1, y_2\right)\right) = V_t\left(\left(x_1, x_2\right), \left(y_1, y_2'\right)\right),$$
$$P_t\left(.|\left(x_1, x_2\right), \left(y_1, y_2\right)\right) = P_t\left(.|\left(x_1, x_2\right), \left(y_1, y_2'\right)\right).$$

Then, for each $x = (x_1, x_2), x' = (x_1, x_2')$, and $y = (y_1, y_2)$,

$$\begin{aligned} &V_t\left(x, y_1\right) - V_t\left(x', y_1\right) \qquad (14)\\ =&\log\frac{\mu_t\left(x, y\right)}{\mu_t\left(x', y\right)} - \log\frac{q_t\left(x, y; \mu\right)}{q_t\left(x', y; \mu\right)}\\ &-\sum_{s>t}\beta^{s-t}\sum_{\xi}\left[P_t^s\left(\xi|x, y_1\right) - P_t^s\left(\xi|x, \emptyset\right) - \left(P_t^s\left(\xi|x', y_1\right) - P_t^s\left(\xi|x', \emptyset\right)\right)\right]\tilde{\mu}_s\left(\xi\right).\end{aligned}$$

Proof. The proof comes from taking the difference between equation (13) for x and the analogous equation for x'. □

The LHS of (14) is a difference (in the second component) of the difference of the individual match surplus with y between two types x and x'. Because the match surplus is defined as the difference between the (dynamic) systematic utility under match and remaining unmatched, the formula identifies a difference of differences, or a second-order property of individual utility.

Example 6. (Continuation of Example 4). In the co-authorship network, neither the preferences nor the transition probabilities depend on the partner's past history of matches. We collect the latter in the second component of the type. The dynamic utility (11) can be written as a function $V_0^*(x_0, s, c')$ of own characteristics x_0, age s, and count $c' = c + \mathbb{1}_a$ of current and past match partners characteristics. Because

$V_t(x,a) = V_0^*(x_0,s,c+\mathbb{1}_a) - V_0^*(x_0,s,c)$, Corollary 1 implies that the second-order property

$$\begin{aligned} & V_t(x_0,s,c+\mathbb{1}_b,a) - V_t(x_0,s,c,a) \\ = & \left[V_0^*(x_0,s,c+\mathbb{1}_a+\mathbb{1}_b) - V_0^*(x_0,s,c+\mathbb{1}_b)\right] \\ - & \left[V_0^*(x_0,s,c+\mathbb{1}_a) - V_0^*(x_0,s,c)\right] \end{aligned}$$

is identified for each $a,b \in X_0$ and each $x_0 \in X_0, s \leq T, c \in \{0,...,T\}^{X_0}$.

4. Welfare

In this section, we discuss equilibrium payoffs. We derive a formula for average welfare as a sum of easy-to-interpret terms. We also show that if the meeting rate has the form given in equation (5), the equilibrium masses maximize a functional that is the sum of welfare and an inefficiency loss term. We use it to discuss the (in)efficiency of the equilibrium.

4.1. Single-agent payoffs.

We start with the payoffs of a single-agent. Let $U_t(x_0;\sigma,r)$ denote the expected continuation payoff of an agent with type x_0 in period t who uses strategy σ given the environment acceptance rates r. Such a strategy induces a distribution probability over future states. Let $\pi_s(x,a)$ be the induced probability that the agent is alive in period s, with type x and realizes match a. We have

$$\begin{aligned} \frac{\pi_s(x,y)}{\pi_s(x)} &= \mathrm{e}^{-\sigma_t(x,y)} r_t(x,y), \text{ and} \\ \pi_s(x) &= \sum_{x\in X, a\in X\cup\{\emptyset\}} \pi_{t-1}(x,a) P_{t-1}(x'|x,a). \end{aligned} \tag{15}$$

The single-agent expected payoffs are equal to

$$\begin{aligned} U_t(x_0;\sigma,r) &= \sum_{s\geq t}\sum_{x\in X, a\in X\cup\{0\}} \beta^{s-t}\left(V(x,a) + \mathbf{1}_{a\in X} E(\varepsilon|\varepsilon \geq \sigma_s(x,a))\right)\pi_s(x,a) \\ &= \sum_{s\geq t}\sum_{x,y\in X} \beta^{s-t}\left(V_s(x,y) + \sigma_s(x,y) + 1\right)\pi_s(x,y) \\ &= \sum_{s\geq t}\sum_{x,y\in X} \beta^{s-t}\left(V_s(x,y) + 1 - \log\frac{\pi_s(x,y)}{\pi_s(x)} + \log r_s(x,y)\right)\pi_s(x,y). \end{aligned} \tag{16}$$

(Recall that, following the discussion in Section 2.6, we assume that the agent's utilities are given by $V_t(x, a)$.) The second equality comes from the fact that $V(x, \emptyset) = 0$ and equation (3). The third equality comes from equation (15).

Formula (16) is closely related to well-known results from static discrete choice models (for instance, McFadden (1978)). It expresses the expected payoffs of an agent as a function of the distribution over types. The payoffs are equal to the sum of easy-to-interpret and compute terms. The first two terms in the brackets of the last line are equal to the systematic utility plus the average value of the payoff shock from accepted matches. The third term measures the selectiveness of the agent strategy. If the agent uses a higher acceptance threshold, this raises the conditional expected value of payoff shock (conditional on forming the match) and decreases the number of matches. The last term measures the impact of environmental acceptance rate r on payoffs. Keeping the numbers of formed matches μ constant, the higher acceptance rate r means that the agent can be more selective when making its own acceptance decision. That means higher thresholds, which raises payoffs.

One can show that formula (16) is strictly concave in π. That would imply that the expected payoffs can be maximized by a unique distribution π. Of course, we have already known that the best response strategy is unique from (4).

4.2. **Welfare.** The payoff formula aggregates well:

Theorem 3. *Given a strategy σ, masses μ that satisfy (7a)-(7c), and acceptance rates r, the aggregate welfare is equal to*

$$\begin{aligned} W(\mu) &= \sum_{t,x} \beta^t Q_t(x) U_t(x; \sigma, r) \qquad (17) \\ &= \sum_t \sum_{x,y \in X} \beta^t \left(V_t(x, y) + 1 - \frac{1}{2} \log \mu_t(x, y) + \frac{1}{2} \log q_t\left(x, y; \mu_t^X\right)\right) \mu_t(x, y). \end{aligned}$$

The proof of the Theorem can be found in the Appendix. The Theorem expresses population welfare as a function of the masses of types and matches. The welfare is decomposed into terms that are analogous to terms from the the single-agent case (equation (16)). In particular, the last term, $\frac{1}{2}\sum_t \sum_{x,y\in X} \beta^t \log q_t\left(x, y; \mu_t^X\right) \mu_t(x, y)$, comes from averaging the impact of the acceptance rates on payoffs. In the population, the last term is a measure of the externality that agents impose on others. We say that μ is *constrained efficient* if it maximizes the right-hand side of (17). Such μ would be chosen by a social planner whose goal is to maximize welfare, but who is constrained by the frictions of the matching process.

4.3. **Interior equilibrium.** The next result provides a characterization of interior equilibrium masses for a special case of the separable meeting function.

Theorem 4. *Suppose that the meeting function is given by (5). If μ^* are interior equilibrium masses, then they are a (constrained) critical point of the extremal problem:*

$$\text{"}\max_{\mu}\text{"}\ W(\mu) - \frac{1}{2}\sum_t \sum_{x,y\in X} \beta^t \mu_t(x,y) \ \ \textit{st.eq. (7b), (7c) and} \tag{18}$$

$$\mu_t(x,y) = \mu_t(y,x) \ \ \textit{for each } x,y \in X, \tag{19}$$

where $W(\mu)$ is defined in (17). Conversely, if μ is a (constrained) critical point of the above optimization problem such that for each t, and each $x,y\in X$,

$$V_t^0(x,\emptyset) - V_t^0(x,y) > \sum_{s>t} \beta^{s-t} \sum_{x'} \left[P_t^s(x'|x,y) - P_t^s(x'|x,\emptyset)\right] \tilde{\mu}_s(x'), \tag{20}$$

then μ are interior equilibrium masses.

Each interior equilibrium mass is a critical point of the extremal problem (18). The constraints come from the definition of equilibrium, with (7a) being replaced by its weaker version (19). The latter is a simple balance identity which says that the mass of x agents who match with ys is equal to the mass of y agents who match with xs. Formally, it is a consequence of the fact that the second line of (7a) is symmetric in x and y.

The objective function in (18) can be rewritten as

$$\sum_t \sum_{x,y\in X} \beta^t \left(V_t(x,y) + \frac{1}{2} + \frac{1}{2}\log q_t^0(x,y)\right)\mu_t(x,y) - \frac{1}{2}\sum_t \sum_{x,y\in X} \beta^t \mu_t(x,y) \log \frac{\mu_t(x,y)}{\mu_t^X(x)\,\mu_t^X(y)}.$$

Because of the last term, the function is not necessarily concave in μ and it may have multiple optima that correspond to different equilibria. The multiplicity of equilibria is due to search complementarities represented by the last term inside the brackets of (17). At the same time, representation (18) simplifies the problem of finding an equilibrium. Instead of looking for a dynamic fixed point, it is sufficient to find solutions to a constrained optimization problem. Computationally, this can be done either directly (for instance, using gradient methods), or by solving for the first-order conditions.

