ONLINE APPENDIX TO "TRACTABLE MODEL OF DYNAMIC MANY-TO-MANY MATCHING"

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This Appendix contains few extensions of the model in Peski (2019). Section 1 describes a model of post-match behavior in the marriage context. Section 2 extends the many-to-one matching model to incorporate endogenous separations. Finally, section 3 shows how to model N-agent interactions for $N \geq 2$. In each case, we describe the model and derive the main identification result: the matching function (equation (13) from Peski (2019)).

Throughout, we use notation consistent with Peski (2019).

1. Marriage with post-match behavior

1.1. **Model.** The main model Peski (2019) assumes that after the match is formed, there is nothing further happening between two matched agents. Often, post-match decisions, like children in the marriage setting, or promotion in the labor setting form an important part of the modeled phenomenon. Here, we show how to incorporate consensual post-match behavior into the main model.

To focus attention, we work with a version of the dynamic model of marriage. The set of types is defined as

$$X = X_0 \cup X_0^2,$$

with the following interpretation: Each agent is either single or married. A single agent has a type $x \in X_0$; the type describes the individual's characteristics (race, age, education level, employment status, etc.). The single characteristics can also contain information about previous marriages, like number of children, etc. A married agent has a type $x = (x_0, y_0) \in X_0^2$, where x_0 and y_0 are characteristics of the agent and her partner in the marriage. Let $\overline{x} = (y_0, x_0)$ denote the married type of the partner.

In period t, an opportunity $a \in A = A_S \cup X_S \cup A_M$ may arrive with a probability $q_t(a|x)$. The opportunity may allow an agent to have a child, to buy a house, to match

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with another single agent, etc. Formally, we distinguish between the following types of opportunities:

- single agent opportunities: $q_t(A_M|x) > 0$ only if $x \in X_0$. Each such opportunity is associated with an i.i.d. exponential payoff shock. The agent decides whether to accept it,
- match opportunity with an agent $a \in X_0$. We assume that only single agents can form a new match: $q_t(X_0|x) > 0$ only if $x \in X_0$. The conditional probability of an agent x_s meeting an agent y_s is given by
$$q_t(y_s|x_s) = \frac{1}{\mu_t^X(x_s)} q_t\left(x_s, y_s; \mu_t^X\right),$$
where $q_t\left(x_s, y_s; \mu_t^X\right) = q_t\left(y_s, x_s; \mu_t^X\right)$ is the meeting function. (The meeting function plays the same role as in the original paper),
- married-pair opportunity $a \in A_M$: $q_t(A_0|x) > 0$ only if $x \in X_0^2$. Each married-pair opportunity arrives with the same probability to each spouse:
$$q_t(a|x) = q_t(a|\overline{x}).$$

The last two opportunities are associated with two independent payoff shocks for each agent. The opportunity is realized only if both agents accept it. If no opportunity $\emptyset$ arrives, which occurs with probability $1 - \sum_{a \in A} q_t(a|x) > 0$, or if the opportunity is not accepted by at least one agent, then we say that the null opportunity $\emptyset$ is realized.

For each existing agent x with realized opportunity a, her next period type becomes x' with probability $P_t(x'|x, a)$ that may depend on her own current type as well as the opportunity. Agent x dies with probability $1 - \sum_{x'} P_t(x'|x, a)$, where $a \in X \cup \{\emptyset\}$. We assume that for all married types $x, x' \in X_0^2$, the probabilities of the next period married type x' agree across the spouses:

$$P_t(x'|x) = P_t\left(\overline{x'}|\overline{x}\right). \tag{1}$$

The married agents may become single either because their partner dies or because of the exogenous separation.

In each period t, a mass $Q_t(x)$ of single type $x \in X_0$ agents enters the market.

In each period t, an agent x with a realized opportunity a receives payoff $v_t(x, a)$ plus the payoff shock (the latter only if e is non-empty). Agents discount the future with a factor $\beta < 1$.

1.2. **Data.** Each variable with subscript t is measured at the beginning of the period.

An econometrician observes masses $\mu_t^X(x)$ of type x agents and $\mu_t(x,a)$ of type x agents with realized opportunity a. This includes the masses of new marriages $\mu_t(x,y)$ for $x,y \in X_0$ and the masses of married types $\mu_t^X(x)$ for $x \in X_0^2$. The assumptions of the model (equation (1)) imply that the mass of types (x_0, y_0) is well-defined and equal to the mass of types (y_0, x_0): for each $x \in X_0^2$,

$$\mu_t^X(x) = \mu_t^X(\overline{x}).$$

Additionally, the econometrician observes transition probabilities P_t and the discount factor β (and if not, that they can be identified in a way described in the main paper.)

1.3. **Equilibrium.** We describe the versions of the equations from Section 2 of the main paper. A (threshold) *strategy* is a mapping $\sigma : \mathbb{N} \times X \times A \to \mathbb{R}^+$, with the interpretation that agent x accepts opportunity a iff $\varepsilon \geq \sigma_t(x,a)$.

Let $r_t(x,a)$ denote the probability that opportunity a arrives to agent x in period t. In the case of meetings between two single agents, this probability is endogenously determined in equilibrium: for each $x_s, y_s \in X_S$,

$$r_t(x_s, y_s) = \frac{1}{\mu_t^X(x_s)} q_t\left(x_s, y_s; \mu_t^X\right) e^{-\sigma_t(y_s, x_s)}, \tag{2}$$

Let $U_t(x;\sigma)$ denote the present value of an agent who begins period t with type x and uses strategy σ. The expected continuation of an agent who realizes an opportunity a (or stays unmatched, if $a = \emptyset$) is equal to

$$U_t(x,a;\sigma) := \upsilon_t(x,a) + \beta \sum_{x'} P_t(x'|x,a) U_{t+1}(x';\sigma). \tag{3}$$

As in Section 2, we compute the present value to be equal to

$$U_t(x) := U_t(x,\emptyset;\sigma) + \sum_a r_t(x,a) e^{-\sigma_t(x,a)} \left[U_t(x,a;\sigma) - U_t(x,\emptyset;\sigma) + 1 + \sigma_t(x,a)\right]. \tag{4}$$

The equilibrium best response threshold is equal to

$$\sigma_t(x,a) = \max\left(0, U_t(x,\emptyset) - U_t(x,y)\right). \tag{5}$$

Let $\mu_t(x,y)$ denote the equilibrium mass of matches between agent types x and y in period t, and let $\mu_t(x,\emptyset)$ denote the mass of type x agents who are unmatched in period t. Let $\mu_t^X(x) = \sum_{a \in X \cup \{\emptyset\}} \mu_t(x,a)$ be the mass of all type x agents.

The masses of agents with realized opportunity a satisfy the following equations:

$$\mu_t(x,a) = \mu_t^X(x)\, r_t(x,a)\, e^{-\sigma_t(x,a)} \tag{6a}$$
$$= \begin{cases} q_t\left(a|x;\mu_t^X\right)\mu_t^X(x)\,\mathrm{e}^{-\sigma_t(x,a)} & \text{if } x\in X_S, a\in A_S,\\ q_t\left(x,a;\mu_t^X\right)\mathrm{e}^{-(\sigma_t(x,a)+\sigma_t(y,x))}, & \text{if } x,a\in X_S,\\ q_t\left(a|x;\mu_t^X\right)\mu_t^X(x)\,\mathrm{e}^{-(\sigma_t(x,a)+\sigma_t(\overline{x},a))} & \text{if } x\in X_M^2 \text{ and } a\in A_M,\end{cases},$$
$$\mu_t(x,\emptyset) = \mu_t^X(x) - \sum_a \mu_t(x,a) \text{ for each } x\in X. \tag{6b}$$

The population dynamics are given by

$$\mu_t^X(x') = Q_t(x') + \sum_{x\in X, a\in X\cup\{\emptyset\}} \mu_{t-1}(x,a)\, P_{t-1}(x'|x,a) \text{ for each } x' \text{ and } t, \tag{6c}$$

Definition 1. We say that tuple $(\mu_t,\sigma_t,U_t(.))_t$ is an *equilibrium* if the continuation values are determined through equations (3) and (4), the strategies satisfy (5) and if the masses evolve according to (6a)-(6c) for some initial distribution μ_0. The equilibrium is interior if $\sigma_t(x,a)>0$ for each x,a.

By (5), the equilibrium strategies are interior if

$$\sigma_t(x,a) = U_t(x,\emptyset) - U_t(x,a) > 0. \tag{7}$$

In the interior equilibrium, equation (4) implies that

$$U_t(x) = U_t(x,\emptyset) + \tilde{\mu}_t(x), \tag{8}$$

where we denote $\tilde{\mu}_t(x) = \frac{\sum_a \mu_t(x,a)}{\mu_t^X(x)} = 1 - \frac{\mu_t(x,\emptyset)}{\mu_t^X(x)}$ as the conditional probability that type x forms a match in period t. The above equation suggests that $\tilde{\mu}_t(x)$ can be interpreted as the (equilibrium) option value of being able to form a match in period t. Our data assumption implies that $\tilde{\mu}_t(x)$ is observed.