Theorems 3 and 4 lead to an intuitive welfare analysis. The equilibrium masses are not constrained efficient. There are two reasons for this. First, an equilibrium mass is a critical point, and not necessarily a maximizer of the functional (18). Second, even if the equilibrium mass is a maximizer, the value of the objective is smaller than welfare (17) by the second term of (18). For any equilibrium, welfare would increase if more matches were formed. The exact value of the welfare loss is due to the fact that individual best responses do not internalize the payoff shocks of the other agents. Indeed, notice that an agent with payoff shock equal to $\sigma_t(x, y)$ is indifferent between accepting or rejecting y. At the same time, conditional on the agent being pivotal, her partner expects a payoff increase of 1. The total welfare loss if the agent rejects the action is equal to -1, or $-\frac{1}{2}$ per agent.

Some welfare can be restored if shared actions are subsidized. Suppose that agent x receives a transfer $\tau_t(x, y)$ if she forms a match with agent y in period t. For any given μ, the subsidy increases the individual systematic utility to $V_t'(x, y) = V_t(x, y) + \tau_t(x, y)$, or the population welfare by $\tau \cdot \mu$. (Of course, total welfare, inclusive of the social planner, is not affected by transfers.) The two theorems imply that if $\tau = \frac{1}{2}$, then, there exists an equilibrium with subsidies that is constrained efficient.

The objective function has three components: the average individual utility, average market friction, and a term that resembles Shannon's entropy. The "utility plus entropy" formulas appear at the intersection of different economic literatures. First, such a representation is a well-known result in static logit models (see McFadden (1978)). Second, a connection between "utility plus entropy" and discrete choice (with a reverse direction of the argument) has been established in the literature on rational inattention (Sims (1998), Sims (2003)). Matejka and McKay (2015) show that the solution of the rational inattention problem is similar in form to the random outcome of the static discrete choice model of McFadden (1978) (this has been extended to dynamic choice in Steiner *et al.* (2015)). Third, the same formula appears in various static cooperative matching models. Choo and Siow (2006) employ a transferable utility model with extreme value type I errors. (Their error term is not entirely idiosyncratic as it is equal across all partners of the same type.) They show that the distribution of matches in stable outcome must satisfy a system of equations - it turns out that the equations are first order conditions to the "welfare plus entropy" maximization problem. An analogous result for the NTU model of Dagsvik (2000) is established in Menzel (2015a) under the assumption that the random utility terms satisfy a certain tail property. (Peski (2017) extends the result further to the roommate problem). Galichon and Salanie (2012) generalize the TU insight of Choo and Siow (2006) to arbitrary random term distributions. In order to obtain their characterization, they replace Shannon entropy by its generalized version. The common thread among all these papers is that the individual preferences are formed in a version of the static discrete choice model.

The characterization of an equilibrium as a solution to an extremal problem bears resemblance to a basic property of potential games (Monderer and Shapley (1996)). In such games, there exists a function of action profiles, called potential, with the property that each equilibrium is a local maximizer of the potential. The potential function, if it exists, has multiple applications beyond finding equilibrium (for instance, in learning and evolutionary theory, global games, etc.) Although our model is not a potential game, functional (18) may play a role of the potential in other applications.

The converse part of Theorem 4 leads to a simple sufficient condition that guarantees the existence of an interior equilibrium:

$$(1-\beta)\,V_t^0\,(x,\emptyset) > 1 + (1-\beta)\,V_t^0\,(x,y)\,,\text{ for each } t \text{ and } x,y \in X. \tag{21}$$

Here, $(1-\beta)\,V_t^0\,(x,a)$ is a normalized present value of the stream of systematic utilities obtained from strategies of never forming a match. In other words, an interior equilibrium exists if each new match has sufficient systematic cost. If condition (21) is satisfied, any constrained critical point of (18) is an interior equilibrium.

5. Transferable utility

In this section, we discuss two modifications (take-it-or-leave-it or Nash bargaining) of the original model of Section 2 to allow for transfers between players. In both cases, we show that the interior density μ is an equilibrium of the strategic model with transfers (TU) if and only if it is an equilibrium of the original NTU model with appropriately modified parameters.

5.1. **Take-it-or-leave-it bargaining.** We modify the model in Section 2 in the following ways. First, when agents x and y meet, one of them becomes an *employer* and the other becomes an *employee.* To fix attention, we assume that agent x becomes an employer with probability $p_t\,(x,y) = 1 - p_t\,(y,x)$, but none of the results nor the identification formula depends on the value of p_t. Second, only the employer observes a single payoff shock ε, which is independently and exponentially distributed. Third, the employee agent y submits a demand $d \in \mathbb{R}$ to the employer, upon which the employer decides whether to accept or reject the match.[3] If the match is accepted, it is formed, the employer receives $\varepsilon - d$ for herself, and the employee receives d. If it is rejected, the match is not formed and the agents go their separate ways. Otherwise, the two models are the same.

Let $d_t(x, y)$ be the demand submitted by employee y in a meeting with employer x and let $\sigma_t(x, y, d)$ be the threshold such that employer x accepts the match with employee y who demands d if $\varepsilon > \sigma_t(x, y, d)$.

Definition 2. A tuple $(\mu_t, (\sigma_t, d_t), U_t)$ is a TU_{TorL}*-equilibrium* if U_t describes the continuation payoffs, if demand d and acceptance strategies σ are best responses given the same strategies used by the others, and if masses satisfy equations (7b)-(7c), and for each $x, y \in X$,

$$\mu_t(x, y) = q_t\left(x, y; \mu_t^X\right)\left(p_t(x, y)\,\mathrm{e}^{-\sigma_t(x,y,d_t(x,y))} + p_t(y, x)\,\mathrm{e}^{-\sigma_t(y,x,d_t(y,x))}\right). \tag{22}$$

(The details of the definition can be found in Appendix C.) A TU-equilibrium is *interior* if actions are accepted only if the net transfers to all agents are strictly positive. (This corresponds to the property of an interior NTU equilibrium, where actions are accepted only if the payoff shock is strictly positive.) In Appendix C, we prove the following result.

Theorem 5. *Masses* (μ_t) *are interior* TU_{TorL}*-equilibrium masses of the model with meeting function* $q_.(.)$ *if and only if they are an interior (NTU) equilibrium of the model with parameters* q^{NTU}*, where for each* t*, each* $x, y \in X$ *and each* μ_t*,*

$$q_t^{NTU}\left(x, y; \mu_t^X\right) = \frac{1}{\mathrm{e}} q_t\left(x, y; \mu_t^X\right). \tag{23}$$

[3]For an example, consider a meeting between a candidate worker and a firm. The firm makes an offer that may depend on its own history as well as whatever the firm learns about the worker. The worker observes privately an idiosyncratic payoff shock and decides whether to accept the offer. The payoff shock may include factors like goodness of fit with unobserved worker abilities, convenience of commute, etc. Such a model of bargaining is very similar to the macro-labor literature, where firms "post" wages, but search is undirected (for instance, Burdett and Judd (1983), Albrecht and Axell (1984), or Burdett and Mortensen (1998)).

The Theorem establishes an equivalence between the two models. All the equilibrium results (the equilibrium characterization of Theorem 4) as well as the the identification from the next section apply to the TU model with modified parameters. An important consequence is that there is no way to distinguish the two models from choice data.

The proof makes clear that the expected payoffs in the two models are identical. For example, if the meeting rate has form (5), the equilibrium welfare of the TU model is given by formula (17) with q replaced by q^{NTU}. Because the equilibrium density is a critical point of (18), we obtain the result that the TU equilibrium density is not efficient. There are two reasons for this inefficiency. As in the NTU case, some of the inefficiency is due to coordination problems in search and the resulting multiplicity of equilibria. But there is a second source of inefficiency due to take-it-or-leave-it bargaining. Because demands are submitted without knowing the value of the payoff shock, the employer is forced to reject some efficient matches. A subsidy of 1 given to either the employer or the employee upon successful formation of the match could push the behavior towards removing the second source of inefficiency.

5.2. **Nash bargaining.** Instead of the above, suppose that an exponential match-specific shock is observed by two agents who, if the match is formed, divide the match surplus through a Nash bargaining with possibly type-dependent bargaining power. The match is formed if the payoff shock exceeds the sum of the utility losses from the match, or when

$$\varepsilon \geq U_t\left(x, \emptyset\right) - U_t\left(x, y\right) + U_t\left(y, \emptyset\right) - U_t\left(y, x\right) =: \sigma_t\left(x, y\right) + \sigma_t\left(y, x\right). \tag{24}$$

In an interior equilibrium, $\sigma_t\left(x, y\right) > 0$, and the probability of the match is equal to

$$\mu_t\left(x, y\right) = q_t\left(x, y; \mu_t^X\right) \mathrm{e}^{-\left(\sigma_t(x,y)+\sigma_t(y,x)\right)}. \tag{25}$$

The match surplus is equal to $\varepsilon - \left(\sigma_t\left(x, y\right) + \sigma_t\left(y, x\right)\right)$. The fraction of the payoff shock that goes to agent x is equal to

$$\tau_t\left(x, y, \varepsilon\right) = U_t\left(x, \emptyset\right) - U_t\left(x, y\right) + \gamma_t\left(x, y\right)\left(\varepsilon - \left(\sigma_t\left(x, y\right) + \sigma_t\left(y, x\right)\right)\right), \tag{26}$$

where parameter $\gamma_t\left(x, y\right)$ measures the bargaining power of agent x in matching with y; we have $\gamma_t\left(x, y\right) + \gamma_t\left(y, x\right) = 1$.

Equation (2) is replaced by

$$
\begin{aligned}
U_t(x) :=& \sum_{y\in X}\frac{\mu_t(x,y)}{\mu_t^X(x)}\left[U_t(x,y)+E\left(\tau_t(x,y,\varepsilon)\,|\varepsilon\geq\sigma_t(x,y)+\sigma_t(y,x)\right)\right] \\
&+\left(1-\sum_{y\in X}\frac{\mu_t(x,y)}{\mu_t^X(x)}\right)U_t(x,\emptyset) \\
=&\sum_{y\in X}\frac{\mu_t(x,y)}{\mu_t^X(x)}\left[U_t(x,y)+U_t(x,\emptyset)-U_t(x,y)+\gamma_t(x,y)\right]+\left(1-\sum_{y\in X}\frac{\mu_t(x,y)}{\mu_t^X(x)}\right)U_t(x,\emptyset) \\
=&U_t(x,\emptyset;\sigma)+\sum_{y\in X}\frac{\gamma_t(x,y)\,\mu_t(x,y)}{\mu_t^X(x)}=U_t(x,\emptyset;\sigma)+\tilde{\mu}_t^{NB}(x)\,,
\end{aligned}
\tag{27}
$$

where $\tilde{\mu}_t^{NB}(x)=\sum_{y\in X}\frac{\gamma_t(x,y)\mu_t(x,y)}{\mu_t^X(x)}$ replaces the probability of forming a match from equation (9).

Definition 3. A tuple (μ_t,σ_t,U_t) is an interior *TU_{NB}-equilibrium* if the continuation values are determined through equations (1) and (2), if the thresholds $\sigma_t(x,y)$ satisfy (8), and if the masses evolve according to (25) and (7b)-(7c) for some initial distribution μ_0.

As in the previous case, we show that a $\mathrm{TU}_{\mathrm{NB}}$-equilibrium is equivalent to the NTU-equilibrium of a model with modified parameters. The proof can be found in Appendix C.

Theorem 6. *Masses (μ_t) are interior* $\mathrm{TU}_{\mathrm{NB}}$*-equilibrium masses of the model with systematic utilities υ_t if and only if they are an interior (NTU) equilibrium of the model with utilities υ_t^{NTU}, where for each t, each $x\in X, a\in X\cup\{\emptyset\}$, and each μ_t,*

$$
\upsilon_t^{NTU}(x,a)=\upsilon_t(x,a)+\tilde{\mu}_t^{NB}(x)-\tilde{\mu}_t(x)\,. \tag{28}
$$

All the equilibrium results from the original NTU model, including identification and the equilibrium characterization of Theorem 4, apply to a TU model with modified parameters.

5.3. **Observed transfers.** In some situations, an econometrician may observe additional data on transfers, like wages in the labor setting or individual profit shares in a partnership[4]. These additional data can be used to improve the scope of identification.

[4]I am grateful to an anonymous referee for this comment.

In the context of take-it-or-leave-it bargaining, suppose that the demands of the employee are observed. In Appendix C, we derive that, in equilibrium, if x is an employee, he demands

$$d_t(x,y) = U_t(x,\emptyset) - U_t(x,y) + 1.$$

Hence, using (10), we obtain an explicit identification equation for employee x individual dynamic surplus (12):

$$V_t(x,y) = 1 - d_t(x,y) - \sum_{s>t} \beta^{s-t} \sum_{x'} \left[P_t^s(x'|x,y) - P_t^s(x'|x,\emptyset)\right] \tilde{\mu}_s(x').$$

Next, identification of the employer's individual dynamic surplus can be obtained the matching function (13).

A similar exercise can be conducted in the context of Nash bargaining. Suppose that the transfers (say, profit shares) to each party are observed. Assume also that the bargaining shares γ_t are known or separately identified. Then, one can use (26) to compute the difference in continuation values $U_t(x,\emptyset) - U_t(x,y)$, which leads to the identification of the individual dynamic surpluses.

6. Dynamic marriage matching

We illustrate the application of our model for the special case of stationary dynamic marriage matching with non-transferable utility. We show that if search frictions are small, all stationary equilibria are interior. In the limit of the model where both search and payoff shock frictions disappear, the equilibrium distribution converges to the distribution of outcomes under stable matching *with transferable utility.* Finally, we consider a special case of one-dimensional types and we show that, in such a case, interior matching is assortative.

We work with Example 1. Thus, $P(x|x,\emptyset) = \delta$ and $P_t(x|x,y) = 0$ for each x and y who are two types of opposite sex.[5] We assume that male agents m and female agents f meet at rate

$$q_t(m,f;\mu) = q^0(m,f)\,\mu^X(f)\,\mu^X(m),$$

where $q^0(m,f) > 0$.

[5]To focus our exposition, we assume that agents of the same sex do not meet. Allowing for smae-sex marriage leads to a consideration of additional cases, but it does not change the results.

Let $Q_F = \sum_{f \in F} Q(f)$ and $Q_M = \sum_{m \in M} Q(m)$. W.l.o.g., we assume that $Q_F \leq Q_M$. If $Q_F = Q_M$, we say that the markets are *balanced*.

6.1. **Search frictions.** Because of search frictions (discounting and probability of death), the agents may accept a match with a low payoff shock rather than continue searching. Under suitable conditions, if search frictions disappear, all stationary equilibria of the dynamic marriage matching model are interior.

Proposition 1. *Suppose that either (a) $Q_F = Q_M$ or (b) $V(m, f) < 0$ for each f, m. Then, there exists $\varepsilon > 0$ such that, if $\beta\delta > 1 - \varepsilon$, then each stationary equilibrium is interior. Moreover, as $1 - \beta\delta \to 0$, then for each $f \in F$,*

$$\sum_{m \in F} \mu(m, f) \to Q(f).$$

If markets are balanced and search frictions are low, we show that each agent can expect a relatively large number of offers before he or she finds a match. As a best response, she waits until she meets a match with a sufficiently high payoff shock. Asymptotically, all agents find their match.

When markets are not balanced, the above remains true for the agents on the short side of the market. Thus, all female equilibrium strategies are interior when search frictions are sufficiently low. On the other hand, only a fraction $\frac{Q_F}{Q_M} < 1$ of males are able to find a match before dying single. At each point in time, each male expects with a non-negligible probability to not receive any more offers. If the systematic utility is sufficiently high, the male is going to accept a match regardless of the payoff shock. To ensure that males use interior strategies, we assume that all systematic utilities are negative. (Note that this assumption has a simple interpretation: each agent on the long-side prefers to remain single rather than to marry the least attractive agent on the short side. That does not seem to be an unreasonable assumption for the marriage market.)

6.2. **Noiseless limit.** The second friction in the model is due to payoff shocks. We can consider reducing this friction by increasing the value of systematic utility relative to the payoff shock, or by replacing utilities with $V_\Lambda(x, y) := \Lambda V(x, y)$ and taking $\Lambda \to \infty$.

The effect of eliminating payoff friction depends on what happens to search frictions. If we directly take $\Lambda \to \infty$, then the model converges to the search NTU model without payoff shocks. Such a model was analyzed in Lauermann and Nöldeke (2014) who show that, as search frictions disappear, stable matches of the NTU cooperative model can be supported by the equilibria of the search model, but that the search model also has other equilibria (some of which can be interpreted as fractional matching).

That observation is not longer true if the order of limits is reversed. Let W^*be defined as the value of the following maximization problem

$$W^* := \max_{\mu:M\times F\to\mathbb{R}^+} \sum_{m,f} V(m,f)\,\mu(m,f) \tag{29}$$
$$\text{s.t. } \sum_m \mu(m,f)\,\text{st.} \ = Q(f)\,\text{for each } f\in F,$$
$$\sum_f \mu(m,f) \leq Q(m)\,\text{for each } m\in M.$$

W^* is the maximum average welfare that can be obtained from the systematic utility only if all agents on the short-side of the market are matched. As is well-known in the matching literature, solutions to problem (29) are match distributions in stable matchings with transferable utility and payoffs $V(x,y)$, and all females are matched.

Proposition 2. *Suppose that either (a) $Q_F = Q_M$ or (b) $V(m,f) < 0$ for each f, m. Then, for each $\varepsilon > 0$, there exists Λ_ε such that if $\Lambda \geq \Lambda_\varepsilon$, and $1-\beta\delta$ are sufficiently small (relative to Λ), then for each mass of matches $\mu(.,.)$ in a stationary equilibrium of the dynamic marriage matching model with payoffs $\Lambda V(x,y)$, we have*

$$\sum_{x,y} W(x,y)\,\mu(x,y) \geq W^* - \varepsilon.$$

The proof can be found in Appendix D.3.