A recursive application of (3) and the above equation leads to the following formula for the continuation values given realized match $a\in X\cup\{\emptyset\}$:

$$U_t(x,a) = V_t^0(x,a) + \sum_{s>t}\beta^{s-t}\sum_{x'} P_t^s(x'|x,a)\,\tilde{\mu}_s(x'), \tag{9}$$

where $P_t^{t+1} = P_t$ and, for each $s > t+1$, we define

$$P_t^s (x_s|x,a) := \sum_{x_{s-1}} P_t^{s-1} (x_{s-1}|x,a) \, P_s (x_s|x_{s-1},\emptyset) \,, \text{ and}$$

$$V_t^0 (x,a) := \upsilon_t (x,a) + \sum_{s>t} \beta^{s-t} \sum_{x'} P_t^s (x'|x,a) \, \upsilon_s (x',\emptyset) \,. \tag{10}$$

The *individual dynamic surplus* is defined as

$$V_t (x,a) := V_t^0 (x,a) - V_t^0 (x,\emptyset) \,. \tag{11}$$

1.4. **Matching function.** We can finally state the matching function. For each type x and opportunity a, define

$$M_t (x,a) = \sum_{s>t} \beta^{s-t} \sum_{x'} \left[P_t^s (x'|x,a) - P_t^s (x'|x,\emptyset)\right] \tilde{\mu}_s (x')$$

as a measure of the impact generated by the opportunity a on the discounted probability of future matches. Notice that $M_t (x,a)$ is directly observed from the data given our assumptions.

Equations (6a), (7), (9), and (11) imply that

- if $x \in X_0$ and $a \in A_S$, then
$$\log \mu_t (x,a) = \log q_t (a|x) + V_t (x,a) + \log \mu_t^X (x) + M_t (x,a) \,,$$
- if $x, y \in X_0$, then
$$\log \mu_t (x,y) = \log q_t \left(x,y;\mu_t^X\right) + V_t (x,y) + V_t (y,x) + M_t (x,y) + M_t (y,x) \,,$$
- if $x \in X_0^2$ and $a \in A_M$, then
$$\log \mu_t (x,a) = \log q_t (a|x) + V_t (x,a) + V_t (\overline{x},a) + \log \mu_t^X (x) + M_t (x,a) + M_t (\overline{x},a) \,.$$

The above equations allow us to identify the individual (in the first case) and joint (in the latter two cases) dynamic surplus confounded by the meeting rate or arrival rate of opportunities.

2. Separation in many-to-one matching

An important example of a post-match behavior is a decision to separate the match. Here, we show that such a decision can be incorporated into a version of the many-to-one matching model.

2.1. **Model.** We consider a version of the many-to-one matching model. Time is discrete $t = 1, 2, ...$ There is a continuum population of firms and workers. Each worker has a permanent characteristic $c \in W_0$ that corresponds to his or her education level, race, gender, etc. Similarly, each firm has a permanent characteristic $f_0 \in F_0$. The sets of characteristics W_0 and F_0 are finite.

Firm type is a tuple $f = \left(f_0, (n_c)_{c \in C}\right) \in F = F_0 \times \mathbb{N}^{W_0}$, where the first coordinate is the characteristic of the firm, and the second is its employment composition. Let $n(f) = \sum_c n_c$ be the total employment of a type f firm; we assume an upper bound $n(f) \leq N_0 < \infty$.

Worker type is a tuple $w = (c, f) \in W = W_0 \times (F \cup \{\emptyset\})$ of the worker's characteristic and her or his employment status f, where $f = \emptyset$ means that the worker is unemployed, and $f \neq \emptyset$ is the type of her or his employer. For simplicity, we assume that workers and firms live forever and no agent is born. (It is easy to allow for endogenous entry and exit - see Section 3 below).

Each period, one of the existing matches can be severed with probability $\delta < 1/N_0$. To simplify the analysis, we assume that for each firm, at most one match can be severed per period. If a match is severed, the agents have an opportunity to renew their relationship. This happens in the same way as the original decision to form a match: they observe exponential shock that they will receive if the match is renewed, and simultaneously decide whether to renew the match. If a match is severed and at least one agent decides not to continue the match, the worker becomes unemployed, and the firm loses a worker.

With the remaining probability, $1 - \delta n(f)$, the firm is on the hiring market. Such a firm can meet an employed worker. The mass of meetings between type f firm and worker with characteristics c is equal to

$$q_t(c, f) := q_t \mu_t^X(c, \emptyset)(1 - \delta n(f)) \mu_t^X(f).$$

Here, $q_t > 0$ is a coefficient. The agents observe an exponential payoff shock and simultaneously decide whether they want to form a match. The match is formed only if both agents accept.

Each worker with type $w \in W$ receives payoff $\upsilon_t^W(w)$; the payoff can depend on the employment status of the worker. Type f firm receives payoff $\upsilon_t^F(f)$. Payoffs are received at the beginning of the period and agents discount the future with factor $\beta < 1$.