The Proposition says that the frictionless limits of the equilibria in the dynamic search matching model with non-transferable utility are also cooperative outcomes of the frictionless model with non-transferable utility. The exact statement of the Proposition 2 heavily depends on the exponential assumption about the payoff shock. However, the intuition behind the connection between the NTU model with payoff shocks and the TU model is more general. The idea is that the acceptance thresholds in the search model play a role of the utility transfers. To see it, suppose that type f women reduce their acceptance threshold from $\sigma(f,m)$ to $\sigma'(f,m) < \sigma(f,m)$. This has two effects. On one hand, accepting less attractive males, reduces type$f$ women expected utility from their formed matches with men m. On the other hand, there are more women f accepting offers of type m men. This increases the chances that an m man finds an f woman above their acceptance threshold, which increases the m men expected payoff from search. In general, the utility transfer does not have to be 1 to 1. However, if the payoff frictions disappear (and the payoff shocks are distributed exponentially), the equilibrium strategies balance each other in exactly the right way to push the outcome towards fully transferable utility. A similar observation about the frictionless limit of the cooperative NTU model with payoff shocks was made in Peski (2017).

6.3. **Assortative one-to-one matching.** Finally, we consider a version of the model with 1-dimensional types, $M, F \subseteq \mathbb{R}$ and that the meetings are uniform $\rho \equiv 1$. Let μ be an equilibrium mass, and, for agents of two opposite sexes x and y, define

$$\hat{\mu}(y|x) = \frac{1}{\mu(x) - \mu(x, \emptyset)} \mu(x, y)$$

Then, $\hat{\mu}(.|x)$ is the conditional probability distribution over agent x's partner types, conditional on x form a matching in period t. The proof of the next result can be found in Appendix D.4.

Proposition 3. *Suppose that the joint surplus function $W(m, f)$ is supermodular. Let μ be an interior equilibrium mass. For each $x < x'$ and each t, $\hat{\mu}(.|x)$ is first-order stochastically dominated by $\hat{\mu}(.|x')$.*

Shimer and Smith (2000) consider a model without payoff shocks (and with transferable utility, but this latter difference is not important here) and argue that supermodularity is not sufficient to ensure assortativeness when the search cost is driven by discounting. (If the search cost is constant in each period and there is no discounting, Atakan (2006) shows that supermodularity is sufficient.) In order to explain the issue, we describe a simple example with two types, l and h (Table 1). Suppose first that, as in Shimer and Smith (2000), there are no payoff shocks and all the payoffs are as described above. In equilibrium, l-types will never say no to h-types. If there are very few h-types on the market, then an h-type will sometimes accept an l-type because it takes too long time to find another h-type. On the other hand, an l-type will always prefer to wait to meet an h-type instead of matching with another l-type. That is because the continuation value of waiting is higher than the value of the match

$W(x,y)$	l	h
l	0	1
h	1	3

TABLE 1.

$W(l,l) = 0$. The resulting matching is not assortative, as l-types always match with a higher type, and h-types sometimes match with a lower type.

The above argument does not apply in our model with payoff shocks. In an interior equilibrium, the marginal type who decides to accept the match has exactly the same continuation payoff regardless of whether she matches with a type l or type h. The difference is in the acceptance probabilities. The relative likelihood of accepting type $y = h$ vs type $y = l$ by type $x = h$ is equal to $\frac{\mathrm{e}^3}{\mathrm{e}^1} = e^2$ which is greater than the analogous likelihood ratio for type $x = l$ of $\frac{\mathrm{e}^1}{\mathrm{e}^0} = e^1$. This implies that type $x = h$ is more likely than type $x = l$ to be matched with type $y = h$, and the matching is (stochastically) assortative.

7. Conclusion

We developed a tractable and dynamic model of many-to-many matching. In the model, agents search for match partners, form matches taking into account both the instantaneous and future dynamic payoff consequences of each decision. We established the existence of equilibrium, showed sharp identification results, and analyzed welfare, the effect of disappearing frictions, and the relations between the NTU and TU versions of the model.

Some natural extensions require no more than cosmetic changes. For instance, as is standard in the matching literature, we assume that all matches occur between pairs of agents. However, there are natural applications (co-authorship models, club formation), where a match is formed between three or more agents. It is not difficult to add such multi-agent matches. Similarly, it is easy to add single-agent decisions into the model. For example, the agent may have an opportunity to switch between different matching markets and makes the choice that optimally balances the present and future consequences of switching the markets. These modifications can be found in earlier versions of the paper or available upon request from the author.

The current model assumes that any two matched agents do not make any other future decisions. That is not a good assumption for an extension of the marriage model to fertility decision making. Another valuable extension would be to add an opportunity to break off the match (i.e., divorce). We leave these extensions for future work.

Appendix A. Existence

Proof. Let $\upsilon_{\max} = \sup_{t,x,a} |\upsilon_t(x,a)|$. Let $Q_{\max} = \sup_{t,x} Q_t(x)$. Let $U_{\max} = \frac{1}{1-\beta}(\upsilon_{\max}+1)$.

Let $\mathcal{U} = \prod_{t\in\mathbb{N}} [0, U_{\max}]^X$ be the space of continuation payoff functions, $\mathcal{Y} = \prod_{t\in\mathbb{N}} [0, Q_{\max}t]^{X\times(X\cup\{\emptyset\})}$ be the space of masses, and $\Sigma = \prod_{t\in\mathbb{N}} [0, U_{\max}]^{X\times X}$ be the space of strategies. The three spaces are compact under the Tychonoff product topology.

We construct a continuous mapping F from $\mathcal{U}\times\mathcal{Y}\times\Sigma$ into itself, with the property that its fixed point is an equilibrium. For each (U,μ,σ), let

$$\begin{aligned} U_t^F(x,a) &:= \upsilon_t(x,a) + \beta\sum_{x'} P_t(x'|x,a)\,U_{t+1}(x;\sigma),\\ r_t^F(x,y) &:= \frac{q_t\left(x,y;\mu_t^X\right)}{\mu_t^X(x)} e^{-\sigma_t(y,x)}. \end{aligned}$$

Then,

$$|U_t(x,a)| \le \upsilon_{\max} + \beta U_{\max} = U_{\max} - 1,$$

for each x and a, and both $U_t(.,.)$ and $r_t(.,.)$ are continuous in (U,μ,σ). Let

$$(F\sigma)_t(x,y) = \max\left(U_t^F(x,\emptyset) - U_t^F(x,y),0\right) \le U_{\max},$$

which is continuous in U_t^F, and, indirectly, in (U,μ,σ). Let □

$$\begin{aligned}
FU_t(x;\sigma) :=& U_t^F(x,\emptyset) + \sum_{y\in X} r_t^F(x,y)\, e^{-F\sigma_t(x,y)} \left[U_t^F(x,\emptyset) - U_t^F(x,y) + (F\sigma)_t(x,y) + 1\right] \\
=& \sum_{y\in X} r_t^F(x,y)\, e^{-F\sigma_t(x,y)} \left[\max\left(U_t^F(x,\emptyset), U_t^F(x,y)\right) + 1\right] \\
&+ \left(1 - \sum_{y\in X} r_t^F(x,y)\, e^{-\sigma_t(x,y)}\right) U_t^F(x,\emptyset) \\
\leq& \sum_{y\in X} r_t^F(x,y)\, e^{-F\sigma_t(x,y)} \left[U_{\max} - 1 + 1\right] + \left(1 - \sum_{y\in X} r_t^F(x,y)\, e^{-F\sigma_t(x,y)}\right) U_{\max} \\
\leq& U_{\max}.
\end{aligned}$$

Clearly, FU_t is continuous in r_t, U_t^F, and $F\sigma_t$, and, hence in (U,μ,σ). Finally, let $F\mu_0^X(x) = Q_0(x)$ and by induction on $t \geq 0$,

$$\begin{aligned}
F\mu_t(x,y) &:= \left(F\mu_t^X\right)(x)\, r_t^F(x,y)\, e^{-F\sigma_t(x,y)} \\
F\mu_t(x,\emptyset) &:= \mu_t^{X,F}(x) - \sum_y F\mu_t(x,y)\,, \text{ and} \\
\mu_{t+1}^{X,F}(x) &:= Q_{t+1}(x) + \sum_{y\in X, a\in X\cup\{\emptyset\}} F\mu_t^X(y,a)\, P_t(x|y,a)\,,
\end{aligned}$$

Then, $\left|\mu_t^{X,F}(x)\right|, |F\mu_t(x,a)| \leq Q_{\max} t$, and $F\mu_t$ is continuous in r_t, U_t^F, and $F\sigma_t$, and hence in (U,μ,σ).

The Kakutani Fixed Point Theorem implies that the mapping F has a fixed point.