2.2. **Data.** Each variable with subscript t is measured at the beginning of the period.

The econometrician observes (a) masses $\mu_t^X(x)$ of agents of type $x \in W \cup F$, (b) masses of new hirings $\mu_t((c, \emptyset), f)$ of unemployed workers c by firms f, and (c) masses of separations $s_t(f, c)$ of firms f and workers c. Let

$$\tilde{\mu}_t(f) = \frac{\sum_c \mu_t(f, (c, \emptyset))}{\mu_t^X(f)},$$
$$\tilde{\mu}_t(c) = \frac{\sum_f \mu_t(f, (c, \emptyset))}{\mu_t^X(c)},$$

be, respectively, the hiring rate of firm f (i.e., the mass of new hires done by firms f divided by the mass of firms f), and the probability that an unemployed worker gets hired. Similarly, define

$$\tilde{s}_t(f) = \frac{\sum_c s_t(f, c)}{\mu_t^X(f)},$$
$$\tilde{s}_t(c, f) = \frac{s_t(f, c)}{\mu_t^X(f)\, n_c(f)}$$

as, respectively, the average number of workers lost by firm f in period t, and the probability that an employed worker c separates from his or her employer f.

2.3. **Interior equilibrium.** A (threshold) *strategy* is a mapping $\sigma_{t,p} : W \times F \to \mathbb{R}^+$ for each $t \in \mathbb{N}$) with the interpretation that agent x accepts the match with y iff $\varepsilon \geq \sigma_t(x, y)$. The subscript $p = h, s$ denotes the threshold used for hiring and separations. Hecnerforth, x and y may refer to both a worker and a firm whenever the equation has a symmetric form for the two sides of the market.

Let $U_t(x; \sigma)$ denote the interior equilibrium present value of an agent who begins period t as type x using strategy σ. Let $U_{t,p}(x) = \max_\sigma U_{t,p}(x; \sigma)$ denote the continuation values. We also denote

$$\begin{aligned}
(c,\emptyset)+f &\to (c,f)\,,\\
(c,\emptyset)+\emptyset &\to (c,\emptyset)\,,\\
(c,f)-f &\to (c,\emptyset)\,,\\
(f_0,(n_{c'}))+c &\to (f_0,(n_{c'},n_c+1))\,,\\
(f_0,(n_{c'}))-c &\to (f_0,(n_{c'},n_c-1))\,.
\end{aligned}$$

In an interior equilibrium,

$$\sigma_{t,h}(x,y)=\sigma_{t,s}(x,y)=\beta\left(U_{t+1}(x)-U_{t+1}(x+y)\right)>0. \tag{12}$$

2.3.1. *Firms.* Let

$$r_{t,m}(f,c)=q_t\mu_t^X(c,\emptyset)\,\mathrm{e}^{-\sigma_t(c,f)}=\frac{\mu_t\left((c,\emptyset),f\right)}{\left(1-\delta n(f)\right)\mu_t^X(f)}e^{\sigma_t(f,c)}$$

denote the probability that, in period t, firm f meets an unemployed worker $(c,\emptyset)$ who accepts the match. For firm f, we have

$$\begin{aligned}
U_t(f):=&v_t^F(f)+\left(1-\delta n(f)\right)\sum_{c\in C}r_{t,m}(f,(c,\emptyset))\,\mathrm{e}^{-\sigma_{t,m}(f,(c,\emptyset))}\left[\beta U_{t+1}(f+c)+E\left(\varepsilon|\varepsilon\geq\sigma_t(f,(c,\emptyset))\right)\right] \qquad (13)\\
&+\left(1-\delta n(f)\right)\left(1-\sum_{c\in C}r_{t,m}(f,(c,\emptyset))\,\mathrm{e}^{-\sigma_{t,m}(f,(c,\emptyset))}\right)\beta U_t(f)\\
&+\delta\sum_c n_c(f)\,\mathrm{e}^{-\sigma_{t,e}(f-c,(c,\emptyset))-\sigma_{t,e}((c,\emptyset),f-c)}\left[\beta U_{t+1}(f)+E\left(\varepsilon|\varepsilon\geq\sigma_t(f-c,c)\right)\right]\\
&+\delta\sum_c n_c(f)\left(1-\mathrm{e}^{-\sigma_{t,e}(f-c,(c,\emptyset))-\sigma_{t,e}((c,\emptyset),f-c)}\right)\beta U_{t+1}(f-c)\\
=&v_t^F(f)+\delta n(f)+\tilde{\mu}_t(f)-\tilde{s}_t(f)+\beta\left(1-\delta n(f)\right)U_t(f)+\delta\sum_c n_c(f)\,\beta U_{t+1}(f-c)\\
=&v_t^F(f)+\delta n(f)+\tilde{\theta}_t(f)+\beta\left(1-\delta n(f)\right)U_t(f)+\delta\sum_c n_c(f)\,\beta U_{t+1}(f-c)\,,
\end{aligned}$$

where we define the net hiring rate of firm f as

$$\tilde{\theta}_t(f)=\tilde{\mu}_t(f)-\tilde{s}_t(f)\,.$$

Define

$$P_t^F(f'|f) = \begin{cases} \beta(1-\delta n(f)) & \text{if } f' = f, \\ \beta\delta n_c(f) & \text{if } f' = f - c, \\ 0 & \text{otherwise.} \end{cases}$$