Appendix B. Equilibrium analysis

B.1. **Proof of Lemma 1.** To shorten the notation, we drop the reference to strategies. Notice that

$$V_t^0(x,a) = \upsilon_t(x,a) + \beta \sum_{x'} P_t(x'|x,a)\, V_t^0(x,\emptyset)\,. \tag{30}$$

Below, we write $p_t(x,y) = r_t(x,y)\, e^{-\sigma_t(x,y)}$ and $p_t(x,\emptyset) = 1 - \sum_{y\in X} p_t(x,y)$ and, in order to simply the notation, we take $\sigma_t(x,\emptyset) = -1$. (Note that $\sigma_t(x,\emptyset)$ is a piece of notation and not a strategy, as there is no threshold to "accept" a null match; in particular, the notation, does not contradict equation (4).) Then, using (1) and (2), the present value of an agent with utility υ can be uniquely characterized by the equation:

$$U_t(x,a;\upsilon) = \upsilon_t(x,a) + \beta \sum_{x'} P_t(x'|x,a) \sum_{a' \in X\{\emptyset\}} p_{t+1}(x',a') \left[U_{t+1}(x',a';\upsilon) + 1 + \sigma_{t+1}(x',a')\right].$$

Using (30), we obtain

$$\begin{aligned} &U_t(x,a;\upsilon) - V_t^0(x,\emptyset) \\ =&V_t(x,a) + \beta \sum_{x'} P_t(x'|x,a) \sum_{a' \in X\{\emptyset\}} p_{t+1}(x',a') \left[U_{t+1}(x',a';\upsilon) - V_{t+1}^0(x',\emptyset) + 1 + \sigma_{t+1}(x',a')\right]. \end{aligned}$$

It follows that $U_t(x,a;\upsilon) - V_t^0(x,\emptyset)$ is the present value of an agent with utility V who uses strategy σ:

$$U_t(x,a;V) = U_t(x,a;\upsilon) - V_t^0(x,\emptyset).$$

The Lemma follows.

B.2. **Proof of Theorem 3.** Let μ be the masses that are induced by σ, i.e., that satisfy (7a)-(7c). Let π^{t,x_0} be the agent's probability distributions that are induced by strategy σ given offer rates r for an agent who is born in period t with type x_0 in the sense defined at the beginning of Section 4.1. In particular, $\pi_t^{t,x_0}(x_0) = 1$, and $\pi_s^{t,x_0}(x,a) = 0$ for each $s < t$, x, and a.

Because the best response behavior is uniquely determined by the agent's current type of the agent and the calendar time, it must be that for each $s \geq t$, each $x_0, x \in X$, each $a \in A$,

$$\frac{\pi_s^{t,x_0}(x,a)}{\pi_s^{t,x_0}(x)} = \frac{\mu_s(x,a)}{\mu_s(x)}.$$

Using (7c), it follows that

$$\mu_s = \sum_{t \leq s,x} Q_t(x_0)\, \pi_s^{t,x_0}.$$

By formula (16), we have

$$\begin{aligned}
&\sum_{t,x} \beta^t Q_t(x) U_t(x;\sigma,r) \\
=&\sum_{t,x_0} Q_t(x_0) \sum_{s\geq t} \beta^s \sum_{x,y\in X} \left(V_s(x,y) + 1 - \log \frac{\pi_s^{t,x_0}(x,y)}{\pi_s^{t,x_0}(x)} + \log r_s(x,y) \right) \pi_s^{t,x_0}(x,y) \\
=&\sum_{t,x_0} Q_t(x_0) \sum_{s\geq t} \beta^s \sum_{x,y\in X} \left(V_s(x,y) + 1 - \log \frac{\mu(x,y)}{\mu_s(x)} + \log r_s(x,y) \right) \pi_s^{t,x_0}(x,y) \\
=&\sum_{s} \beta^s \sum_{x,y\in X} \left(V_s(x,y) + 1 - \log \frac{\mu(x,y)}{\mu_s(x)} + \log r_s(x,y) \right) \left(\sum_{t\leq s,x_0} Q_t(x_0) \pi_s^{t,x_0}(x,y) \right) \\
=&\sum_{s} \beta^s \sum_{x,y\in X} \left(V_s(x,y) + 1 - \log \frac{\mu(x,y)}{\mu_s(x)} + \log r_s(x,y) \right) \mu_s(x,y) .
\end{aligned}$$

Further, using the definition of acceptance rates (6) as well as (7a), we obtain that, for each t, and each $x, y \in X$,

$$\begin{aligned}
&\log r_t(x,y) + \log r_t(y,x) \\
=&\log q_t(x,y,\mu_t) - \log \mu_t(x) - \sigma_t(y,x) + \log q_t(y,x,\mu_t) - \log \mu_t(y) - \sigma_t(x,y) \\
=&\log \mu_t(x,y) + \log q_t(x,y,\mu_t) - \log \mu_t(x) - \log \mu_t(y) .
\end{aligned}$$

Because in equilibrium $\mu_t(x,y) = \mu_t(y,x)$, we have

$$\begin{aligned}
\log r_t(x,y)\mu_t(x,y) + \log r_t(y,x)\mu_t(y,x) =& \mu_t(x,y)\log \mu_t(x,y) + \mu_t(x,y)\log q_t(x,y,\mu_t) \\
&- \mu_t(x,y)\log \mu_t(x) - \mu_t(y,x)\log \mu_t(y) .
\end{aligned}$$

After multiplying by β^t, and taking the sum over all $x, y \in X$, and t, we have

$$\begin{aligned}
&2\sum_{t,x,y} \beta^t \log r_t(x,y)\mu_t(x,y) \\
=&\sum_{t,x,y} \beta^t \left(\log \mu_t(x,y) + \log q_t(x,y) - \log \mu_t(x) - \log \mu_t(y)\right)\mu_t(x,y) .
\end{aligned}$$

Thus,

$$\begin{aligned}&\sum_{t,x}\beta^t Q_t(x)\,U(\sigma,x;F(.;\sigma,\mu))\\&=\sum_t\beta^t\sum_{x,y\in X}\left(V_t(x,y)+1-\log\frac{\mu_t(x,y)}{\mu_t(x)}+\frac{1}{2}\log\mu_t(x,y)+\frac{1}{2}\log q_t(x,y;\mu_t)-\log\mu_t(x)\right)\mu_s(x,y)\\&=\sum_t\beta^t\sum_{x,y\in X}\left(V_t(x,y)+1-\frac{1}{2}\log\mu_t(x,y)+\frac{1}{2}\log q_t(x,y;\mu_t)\right)\mu_s(x,y).\end{aligned}$$

B.3. **Proof of Theorem 4.** Formula (16) and the first part of the proof of Theorem 3 imply that, if mass μ^* is an interior equilibrium mass, then there are acceptance rates r^* that satisfy (6) for μ^* instead of μ, and such that

$$\mu^*\in\arg\max_{\mu\text{ st. eq. (7b),(7c)}}\sum_s\sum_{x,y\in X}\beta^s\left(V_s(x,y)+1-\log\frac{\mu_s(x,y)}{\mu_s(x)}+\log r^*_s(x,y)\right)\mu_s(x,y).\tag{31}$$

Additionally, if μ^* is an interior equilibrium, then it satisfies (19). The latter can be added as a constraint to the optimization problem:

$$\mu^*\in\arg\max_{\mu\text{ st. eq. (7b),(7c), and (19)}}\sum_s\sum_{x,y\in X}\beta^s\left(V_s(x,y)+1-\log\frac{\mu_s(x,y)}{\mu_s(x)}+\log r^*_s(x,y)\right)\mu_s(x,y).\tag{32}$$

As in the proof of Theorem 3, for each μ that satisfies (19), we have

$$\begin{aligned}&\sum_{t,x,y}\beta^t\log r_t(x,y)\,\mu_t(x,y)\\&=\frac{1}{2}\sum_{t,x,y}\beta^t\left(\log\mu^*_t(x,y)+\log q_t(x,y;\mu^*)-\log\mu^*_t(x)-\log\mu^*_t(y)\right)\mu_t(x,y)\\&=\sum_{t,x,y}\beta^t\frac{1}{2}\log\mu^*_t(x,y)\,\mu_t(x,y),\end{aligned}$$

where the last equality comes from the assumption that $q_t(x,y;\mu^*)=q^0_t(x,y)\,\mu^*_t(x)\,\mu^*_t(y)$.

Hence,

$$\mu^* \in \arg \max_{\mu \text{ st. eq. (7c), and (19) hold}} \tag{33}$$
$$\sum_s \sum_{x,y\in X} \beta^s \left(V_s(x,y) + 1 - \log \mu_s(x,y) + \log \mu_s(x) + \frac{1}{2} \log \mu_s^*(x,y) \right) \mu_s(x,y)$$
$$= \arg \max_{\mu \text{ st. eq. (7c), and (19) hold}} W(\mu) - \frac{1}{2}\mathbf{1} \cdot \mu + V(\mu;\mu^*),$$

where

$$V(\mu;\mu^*)$$
$$:= \sum_s \sum_{x,y\in X} \beta^s \frac{1}{2} \left(1 + (\log \mu_s^*(x,y) - \log \mu_s(x,y)) + \log q_t(x,y;\mu) - 2\log \mu_s(x)\right) \mu_s(x,y)$$
$$= \sum_s \sum_{x,y\in X} \beta^s \frac{1}{2} \left(1 + (\log \mu_s^*(x,y) - \log \mu_s(x,y))\right) \mu_s(x,y),$$

where, in the last equality, we used the balance identities, and the assumption that $q_t(x,y;\mu) = q_t^0(x,y)\,\mu_t(x)\,\mu_t(y)$.