Define $P_t^{F,s}(f'|f)$ as in thePeski (2019). For any sequence of functions $g_t : F \to R$, let

$$\left(\Pi_t^F g\right)(f) = \sum_{s\geq t}\sum_{f'} P_t^{F,s}(f'|f)\, g_s(f')$$

be the present value of the stream of payoffs from function g_t if the firm never hires and never re-hires any worker. Then, recursively applying (13), we obtain

$$U_t(f) = \Pi_t^F\left(\upsilon^F + \delta n\right)(f) + \Pi_t^F\tilde{\theta}(f). \tag{14}$$

2.3.2. *Workers.* For an unemployed worker $(c, \emptyset)$, we have

$$U_t(c,\emptyset) = \upsilon_t^W(c,\emptyset) + \beta U_{t+1}(c,\emptyset) + \tilde{\mu}_t(c), \tag{15}$$

where $\tilde{\mu}_t(c)$ is the probability of being hired. For an employed worker (c, f), we have

$$\begin{aligned} U_t(c,f) &= \upsilon_t^W(c,f) + (1-\delta)\beta U_{t+1}(c,f) \\ &\quad + \delta e^{-\sigma_{t,e}(f-c,(c,\emptyset))-\sigma_{t,e}((c,\emptyset),f-c)}\left(U_{t+1}(c,\emptyset) + E(\varepsilon|\varepsilon \geq \sigma_t(c,f-c))\right) \\ &\quad + \delta\left(1 - e^{-\sigma_{t,e}(f-c,(c,\emptyset))-\sigma_{t,e}((c,\emptyset),f-c)}\right)U_{t+1}(c,\emptyset) \\ &= \upsilon_t^W(c,f) + (1-\delta)\beta U_{t+1}(c,f) + \delta\beta U_{t+1}(c,\emptyset) + \delta e^{-\sigma_{t,e}(f-c,(c,\emptyset))-\sigma_{t,e}((c,\emptyset),f-c)} \\ &= \upsilon_t^W(c,f) + \delta - \tilde{s}_t(c,f) + \beta(1-\delta)U_{t+1}(c,f) + \beta\delta U_{t+1}(c,\emptyset), \end{aligned} \tag{16}$$

where we used equations (18) below.

Define

$$P_t^W(w'|w) = \begin{cases} \beta(1-\delta) & \text{if } w' = w = (c,f) \text{ and } f \neq \emptyset, \\ \beta\delta & \text{if } w' = (c,\emptyset) \neq w = (c,f), \\ 1 & \text{if } w' = w = (c,\emptyset), \\ 0 & \text{otherwise.} \end{cases}$$

Define $P_t^{W,s}(f'|f)$ and, for any function $g_t(f)$, define

$$\Pi_t^W g(f) = \sum_{s\geq t}\sum_{f'} P_t^{W,s}(f'|f)\, g_s(f'),$$

as above. Let

$$\tilde{\theta}_t^W(c,f) = \begin{cases} -\tilde{s}_t(c,f) & \text{if } f \neq \emptyset, \\ \tilde{\mu}_t(c) & \text{if } f = \emptyset. \end{cases}$$

Then, recursively applying (15) and (16), we obtain

$$U_t(c,f) = \Pi_t^W\left(\upsilon^W + \delta\right)(c,f) + \Pi_t^W \tilde{\theta}_t^W(c,f). \tag{17}$$

2.3.3. *Population.* In the population, we have

$$\begin{aligned} \mu_t((c,\emptyset),f) &= q_t \mu_t^X(c,\emptyset)(1-\delta n(f))\,\mu_t^X(f)\,\mathrm{e}^{-\sigma_t((c,\emptyset),f)-\sigma_t(f,(c,\emptyset))}, \\ s_t(f,c) &= \delta \mu_t^X(f)\,n_c(f)\left(1-\mathrm{e}^{-\sigma_t(f-c,(c,\emptyset))-\sigma_t((c,\emptyset),f-c)}\right). \end{aligned} \tag{18}$$

The first equation describes the evolution of formed matches; the second equation describes the separations. Some algebra leads to the following equality

$$\mathrm{e}^{-\sigma_t((c,\emptyset),f)-\sigma_t(f,(c,\emptyset))} = \frac{\mu_t((c,\emptyset),f)}{q_t \mu_t^X(c,\emptyset)(1-\delta n(f))\,\mu_t^X(f)} = 1 - \frac{1}{\delta}\frac{s_t(f+c,c)}{\mu_t^X(f+c)\,n_c(f+c)}. \tag{19}$$

2.4. **Matching function.** The evolution equations (18) and the best response equation (12) imply that

$$\begin{aligned} &\log \mu_t((c,\emptyset),f) \\ =&\log \mu_t^X(c,\emptyset) + \log(1-\delta n(f)) + \log \mu_t^X(f) + \log q_t \\ &+ \beta\left(U_{t+1}(f+c) - U_{t+1}(f) + U_{t+1}(c,f) - U_{t+1}(c,\emptyset)\right). \end{aligned}$$

Let

$$\begin{aligned} V_t^F(f,c) &= \beta\left(\Pi_{t+1}^F(\upsilon+\delta n)(f+c) - \Pi_{t+1}^W(\upsilon+\delta n)(f)\right) \\ V_t^W(c,f) &= \beta\left(\Pi_{t+1}^W(\upsilon+\delta)(c,f) - \Pi_{t+1}^W(\upsilon+\delta)(c,\emptyset)\right). \end{aligned}$$

Then, V_t^F and V_t^W are the reduced parameters of the model in the sense of Section 2.6 of Peski (2019) and Lemma 1 therein.

Using the recursive formulas for the payoffs of firm (14) and worker (17), we obtain the matching function:

$$\begin{aligned}
&\log \mu_t \left(\left(c,\emptyset\right),f\right) \qquad (20)\\
=&\log \mu_t^X \left(c,\emptyset\right) + \log\left(1-\delta n\left(f\right)\right) + \log \mu_t^X\left(f\right) + \log q_t \\
&+\beta \Pi_{t+1}^W \tilde{\theta}\left(f+c\right) - \beta \Pi_{t+1}^W \tilde{\theta}\left(f\right) + \beta \Pi_{t+1}^W \tilde{\theta}_t^W\left(c,f\right) - \beta \Pi_{t+1}^W \tilde{\theta}_t^W\left(c,\emptyset\right) \\
&+\beta\left(V_t^F\left(f,c\right) + V_t^W\left(c,f\right)\right).
\end{aligned}$$

The interpretation of the function is the same as in the original model.

If q_t and δ are known, the matching function can be used to identify the reduced form parameters. Goussé *et al.* (2017) suggest to identify these two variables from equality (19).

3. Model with $k \geq 1$ agent events

Here, we consider an extension in which, additionally to 2-agent matches, the agents are allowed to form multi-agent partnerships, and also the agents make single-agent decisions. An important example of the latter is a decision to enter or exit from the market.

3.1. **Model.** Time is discrete $t = 1, 2, \ldots$.. There is a continuum population of agents. In each period, each agent is characterized by type $x \in X$, where X is a finite set. The type is typically not permanent, and it may change depending on the agent's behavior. In each period, a mass $Q_t(x)$ of agents are born.

In each period, a type x agent may encounter the following opportunities.

- *single-agent event:* an opportunity $\alpha \in A_0$ drawn from a finite set with a probability distribution $q_t(.|x) \in \Delta A_0$. The agent observes an exponential payoff shock that he or she will receive only if she or he accepts the opportunity. The agent decides whether or not to accept the opportunity. If the agent accepts the opportunity, we denote the realized opportunity as $a = \alpha$; otherwise, we take $a = \emptyset$.
- *multi-agent event*: The agent can meet one or more other agents at random. The mass meetings between agents of type x_1 and $x_2, \ldots, x_k$ is given by meeting function $q_t^k\left(x_1, x_2, \ldots, x_k; \mu^X\right)$, where we assume that the value of the meeting function does not depend on the ordering of agents (i.e., $q_t^k\left(.; \mu_t^X\right)$ is invariant

to permutations). All agents observe i.i.d. exponential payoff shocks that they will receive only if all of them accept the match. Each agent simultaneously decides whether to accept the match. If the match is accepted by all agents, we take $a = (x_2, ..., x_k)$; otherwise, we take $a = \emptyset$. We assume an upper bound K on the size of the multi-agent event.