Function $V(\mu;\mu^*)$ is concave with a unique maximum at $\mu = \mu^*$. (To see that, take $f(x) = x(\log x_0 + 1 - \log x)$, and notice that it is a concave function with the first derivative equal to $f'(x) = \log x_0 - \log x$.). It follows that any optimal solution to the problem (33) must be a critical point of

$$W(\mu) - \frac{1}{2}\mathbf{1} \cdot \mu \text{ st. (7c), and (19).}$$

(Note that it is possible that μ is a not necessarily a maximum if function $W(\mu)$ is not concave in the neighborhood of μ^*.)

To prove the converse, trace back each of the above assertions from the end to the beginning. The inequality (20) together with the equilibrium characterization (10) implies that the acceptance thresholds are strictly positive and the equilibrium strategies are interior.

Appendix C. Transferable utility

C.1. **Proof of Theorem 5.**

C.1.1. *Best responses.* We start with describing the (interior) best response behavior. Let $U_t(x)$ denote the t-period continuation value of agent x.

Suppose that agents x and y meet with x playing the role of the employer. The employer's best response is to accept if the payoff shock plus the continuation payoff minus the demand is larger than the continuation payoff after not accepting the action,

$$U_t(x,y) - d + \varepsilon \geq U_t(x,\emptyset).$$

It follows that the payoff shock must be larger than the threshold

$$\sigma_t(x,y,d) = d + \underbrace{U_t(x,\emptyset) - U_t(x,y)}_{\sigma_t^0(x,y)}, \tag{34}$$

where $\sigma_t^0(x,y)$ is a convenient notation. The probability that the employer accepts the action is equal to

$$\mathrm{e}^{-\sigma_t(x,y,d)} = \mathrm{e}^{-d}\mathrm{e}^{-\sigma_t^0(x,y)}. \tag{35}$$

The expected payoff of the employee is equal to

$$\begin{aligned} &\mathrm{e}^{-d}\mathrm{e}^{-\sigma_t^0(x,y)}\left(U_t(y,x)+d\right) + \left(1-\mathrm{e}^{-d}\mathrm{e}^{-\sigma_t^0(x,y)}\right)\left(U_t(y,\emptyset)\right) \\ =&U_t(y,\emptyset) + \mathrm{e}^{-d}\mathrm{e}^{-\sigma_t^0(x,y)}\left(d-\sigma_t^0(y,x)\right). \end{aligned}$$

In the interior case, the expected payoff is maximized by

$$d_t(y,x) = \sigma_t^0(y,x) + 1.$$

Notice that if two agents x and y meet, the probability that the match is formed, computed before the employer is chosen, is equal to

$$p_t(x,y)\,\mathrm{e}^{-\sigma_t(x,y,d_t(y,x))} + p_t(y,x)\,\mathrm{e}^{-\sigma_t(y,x,d_t(y,x))} = \mathrm{e}^{-1-\sigma_t^0(x,y)-\sigma_t^0(y,x)},$$

and it does not depend on p.

C.1.2. *Proof of the Theorem.* Suppose that μ is an interior TU equilibrium together with strategies σ, d. Suppose that σ^0 is defined as in equation (34). We are going to show that μ and σ^0 form an interior (NTU) equilibrium of the model with parameters υ and ρ^{NTU}. Notice that equation (22) becomes

$$\begin{aligned} \mu_t(x,y) &= q_t(x,y;\mu)\,\mathrm{e}^{-1-\sigma_t^0(x,y)-\sigma_t^0(y,x)} \\ &= q_t^{NTU}(x,y;\mu)\,\mathrm{e}^{-\sigma_t^0(x,y)-\sigma_t^0(y,x)}. \end{aligned}$$

Thus, μ satisfies equations (7a) and, hence, the condition (b) of Definition 1.

Next, we show that the continuation payoffs in the two models are identical. If so, then the definition of σ^0 in equation (34) and property (8) show that σ^0 is the best response in the NTU model. Because the mass densities and systematic utility are the same in the two models, it is enough to show that the expected payoff from shocks in the NTU model are equal to the payoffs from demands and shocks in the TU model. Indeed, if x is an employer meeting with y, her expected net payoff shock conditional on accepting the action is equal to the (gross) expected payoff shock conditional on acceptance minus the demands

$$E\left(\varepsilon|\varepsilon \geq \sigma_t^0\left(x,y\right)+d_t\left(y,x\right)\right)-d_t\left(y,x\right)=\sigma_t^0\left(x,y\right)+1.$$

The expected conditional instantaneous payoff of employee y is equal to her demand $d_t\left(y,x\right)=\sigma_t^0\left(y,x\right)+1$. In both cases, the expected payoff conditional on acceptance is equal to the same value as the conditional expectation of the payoff shock given that the action is accepted in the NTU model. The result follows.

C.2. **Proof of Theorem 6.** Suppose that (μ_t, σ_t, U_t) is an interior *TU_{NB}-equilibrium* with systematic utility υ_t. For each $a \in X \cup \{\emptyset\}$, define

$$U_t^{NTU}\left(x,a\right)=U_t\left(x,a\right)+\tilde{\mu}_t^{NB}\left(x\right)-\tilde{\mu}_t\left(x\right) \text{ and } U_t^{NTU}\left(x\right)=U_t\left(x\right).$$

Then, we check that $\left(\mu_t, \sigma_t, U_t^{NTU}\right)$ is an interior *(NTU) equilibrium* with utilities υ_t^{NTU} (i.e., the tuple satisfies equations (1), (2), (4) and the masses evolve according to (6) and (7a)-(7c). The reverse direction is analogous.

Appendix D. Dynamic marriage matching

D.1. **Stationary equilibrium.** Here, we develop notation and derive some equations and bounds that hold in a (not necessarily interior) stationary equilibrium of a dynamic marriage matching model. Let

$$\begin{aligned}
V_{\max} &= \max_{x,y} V\left(x,y\right), V_{\min} = \min_{x,y} V\left(x,y\right),\\
\Delta_V &= V_{\max} - V_{\min},\\
q_{\min} &= \min_{m,f} q_0\left(m,f\right) > 0, q_{\max} = \max_{m,f} q_0\left(m,f\right).
\end{aligned}$$

Let $\mu(.,.), r(.,.), U(.)$ be the stationary equilibrium objects. Recall that

$$\tilde{\mu}(x) = \sum_{y \in X} r(x,y)\, e^{-\sigma(x,y)} = \frac{1}{\mu(x)} \sum_y \mu(x,y)$$

is the probability with which agent x finds a match in any given period.

The steady state version of flow equation (7c) implies that

$$\begin{aligned}\mu(x) &= Q(x) + \delta\left(\mu(x) - \sum_{x \in X, y \in X} \mu(x,y)\right)\\ &= Q(x) + \delta(1 - \tilde{\mu}(x))\,\mu(x),\end{aligned}$$

or

$$Q(x) = (1 - \delta(1 - \tilde{\mu}(x)))\,\mu(x). \tag{36}$$

In particular, $\mu(x) \geq Q(x)$ for each x. Using equation (36), we can derive a formula for the mass of formed matches in a given period as

$$\sum_{x,y} \mu(x,y) = \sum_x \mu(x)\,\tilde{\mu}(x) = \sum_x \frac{Q(x)}{1 - \delta + \delta\tilde{\mu}(x)}\tilde{\mu}(x) = \sum_x \frac{1}{\frac{1-\delta}{\delta\tilde{\mu}(x)} + 1} Q(x). \tag{37}$$

Let

$$\pi(x) = \frac{(1-\delta)\,\mu(x)}{Q(x)} = 1 - \frac{\sum_y \mu(x,y)}{Q(x)} \tag{38}$$

be the probability that type x agent dies before finding a match. Then, using (36), we have

$$\tilde{\mu}(x) = \frac{1-\delta}{\delta}\left(\frac{1}{\pi(x)} - 1\right) \text{ and } \frac{\mu(x)}{Q(x)} = \frac{\pi(x)}{1-\delta}. \tag{39}$$

In any stationary equilibrium, equation (1) implies that $U(x, \emptyset) = \beta\delta U(x)$. Using (2) and (4), we obtain

$$U(x) = \frac{1}{1-\beta\delta} \sum_{y \in X} r(x,y)\, e^{-\sigma(x,y)} \left[1 + \max\left(V(x,y) - \beta\delta U(x), 0\right)\right]. \tag{40}$$

Then, (40) leads to bounds

$$\frac{1}{1-\beta\delta}\tilde{\mu}(x) \leq U(x) \leq \frac{1}{1-\beta\delta}\tilde{\mu}(x)\left(1 + V_{\max}\right). \tag{41}$$

The best response equation (4) implies that for each x, y, y',

$$U(x) - V_{\max} \leq \sigma(x,y) \leq U(x) - V_{\min}. \tag{42}$$

D.2. **Proof of Proposition 1.**

D.2.1. *Probability bounds.* Next, we show that the probabilities of finding a match cannot differ too much between agents on the same side of the market. Let

$$\tilde{\mu}_M = \min_{m\in M} \tilde{\mu}(m), \tilde{\mu}_F = \min_{f\in F} \tilde{\mu}(f).$$

Lemma 2. *There exists constant C (that does not depend on β and δ) such that for any m and f,*

$$\tilde{\mu}(m) < C\tilde{\mu}_M \text{ and } \tilde{\mu}(f) < C\tilde{\mu}_F.$$

Proof. Take $m_0 \in \arg\max_{m\in M} U(m)$. Then, for each male m, we have

$$\begin{aligned}
\tilde{\mu}(m) &= \sum_f q_0(m,f)\,\mu(f)\,e^{-\sigma(m,f)-\sigma(f,m)} \\
&\leq \sum_y q_{\max}\mu(f)\,e^{-U(m)-U(f)+2V_{\max}} \\
&= e^{2(V_{\max}-V_{\min})}e^{U(m_0)-U(m)}\sum_f q_{\max}\mu(f)\,e^{-(U(m_0)-V_{\min})-(U(f)-V_{\min})} \\
&\leq e^{2\Delta_V}\sum_y q_{\max}\mu(f)\,e^{-\sigma(m_0,f)-\sigma(f,m_0)} \\
&\leq e^{2\Delta_V}\frac{q_{\max}}{q_{\min}}\tilde{\mu}(m_0).
\end{aligned}$$

The first inequality comes from (42). The second inequality is a consequence of (42) as well as the fact that $U(m_0) \leq U(m)$ for each m.