The set of all possible (realized) opportunities is equal to $A = A_0 \cup \bigcup_{2 \leq k \leq K} X^{k-1} \cup \{\emptyset\}$. Let

$$q_t\left(x, a; \mu_t^X\right) = \begin{cases} q_t\left(a|x\right) \mu_t^X\left(x\right) & \text{if } a \in A_0, \\ q_t\left(x, y_1, ..., y_{k-1}; \mu_t^X\right) & \text{if } a = y_1, ..., y_{k-1} \in X^{k-1}. \end{cases}$$

The agent receives a payoff $\upsilon_t(x, a)$ plus, if $a \neq \emptyset$, the observed payoff shock. Next, the agent's new type is drawn from a distribution $P_t(.|x, a)$ that may depend on her own current type as well as the type of her match partner. Agent x dies with probability $1 - \sum_{x'} P_t(x'|x, a)$, where $a \in A$. For each $a = (x_2, ..., x_k) \in X^{k-1}$, neither the payoffs nor the transition probabilities depend on the ordering in the tuple. Agents discount the future with a factor $\beta < 1$.

The model is completely characterized by the systematic utility υ, birth rates Q, transition probabilities P, and the meeting function q_t.

3.2. **Data.** Each variable with subscript t is measured at the beginning of the period.

The econometrician observes (a) masses $\mu_t^X(x)$ of agents of type $x \in X$ and (b) masses $\mu_t(x, a)$ of newly accepted opportunities.

3.3. **Equilibrium.** A (threshold) *strategy* is a mapping $\sigma : \mathbb{N} \times X \times A \to \mathbb{R}^+$, with the interpretation that agent x accepts opportunity a iff $\varepsilon \geq \sigma_t(x, a)$. The restriction to threshold strategies is w.l.o.g.

Let $r_t(x, a)$ denote the probability that opportunity $a \in A$ arrives. This probability is equal to

$$r_t(x, a) = \begin{cases} \frac{1}{\mu_t^X(x)} q_t\left(x, a; \mu_t^X\right) & \text{if } a \in A_0, \\ \frac{1}{\mu_t^X(x)} q_t\left(x, a; \mu_t^X\right) e^{-\sum_{l \leq k} \sigma_t(y_l, (x, y_{-l}))} & \text{if } a = (y_1, ..., y_{k-1}). \end{cases} \tag{21}$$

Let $U_t(x; \sigma)$ (resp., $U_t(x, a; \sigma)$) denote the present value of an agent who begins period t in state x (resp., the present value of an agent who is state x and who realizes opportunity a in period t) and uses strategy σ. Then,

$$U_t(x,a;\sigma) := \upsilon_t(x,a) + \beta \sum_{x'} P_t(x'|x,a)\, U_{t+1}(x';\sigma), \tag{22}$$

and

$$\begin{aligned} U_t(x;\sigma) :=& \sum_{y\in X} r_t(x,y)\, e^{-\sigma_t(x,y)} \left[U_t(x,y;\sigma) + E(\varepsilon|\varepsilon \geq \sigma_t(x,y))\right] \\ &+ \left(1 - \sum_{y\in X} r_t(x,y)\, e^{-\sigma_t(x,y)} U_t(x,\emptyset;\sigma)\right) \\ =& U_t(x,\emptyset;\sigma) + \sum_{y\in X} r_t(x,y)\, e^{-\sigma_t(x,y)} \left[U_t(x,y;\sigma) - U_t(x,\emptyset;\sigma) + 1 + \sigma_t(x,y)\right], \end{aligned} \tag{23}$$

where the *best response* strategy has the form:

$$\sigma_t(x,a) = \max\left(0, U_t(x,\emptyset) - U_t(x,a)\right). \tag{24}$$

The symmetry of the payoffs and transition rates imply that, for $a = (x_2, ..., x_k)$, neither the continuation payoff $U_t(x,a)$ nor the best response depends on the ordering in the tuple.

The masses of agents and opportunities are determined according to the following equations: for each $x_1 \in X$,

$$\begin{aligned} \mu_t(x_1,a) &= \mu_t^X(x_1)\, r_t(x_1,a)\, e^{-\sigma_t(x_1,a)} \qquad (25a) \\ &= \begin{cases} q_t\left(x_1,a;\mu_t^X\right) e^{-\sigma_t(x_1,a)} & \text{if } a \in A_0, \\ q_t\left(x_1,...,x_k;\mu_t^X\right) e^{-\sum_{l\leq k}\sigma_t(x_l,(x_{-l}))} & \text{if } a = (x_2,...,x_k) \in X^{k-1} \end{cases} \\ \mu_t(x,\emptyset) &= \mu_t(x) - \sum_y \mu_t(x,y) \ \text{ for each } x \in X. \qquad (25b) \end{aligned}$$

The population dynamics are given by

$$\mu_t^X(x') = Q_t(x') + \sum_{x\in X, a\in A} \mu_{t-1}(x,a)\, P_{t-1}(x'|x,a) \ \text{ for each } x' \text{ and } t. \tag{25c}$$

Definition 2. We say that $(\mu_t, \sigma_t, U_t(.))_t$ are an *equilibrium* if equations (22), (23), (24), and (25a)-(25c) hold for some initial distribution μ_0. The equilibrium is stationary if the equilibrium variables do not depend on time, with $\mu_0 = \mu_t$ determined in equilibrium. An equilibrium is interior if $\sigma_t(x,y) > 0$ for all $x, y \in X$ and $t \geq 0$. An equilibrium is *interior* if σ is interior.

In the interior equilibrium, $\sigma_t(x,y) = U_t(x,\emptyset) - U_t(x,y) > 0$ and

$$\begin{aligned} U_t(x) =& U_t(x,\emptyset) + \sum_{y\in X} r_t(x,y)\, e^{-\sigma_t(x,y)} \\ =& U_t(x,\emptyset) + \frac{\sum_{y\in X}\mu_t(x,y)}{\mu_t^X(x)} = U_t(x,\emptyset) + \tilde{\mu}_t(x), \end{aligned}$$

where we denote $\tilde{\mu}_t(x) = \sum_{a\in A} \frac{\mu_t(x,a)}{\mu_t(x)} = 1 - \frac{\mu_t(x,\emptyset)}{\mu_t(x)}$ as the conditional probability that type x realizes some opportunity in period t.

A recursive application of (22) and the above equation leads to the following formula for the continuation values given realized match $a \in A$:

$$\begin{aligned} U_t(x,a) =& \upsilon_t(x,a) + \beta \sum_{x_{t+1}} P_t(x_{t+1}|x,a)\left(U_{t+1}(x_{t+1},\emptyset) + \tilde{\mu}_{t+1}(x_{t+1})\right) \qquad (26) \\ =& \upsilon_t(x,a) + \beta \sum_{x_{t+1}} P_t(x_{t+1}|x,a)\left(\upsilon_{t+1}(x_{t+1},\emptyset) + \tilde{\mu}_{t+1}(x_{t+1})\right) \\ &+ \beta^2 \sum_{x_{t+1},x_{t+2}} P_t(x_{t+1}|x_t,a)\, P_t(x_{t+2}|x_{t+1},\emptyset)\left(U_{t+2}(x_{t+2},\emptyset) + \tilde{\mu}_{t+2}(x')\right) \\ =& \ldots \\ =& V_t^0(x,a) + \sum_{s>t}\beta^{s-t}\sum_{x'} P_t^s(x'|x,a)\,\tilde{\mu}_s(x'), \end{aligned}$$

where $P_t^{t+1} = P_t$ and, for each $s > t+1$, we recursively define

$$P_t^s(x_s|x,a) := \sum_{x_{s-1}} P_t^{s-1}(x_{s-1}|x,a)\, P_s(x_s|x_{s-1},\emptyset)$$

as the s-period probability distribution of agent types given that she or he has type x in period t, forms a realized match a, and does not forms any other match before period s. Further, define

$$V_t^0(x,a) := \upsilon_t(x,a) + \sum_{s>t}\beta^{s-t}\sum_{x'} P_t^s(x'|x,a)\,\upsilon_s(x',\emptyset)$$

as the present expected value of the stream of systematic utility generated by a strategy of never forming a match. For each $x \in X$ and $a \in X \cup \{\emptyset\}$, let

$$V_t(x,a) := V_t^0(x,a) - V_t^0(x,\emptyset)$$

be the increase in the present value due to match a. The symmetry assumptions imply that $V_t(x,a)$ does not depend on the ordering of agents in a multi-agent event.

3.4. **Matching function.** The above equations lead to two kinds of "matching function:

- for single-agent event $a \in A_0$, we obtain

$$\log \mu_t (x, a) = \log q_t \left(x, a; \mu_t^X\right) + V_t (x, a) + \sum_{s>t} \beta^{s-t} \sum_{x'} \left[P_t^s (x'|x, a) - P_t^s (x'|x, \emptyset)\right] \tilde{\mu}_s (x') .$$

  This allows us to identify the individual dynamic surplus $V_t (x, a)$,

- for multi-agent events $x_1 \in X, a \in X^{k-1}$, we obtain

$$\log \mu_t (x_1, (x_2, ..., x_k)) = \log q_t \left(x_1, ..., x_k; \mu_t^X\right) + \sum_l V_t (x_l, x_{-l}) + \sum_l \sum_{s>t} \beta^{s-t} \sum_{x'} \left[P_t^s (x'|x_l, x_{-l}) - P_t^s (x'|x, \emptyset)\right] \tilde{\mu}_s (x') .$$

  The above equation allows us to identify the joint dynamic surplus $\sum_l V_t (x_l, x_{-l})$.

References

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