An analogous argument holds for female types. □

Lemma 3. *We have*

$$\tilde{\mu}_M \leq C\tilde{\mu}_F. \tag{43}$$

Proof. Using Lemma 2, the number of formed matches (37) can be bounded by

$$Q_M\frac{1}{1+\frac{1-\delta}{\delta\tilde{\mu}_M}} \leq \sum_{x,y}\mu(x,y) \leq Q_F\frac{1}{1+\frac{1-\delta}{\delta C\tilde{\mu}_F}}.$$

The claim follows from the two inequalities and the fact that $Q_M \geq Q_F$. □

Lemma 4. *For each $A > 0$, if $1-\beta\delta$ is sufficiently small, then $\tilde{\mu}_F(1-\beta\delta)^{-1} \geq A$.*

Proof. Suppose that $\tilde{\mu}_F(1-\beta\delta)^{-1} < A$. Then, (43) and Lemma 2 imply that for each type x, $\tilde{\mu}(x)(1-\beta\delta)^{-1} \leq C^2A$. The second inequality in (41) implies further that

$$U(f) \leq C^2A(1+V_{\max}) =: U_{\max}(A).$$

Further, bounds (42) imply that for each y, $\sigma(x,y) \leq U_{\max}(A) - V_{\min} =: \sigma_{\max}(A)$. Then, for each female,

$$\tilde{\mu}(f) = \sum_m q_0(m,f)\,\mu(m)\,e^{-\sigma(m,f)}e^{-\sigma(f,m)} \geq \sum_m q_{\min} Q(m)\,e^{-2\sigma_{\max}(A)} \geq q_{\min} Q_M e^{-2\sigma_{\max}(A)}.$$

It follows that

$$A(1-\beta\delta) \geq \tilde{\mu}_F \geq q_{\min} Q_M e^{-2\sigma_{\max}(A)}.$$

Together with $A(1-\beta\delta) \geq \tilde{\mu}_F$, we obtain that

$$1 - \beta\delta \geq \frac{1}{A} q_{\min} Q_M e^{-2\sigma_{\max}(A)}.$$

□

D.2.2. *Payoff bound.* We have the following useful payoff bounds. Recall that $\pi(x)$ is the probability of dying single (39).

Lemma 5. *In any stationary equilibrium of the dynamic matching model, for each type x, $U(x) \leq \max\left(V_{\max}, \frac{1-\pi(x)}{\pi(x)}\right)$.*

Proof. The expected continuation value is not higher than the probability of finding a match plus the systematic utility from the match plus the (conditional) expected payoff shock. Hence,

$$U(x) \leq (1-\pi(x)) \max_y (V(x,y) + \sigma(x,y) + 1) \leq (1-\pi(x))(\max(U(x), V_{\max}) + 1).$$

If $U(x) \geq V_{\max}$, then $U(x) \leq (1-\pi(x))\,U(x) + (1-\pi(x))$, which implies that $U(x) \leq \frac{1-\pi(x)}{\pi(x)}$. □

Lemma 6. *Suppose that markets are imbalanced: $Q_M > Q_F$. Then, there is a constant U_0 such that for each m, $U(m) \leq U_0$.*

Proof. Because in a stationary equilibrium, it must be that

$$\sum_{m,f} \mu(m,f) \leq Q_F,$$

there is a type m_0 such that $a(m_0) = \frac{\sum_f \mu(m_0,f)}{Q(m_0)} \leq \frac{Q_F}{Q_M}$. By Lemma 5, $U(m_0) \leq \max\left(V_{\max}, \frac{Q_F}{Q_M - Q_F}\right)$. By an application of bounds (41) and Lemma 2, we obtain for each male type m,

$$\begin{aligned} U(m) &\leq \frac{1}{1-\beta\delta} V_{\max}\tilde{\mu}(m) \leq V_{\max} C \frac{1}{1-\beta\delta}\tilde{\mu}(m_0) \\ &\leq V_{\max} C U(m_0) \\ &\leq V_{\max} C \max\left(V_{\max}, \frac{Q_M}{Q_F - Q_M}\right) =: U_0. \end{aligned}$$

□

D.2.3. *Proof of Proposition 1.* By Lemma 4, for arbitrary $\varepsilon > 0$, if $1-\beta\delta$ is sufficiently small, then $\tilde{\mu}_F (1-\beta\delta)^{-1} > \frac{1}{\varepsilon}$. If $\varepsilon^{-1} > V_{\max}$, then (41) implies that $U(f,\emptyset) = \beta\delta U(f) \geq V_{\max}$. In such a case, equilibrium strategies of all female types are interior.

If the markets are balanced, an analogous argument implies that male strategies are interior. If the markets are imbalanced, then the male's utility from the match is negative by assumption. Because the continuation is always positive (as any male can always reject any offer), male strategies must be interior.

Finally, notice that the number of matches formed by female f in each period is bounded from below by

$$\frac{Q(f)}{1+\frac{1-\delta}{\delta\tilde{\mu}(f)}} \geq \frac{Q(f)}{1+\frac{1-\beta\delta}{\delta\tilde{\mu}_F}} \geq \frac{Q(f)}{1+\frac{1}{\delta}\varepsilon} \nearrow Q(f),$$

where the convergence holds as $1-\beta\delta \to 0$.

D.3. **Proof of Proposition 2.**

D.3.1. *Matching function.* Suppose that μ is a distribution of masses in an interior stationary equilibrium. Substituting (8) into the evolution equations (7a) yields for each m, f

$$\begin{aligned} \log\mu(m,f) =& W(m,f) + \log q(m,f;\mu) - U(m,\emptyset) - U(f,\emptyset) \\ =& W(m,f) + \log q_0(m,f) Q(m) Q(f) + \log\frac{\mu(m)}{Q(m)} + \log\frac{\mu(f)}{Q(f)} \\ & - \beta\delta U(m) - \beta\delta U(f). \end{aligned}$$

Because the equilibrium is interior, (40) implies that $U(x) = \frac{1}{1-\beta\delta}\tilde{\mu}(x) = \frac{1}{\delta}\frac{1-\delta}{1-\beta\delta}\left(\frac{1}{\pi(x)} - 1\right)$. Together with (39), we obtain a version of the matching function:

$$\begin{aligned}\log\mu(m,f) =& \Lambda W(m,f) + \log q_0(m,f)\,Q(m)\,Q(f) - 2\log(1-\delta) \\ & - \gamma_{\beta,\delta}(\pi(x)) - \gamma_{\beta,\delta}(\pi(y)),\end{aligned} \quad (44)$$

where

$$\gamma_{\beta,\delta}(\pi) := \frac{\beta(1-\delta)}{1-\beta\delta}\left(\frac{1}{\pi} - 1\right) - \log\pi \geq 0 \text{ for each } \pi \leq 1.$$

D.3.2. *TU problem.* Let $\mathcal{M}_0$ be the space of positive measures over $M \times F$ such that for each x, $\sum_y \mu(x,y) \leq Q(x)$. Let $\mathcal{M} \subseteq \mathcal{M}_0$ be the subspace of measures μ such that

$$\begin{aligned}&\sum_m \mu(m,f) = Q(f) \text{ for each } f, \text{ and} \\ &\sum_f \mu(m,f) \leq Q(m) \text{ for each } m.\end{aligned}$$

Let

$$\mathcal{M}^* = \arg\max_{\mu\in\mathcal{M}} \sum W(m,f)\,\mu(m,f).$$

Notice that $\mathcal{M}_0, \mathcal{M}$, and $\mathcal{M}^*$ are compact. A standard argument implies the following:

Lemma 7. *For any $\mu \in \mathcal{M}\backslash\mathcal{M}^*$, there is a sequence $m_0, f_0, , ..., f_n, m_{n+1}$ such that*

$$\begin{aligned}\mu(m_l, f_l) > 0 \textit{ for each } l \leq n, \\ \Delta_W = \sum_l W(m_{l+1}, f_l) - \sum_l W(m_l, f_l) > 0,\end{aligned} \quad (45)$$

and, either (a) $m_{n+1} = m_0$ or (b) $m_{n+1} \neq m_0$ and $\sum_f \mu(m_{n+1}, f) < Q(m_{n+1})$.

Proof. We include the proof for the sake of completeness. By the Kuhn-Tucker conditions, there are $\lambda(f)$ and $\lambda(m) \geq 0$ such that $\mu^* \in \mathcal{M}^*$ if and only if for each m, f:

$$w(m,f) := W(m,f) - \lambda(f) - \lambda(m) \leq 0 \text{ with equality if } \mu^*(m,f) > 0,$$

and $\lambda(m) > 0$ only if $\sum_f \mu(m,f) = Q(m)$.

Fix $\mu^* \in \mathcal{M}^*$ and $\mu \in \mathcal{M}\backslash\mathcal{M}^*$. Consider a directed graph, where there is an arrow from m to f if and only if $\mu(m,f) > \mu^*(m,f)$ and an arrow from f to m if and only if $\mu(m,f) < \mu^*(m,f)$. It is easy to see that □

- for each f, f has outgoing arrows if and only if it has incoming arrows,

- if m has only incoming arrows, then $\sum_f \mu(m,f) < Q(m)$, and
- if m has only outgoing arrows, then $\sum_f \mu^*(m,f) < Q(m)$.

Moreover, there is at least one arrow from m to f such that $w(m,f) < 0$. Using these properties, we can find a chain $m_0, f_0, ,..., f_n, m_{n+1}$ such that

- $\mu(m_{l+1}, f_l) < \mu^*(m_{l+1}, f_l)$,
- $\mu(m_l, f_l) > \mu^*(m_l, f_l)$,
- for at least one l^*, $w(m_{l^*}, f_{l^*}) < 0$, and $\mu(m_{l^*}, f_{l^*}) > 0 = \mu^*(m_{l^*}, f_{l^*})$, and
- either (a) $m_{n+1} = m_0$ or (b) $m_{n+1} \neq m_0$, $\sum_f \mu^*(m_0, f) < Q(m_0)$, and $\sum_f \mu(m_{n+1}, f) < Q(m_{n+1})$.

Notice that $\mu^*(m_{l+1}, f_l) > 0$ for each l, which implies that

$$\sum_l w(m_{l+1}, f_l) = 0 > w(m_{l^*}, f_{l^*}) + \sum_{l \neq l^*} w(m_l, f_l).$$

However,

$$0 < \sum_l w(m_{l+1}, f_l) - \sum_l w(m_l, f_l) = \Delta_W + \lambda(m_0) - \lambda(m_{n+1}) \leq \Delta_W,$$

where the last inequality comes from the fact that $\lambda(m_0) = 0$ and $\lambda(m_{n+1}) \geq 0$.

D.3.3. *Proof of Proposition 2.* Let $\mathcal{M}_{\Lambda,\beta,\delta} \subseteq \mathcal{M}_0$ be the set of positive measures in the stationary equilibria of a model with parameters $\Lambda V(.,.), \beta$, and δ. Each such set is closed and contained in a compact set, and hence compact. Let $\mathcal{M}_\Lambda = \limsup_{\beta,\delta,\to 1} \mathcal{M}_{\Lambda,\beta,\delta}$.[6] Let $\mathcal{M}_{eq} = \limsup_{\Lambda\to\infty} \mathcal{M}_\Lambda$.

By Proposition 1, $\mathcal{M}_{\text{eq}} \subseteq \mathcal{M}$. Suppose that there is $\mu \in \mathcal{M}_{\text{eq}}$ such that $\mu \notin \mathcal{M}^*$. Find a sequence $\mu_\Lambda \in \mathcal{M}_\Lambda$ such that $\mu_\Lambda \to \mu$. Find $\mu_{\Lambda,\beta,\delta} \in \mathcal{M}_{\Lambda,\beta\delta}$ such that $\mu_{\Lambda,\beta\delta} \to \mu_\Lambda$. Find a sequence $m_0, ..., m_{n+1}$ with properties as in Lemma 7. Using the matching function (44), we obtain

$$\begin{aligned}\log\left(\frac{\prod_l \mu_{\Lambda,\beta,\delta}(m_{l+1}, f_l)}{\prod_l \mu_{\Lambda,\beta,\delta}(m_l, f_l)}\right) =& \Lambda\Delta_W + \left(\log \frac{q_0(m_{n+1}, f_0) Q(m_{n+1})}{q_0(m_0, f_n) Q(m_0)}\right) \\ & - \gamma_{\beta,\delta}(\pi(m_{l+1}; \mu_{\Lambda,\beta,\delta})) + \gamma_{\beta,\delta}(\pi(m_0; \mu_{\Lambda,\beta,\delta})).\end{aligned}$$

If $m_0 = m_{n+1}$, then the last two terms cancel out. If $m_0 \neq m_{n+1}$, then $\sum_f \mu(m_{n+1}, f) < Q(m_{n+1})$, which implies that $\pi(m_{l+1}) > 0$. Because $\gamma_{\beta,\delta}(\pi(m_{l+1}; \mu_{\Lambda,\beta,\delta})) \geq 0$, there

[6] For a sequence of sets $A_\varepsilon \subseteq A_0$, we define $\limsup_{\varepsilon\to 0} A_\varepsilon$ as the set of all limit points of convergent sequences $a_n \in A_{\varepsilon_n}$, where $\varepsilon_n \to 0$.

exists a bound

$$\lim_{\Lambda\to\infty}\lim_{\beta,\delta\to1}\left[-\gamma_{\beta,\delta}\left(\pi\left(m_{l+1};\mu_{\Lambda,\beta,\delta}\right)\right)+\gamma_{\beta,\delta}\left(\pi\left(m_0;\mu_{\Lambda,\beta,\delta}\right)\right)\right]$$
$$\geq\lim_{\Lambda\to\infty}\lim_{\beta,\delta\to1}\left[-\gamma_{\beta,\delta}\left(\pi\left(m_{l+1};\mu_{\Lambda,\beta,\delta}\right)\right)\right]=-\gamma_{\beta,\delta}\left(\pi\left(m_{l+1}\right)\right)<\infty.$$

Thus, in both cases, we have

$$\lim_{\Lambda\to\infty}\lim_{\beta,\delta\to1}\frac{1}{\Lambda}\log\left(\frac{\prod_l\mu_{\Lambda,\beta,\delta}\left(m_l,f_{l+1}\right)}{\prod_l\mu_{\Lambda,\beta,\delta}\left(m_l,f_l\right)}\right)\geq\Delta_W>0.$$

Because of (45),

$$\lim_{\Lambda\to\infty}\lim_{\beta,\delta\to1}\log\left(\prod_l\mu_{\Lambda,\beta,\delta}\left(m_l,f_l\right)\right)=\sum_l\log\mu\left(m_l,f_l\right)>-\infty.$$

The two inequalities together imply that

$$\lim_{\Lambda\to\infty}\lim_{\beta,\delta\to1}\log\left(\prod_l\mu_{\Lambda,\beta,\delta}\left(m_l,f_{l+1}\right)\right)\to\infty,$$

which contradicts the fact that in a stationary equilibrium, for each l,

$$\mu_{\Lambda,\beta,\delta}\left(m_l,f_{l+1}\right)\leq Q\left(f\right).$$

D.4. **Proof of Proposition 3.** Let $U_t\left(x,t\right)$ be the period t continuation value of the type x agent who is still looking at the end of period t. In an interior equilibrium, the best response thresholds are equal to

$$\sigma\left(x,y,t\right)=U_t\left(x\right)-\upsilon\left(x,y\right). \tag{46}$$

If μ is an equilibrium mass, then equations (7a) imply that for each y and i,

$$\begin{aligned}\mu_t\left(x,y\right)&=\mathrm{e}^{-\sigma_t(x,y)}\mathrm{e}^{-\sigma_t(x,y)}\mu_t^X\left(x\right)\mu_t^X\left(y\right)\frac{1}{\mu_t^0}\\&=\mu_t^0\mathrm{e}^{-U_t(y)}\mu_t^X\left(y\right)\mathrm{e}^{-U_t(x)}\mu_t^X\left(x\right)\mathrm{e}^{f(x,y)}\\&=c_t\left(y\right)\mathrm{e}^{-U_t(x)}\mu_t^X\left(x\right)\mathrm{e}^{f(x,y)},\end{aligned}$$

where $c_t\left(y\right)$ is defined though the last equality. It follows that, for each $x<x'$ and each $y<y'$,

$$\frac{\hat{\mu}\left(y'|x,t\right)}{\hat{\mu}\left(y|x,t\right)}=\mathrm{e}^{f(x,y')-f(x,y)}\frac{c\left(y',t\right)}{c\left(y,t\right)}<\mathrm{e}^{f(x',y')-f(x',y)}\frac{c\left(y',t\right)}{c\left(y,t\right)}=\frac{\hat{\mu}\left(y'|x',t\right)}{\hat{\mu}\left(y|x',t\right)}.$$

Thus, the conditional distributions are ordered by the Monotone Likelihood Ratio Property, which implies first-order stochastic dominance.

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